Algebra

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Algebra is often thought as a series of 'rules' which one must learn off or practise until they become second nature. However in reality this need not be the case. Mathematics is the study of the consequences of a set of axioms. Axioms are statements we *assume without proof*. As long as the axioms are well defined and not self-contradictory we can deduce the consequences of them. Algebra is simply the consequences of the following axioms. The set of real numbers, $\mathbb{R}$, is just all the numbers on the number line.

The Axioms of Arithmetic in $\mathbb{R}$

Closure

For any x, $y \in \mathbb{R}$;

$$x + y \in \mathbb{R},$$
$$x \times y \in \mathbb{R}.$$

Commutivity

For any x, $y \in \mathbb{R}$;

$$x + y = y + x,$$
$$x \times y = y \times x.$$

Associativity

For any x, y, $z \in \mathbb{R}$;

$$x + (y + z) = (x + y) + z,$$
$$x \times (y \times z) = (x \times y) \times z.$$

Identity

There is a special real number $0 \in \mathbb{R}$ such that:

$$0 + x = x,$$

for every $x \in \mathbb{R}$. Also there is a special number $1 \in \mathbb{R}$ $(1 \neq 0)$ such that:

$$1 \times x = x,$$

for every $x \in \mathbb{R}$.

Subtraction and Division

For every number x there corresponds a number $-x \in \mathbb{R}$ such that:

$$x + (-x) = 0.$$

Also if $x \neq 0$ there is a number $x^{-1} \in \mathbb{R}$ ($x^{-1} = 1/x$) such that:

$$x \times x^{-1} = 1.$$

Distributive Law

For x, y, $z \in \mathbb{R}$,

$$x \times (y + z) = (x \times y) + (x \times z).$$

We can deduce any and all of the 'rules of algebra' from these axioms. Some of the rules which you think are 'obvious' may be proved to be a consequence of the axioms ($0 \times x = 0$). Some examples:

Proposition (Multiplication by Zero)

For all $x \in \mathbb{R}$, $0 \times x = 0$.

Proof

By identity, $0 = 0{+}0$. Therefore $x{\times}0 = x{\times}(0{+}0)$. By distributivity, $x{\times}(0{+}0) = x{\times}0{+}x{\times}0$. $x \times 0$ has an additive inverse $-(x \times 0)$:

$$\begin{aligned} x \times 0 &= x \times 0 + x \times 0 \\ \underset{+(-(x\times 0))}{\Rightarrow} \underbrace{x \times 0 + (-(x \times 0))}_{=0} &= x \times 0 + \underbrace{x \times 0 + (-(x \times 0))}_{=0} \\ \Rightarrow 0 &= x \times 0 + 0 = x \times 0 \\ \Rightarrow x \times 0 &= 0 \ \bullet \end{aligned}$$

Proposition (Division by Zero is Contradictory)

There does not equal a real number equal to $1/0$. (Division by any number y may is realised as multiplication by $1/y$.)

Proof

Assume there exists a real number $1/0$. Now by the previous theorem:

$$0 \times \frac{1}{0} = 0 \tag{1}$$

However, by inverses:

$$0 \times \frac{1}{0} = 1 \tag{2}$$

Hence if $1/0$ exists $1 = 0$. This contradicts the assumption that $1 \neq 0$. Hence $1/0$ does not exist. (i.e. division by zero is not allowed) $\bullet$

Proposition (Anything over anything (!) is equal to 1)

For all $a \neq 0$,

$$\frac{a}{a} = 1 \tag{3}$$

Proof

Now

$$\frac{a}{a} = a \times \frac{1}{a} = 1 \bullet$$

Proposition (Canceling above and below)

Suppose $a, b, c \in \mathbb{R}$, $b, c \neq 0$. Then

$$\frac{ac}{bc} = \frac{a}{b} \tag{4}$$

Proof

Now $c \neq 0$ so there exists a number $1/c$ such that $c \times 1/c = 1$.From the last proposition:

$$\frac{1/c}{1/c} = 1.$$

Now by the axioms, $x \times 1 = x$:

$$\frac{ac}{bc} = \frac{ac}{bc} \times \underbrace{\frac{1/c}{1/c}}_{=1}$$

$$= \frac{a\,\overbrace{c \times 1/c}^{=1}}{b\,\underbrace{c \times 1/c}_{=1}}$$

$$= \frac{a \times 1}{b \times 1} = \frac{a}{b} \bullet$$

Remark

Careful!! Let $a, b, c, d \in \mathbb{R}$, $b, c \neq 0$. The following move is nonsense (unless $d = 0$):

$$\frac{ac + d}{bc} = \frac{a\not{c} + d}{b\not{c}} = \frac{a + d}{b}$$

Nowhere in the axioms does it say we can make this cancelation. The above theorem is very precise — it only allows cancelations as above.

Proposition

For all $x \in \mathbb{R}$*:*

$$-x = (-1)x \tag{5}$$

The notation ab *of course means* $a \times b$*.*

Proof

Now

$$\begin{aligned} x + (-1)x &= x(1) + x(-1) \\ &= x(1 + (-1)) \\ &= x(0) = 0 \end{aligned}$$

But $x + (-x) = 0$ also:

$$x + (-1)x = x + (-x)$$

Add $(-x)$ to both sides:

$$\begin{aligned} \underbrace{(-x) + x}_{=0} + (-1)x &= \underbrace{(-x) + x}_{=0} + (-x) \\ 0 + (-1)x &= 0 + (-x) \\ (-1)x &= -x \ \bullet \end{aligned}$$

Proposition (Minus by Minus is Plus)

$-(-1) = 1$*. Now* $-a \times -b = (-1)a(-1)b = (-1)(-1)ab = ab$*.*

Proof

What is the additive inverse of -1? We need to find a number $[-(-1)]$ such that:

$$(-1) + [-(-1)] = 0. \tag{6}$$

However we know that $1 + (-1) = 0 = (-1) + 1$ also:

$$(-1) + [-(-1)] = (-1) + 1$$

Add 1 to both sides:

$$\begin{aligned} \underbrace{1 + (-1)}_{=0} + [-(-1)] &= \underbrace{1 + (-1)}_{=0} + 1 \\ 0 + [-(-1)] &= 0 + 1 \\ -(-1) &= 1 \ \bullet \end{aligned}$$

Example

Simplify

$$\left[\frac{1+x}{x-1}-1\right]\div\frac{1}{1-x} \tag{7}$$

Solution: *This would be an absolutely perfect way to do this but most markers/ teachers/ etc. wouldn't do it like this.*

$$\begin{aligned}\left[\frac{1+x}{x-1}-1\right]\div\frac{1}{1-x} &= \frac{\frac{1+x}{x-1}-1}{\frac{1}{1-x}}\times\overbrace{\frac{1-x}{1-x}}^{=1}\\ &= \frac{\frac{1+x}{x-1}(1-x)-(1-x)}{\underbrace{\frac{1-x}{1-x}}_{=1}}\end{aligned}$$

Now $1-x=(-1)(x-1)$,

$$\begin{aligned}\left[\frac{1+x}{x-1}-1\right]\div\frac{1}{1-x} &= \frac{1+x}{\cancel{x-1}}(-1)\cancel{(x-1)}-(1-x)\\ &= -1-x-1+x=-2\end{aligned}$$

Going slightly beyond these 'facts', we have *identities*. Again these should not be thought of as rules but rather as provable consequences of the axioms. Now $a^n=\underbrace{a\times\cdots\times a}_{n\text{ times}}$. So $a^2=a\times a$. The second identity here is *not*

$$(x-y)^2=x^2-y^2 \tag{8}$$

The notation $(x-y)^2$ means (by convention) $(x-y)(x-y)$. If we want to say x^2-y^2 we will write x^2-y^2!

Proposition (Three Identities)

(i) *If $x,y\in\mathbb{R}$, neither equal to zero, then*

$$\frac{1}{x}+\frac{1}{y}=\frac{x+y}{xy} \tag{9}$$

(ii) For any $x,y\in\mathbb{R}$;

$$(x-y)^2=x^2-2xy+y^2 \tag{10}$$

(iii) Difference of two squares:

$$x^2-y^2=(x-y)(x+y) \tag{11}$$

Proof

(i)

$$\begin{aligned}\frac{1}{x}+\frac{1}{y} &= \frac{1}{x}\underbrace{\frac{y}{y}}_{=1}+\frac{1}{y}\underbrace{\frac{x}{x}}_{=1}\\ &= \frac{y}{xy}+\frac{x}{yx}\\ &= \frac{y}{xy}+\frac{x}{xy}\\ &= \frac{1}{xy}y+\frac{1}{xy}x\\ &= \frac{1}{xy}(y+x)\\ &= \frac{x+y}{xy}\ \bullet\end{aligned}$$

(ii)

$$\begin{aligned}(x-y)^2 &= (x+(-y))(x_(-y))\\ &= x(x+(-y))+(-y)(x+(-y))\\ &= x^2+x(-y)+(-y)x+(-y)(-y)\\ &= x^2+x(-1)y+x(-1)y+\underbrace{(-1)(-1)}_{=1}y^2\\ &= x^2-xy-xy+y^2\\ &= x^2-2xy+y^2\ \bullet\end{aligned}$$

(iii)

$$\begin{aligned}(x-y)(x+y) &= x(x+y)+(-y)(x+y)\\ &= x^2+xy+(-1)yx+(-y)y\\ &= x^2+xy-yx-y^2\\ &= x^2-y^2\ \bullet\end{aligned}$$

We have to be careful in what we think are identities. Alot of things...