MS 2001: Test 2 A

Name:
Student Number:

Answer all questions. Marks may be lost if necessary work is not clearly shown.
Remarks by me in italics and would not be required in a test - J.P.

Question 1

(a) Where are the following functions differentiable on $\mathbb{R}$? Please quote theorems/ rules used:

(i)
$$f_1(x) = \frac{x^2+2}{x^3-1}$$

(ii)
$$f_2(x) = \cos(x^2)$$

(iii)
$$f_3(x) = \sqrt{x-3} + \log x$$

(b) Consider the curve
$$x^3 + y^3 + xy = 3 \qquad (1)$$
Show that the point $(1, 1)$ is on the curve. Find the slope of the tangent to the curve at the point $(1, 1)$.

Solution

(a)

(i) x^2+2 and x^3-1 are both differentiable everywhere as polynomials. By the Quotient Rule f_1 is differentiable where $x^3-1 \neq 0 \Leftrightarrow x \neq 1$. Ans: $\mathbb{R}\backslash\{1\}$.

(ii) $\cos x$ is differentiable everywhere and x^2 is differentiable everywhere as a polynomial. f_2 is the composition of differentiable functions and hence by the Chain Rules is differentiable. Ans: $\mathbb{R}$.

(iii) $\sqrt{x-3}$ is the composition of the function $g(x) = \sqrt{x}$ and the differentiable everywhere function $h(x) = x-3$. Now $g(x)$ is differentiable for all $x > 0$ hence by the Chain Rule $g \circ h(x) = \sqrt{x-3}$ is differentiable for all $x - 3 > 0 \Rightarrow x > 3$. $\log x$ is differentiable for $x > 0$. Hence by the Sum Rule $f_3(x)$ is differentiable for $x > 3$. Ans: $(3, \infty)$.

(b) Firstly,

$$(1)^3 + (1)^3 + (1)(1) = 3$$

Hence $(1, 1)$ is on the curve.

Now, differentiating with respect to x:

$$\begin{aligned}
3x^2 + 2[y(x)]^2\frac{dy}{dx} + x.\frac{dy}{dx} + 1.[y(x)] = 0 \\
\frac{dy}{dx}\left[2y^2 + x\right] = -3x^2 - y \\
\frac{dy}{dx} = \left.\frac{-3x^2 - y}{2y^2 + x}\right|_{(x,y)=(1,1)} = \frac{-3(1)^2 - 1}{2(1)^2 + (1)} = -1
\end{aligned}$$

Question 2

Let $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x) = \begin{cases} x^2 + 1 & \text{if } x > 0 \\ x^3 + 1 & \text{if } x \leq 0 \end{cases} \tag{2}$$

You may assume that g is continuous. Show that g is differentiable on $\mathbb{R}$. Is g twice differentiable? Justify your answer.

Solution

The answer we give here is simpler than has been done in the past but requires two additional hypothesis. Please see the webpage for a proof that our method still works.
Away from 0 (*on the intervals* $(-\infty, 0), (0, \infty)$) g is a polynomial and hence differentiable:

$$g'(x) = \begin{cases} 2x & \text{if } x > 0 \\ 3x^2 & \text{if } x < 0 \end{cases} \tag{3}$$

Now

$$\lim_{x \to 0^+} g'(x) = \lim_{x \to 0^+} 2x = 0$$
$$\lim_{x \to 0^-} g'(x) = \lim_{x \to 0^-} 3x^2 = 0$$

Hence $\lim_{x \to 0} g'(x)$ exists (*and because it is bounded and g continuous at $x = 0$*), and thus g is differentiable at $x = 0$ and so differentiable for all $x \in \mathbb{R}$.

$$g''(x) = \begin{cases} 2 & \text{if } x > 0 \\ 6x & \text{if } x < 0 \end{cases} \tag{4}$$

Now

$$\lim_{x \to 0^+} g''(x) = \lim_{x \to 0^+} 2 = 2$$
$$\lim_{x \to 0^-} g''(x) = \lim_{x \to 0^-} 6x = 0$$

Hence $\lim_{x \to 0} g''(x)$ does not exist and hence g is not twice differentiable.

Question 3

1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is *differentiable* at a point $a \in \mathbb{R}$. Which of the following statements are true? (Circle the correct statement)

   (a) $f'(x) = 0$ for some $x \in \mathbb{R}$.
   $f(x) = x$ is differentiable at $x = 0$ but has derivative 1, $\forall\, x \in \mathbb{R}$.

   (b) $f'(a) = 0$.
   $f(x) = x$ is differentiable at $x = 0$ but has derivative $f'(0) = 1 \neq 0$.

   (c) f is continuous at $a \in \mathbb{R}$. ✓

   (d) f is not continuous at $a \in \mathbb{R}$.
   $f(x) = x$ is differentiable at $x = 0$ and is also continuous there.

2. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function and has a local maximum at $c \in \mathbb{R}$. Which of the following are true? (Circle the correct statement)

   (a) There exists an interval $I \subset \mathbb{R}$ such that $c \in I$ and $f(x) \leq f(c)$, $\forall\, x \in I$. ✓

   (b) $f'(c) = 0$
   $f(x) = -|x|$ has a local maximum at 0 but f' is undefined at 0 let alone equal to 0.

   (c) $f'(c) = 0$ *and* $f''(c) < 0$.

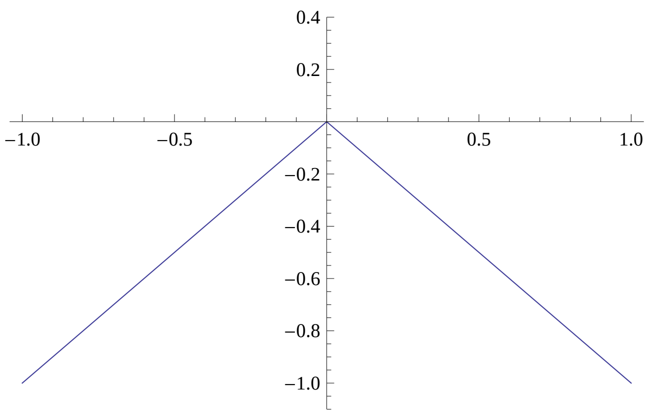

Figure 1: $f(x) = -|x|$ has a local maximum at $x = 0$, but $f'(0) \neq 0$.

   $f(x) = -|x|$ has a local maximum at 0 but f' and f'' are undefined at 0 let alone equal to 0 and negative.

   (d) None of the above.
   (a) is correct.

3. Which of the following functions satisfy the hypothesis of *Mean Value Theorem* on the closed interval $[0,1]$? (Circle the correct statement)

(a) f is differentiable on $(0,1)$ and $f(0) = f(1)$.
Let

$$f(x) = \begin{cases} x+1 & \textit{if } x \in (0,1) \\ 0 & \textit{otherwise} \end{cases}$$

Then for all $x \in (0,1)$, f *is differentiable, with* $f' = 1$. *Hence there is no point in* $c \in (0,1)$ *such that*

$$f'(c) = \frac{f(1) - f(0)}{1 - 0} = 0. \tag{5}$$

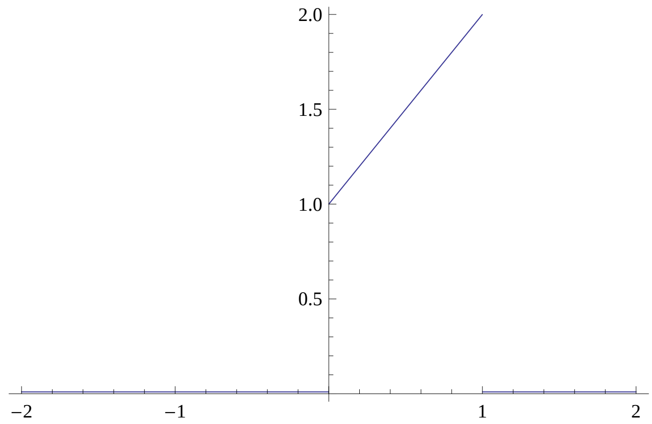

Figure 2: $f(x)$ is differentiable on $(0,1)$ but does not have a derivative equal to the 'average' ($\bar{f}'(x) = 0$) here.

(b) f is differentiable on $[0,1]$. ✓

(c) f is continuous at 0 and at 1 and $f(0) = f(1)$.
$f(x) = 1 - |2x - 1|$ *is continuous at* 0 *and* 1 *and* $f(0) = 0 = f(1)$ *but there does not exist a point* $c \in (0,1)$ *such that*

$$f'(c) = \frac{f(1) - f(0)}{1 - 0} = 0.$$

In fact on $(0, 1/2)$, $f' = 2$ *and on* $(1/2, 1)$, $f' = -2$.

(d) f is continuous on $[0,1]$.
$f(x) = 1 - |2x - 1|$ *is continuous on* $[0,1]$ *but there does not exist a point* $c \in (0,1)$ *such that*

$$f'(c) = \frac{f(1) - f(0)}{1 - 0} = 0. \tag{6}$$

In fact on $(0, 1/2)$, $f' = 2$ *and on* $(1/2, 1)$, $f' = -2$.

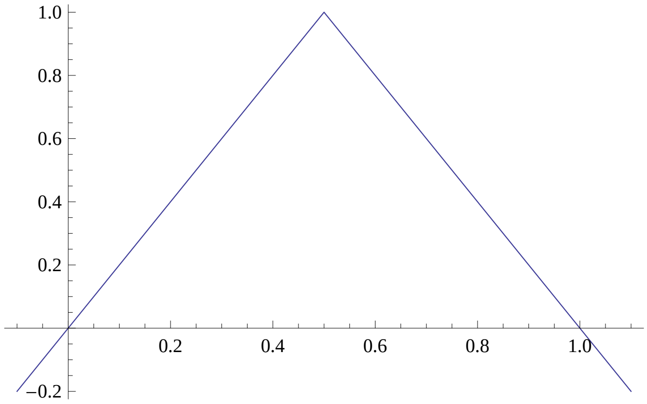

Figure 3: $f(x) = 1 - |2x - 1|$ is continuous on $[0, 1]$ but does not have a derivative equal to the 'average' ($\bar{f}'(x) = 0$) here.