MATH6014 - Technological Mathematics 1

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Contents

0.1 Introduction

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This page will comprise the webpage for this module and as such shall be the venue for course announcements including definitive dates for continuous assessments. This page shall also house such resources as a copy of these initial handouts, the exercises, a copy of the course notes, links, as well as supplementary material. Please note that not all items here are relevant to MATH6014; only those in the category 'MATH6014'. Feel free to use the comment function therein as a point of contact.

Module Objective

This module is designed to consolidate and develop student competence in using mathematical techniques for scientific and engineering programs.

Module Content

Basic Mathematics

Indices, logarithms, scientific notation, units. Transposition and evaluation of formulae.

Algebra

Formulation and solution of equations: linear, quadratic, cubic equations. Complex roots. Linear simultaneous equations. Partial fractions. Polynomial functions.

Linear Graphs

Reduction of non-linear relationships to linear form. Manipulation of data and plotting of graphs, evaluation and interpretation of constants.

Trigonometry

Trigonometric ratios, unit circle. Graphs of trigonometric functions (waveforms). Solution of trigonometric equations, sine and cosine rules.

Assessment

Total Marks 100: End of Year Written Examination 70 marks; Continuous Assessment 30 marks.

Continuous Assessment

I am considering having two equal weighted in-class tests to comprise the continuous assessment. The first in-class test would probably take place in Week 6 of term and the second in the final week. The webpage will contain the latest and definitive information about these. Absence from a test will not be considered accept in truly extraordinary cases. Plenty of notice will be given of the test date. For example, routine medical and dental appointments will not be considered an adequate excuse for missing the test.

Lectures

It will be vital to attend all lectures as although I intend that there will be a copy of the course notes available within the month, many of the examples, proofs, etc. will be completed by us in class.

Tutorials

There will be a tutorial every fortnight starting in the second week of term. There are many ways to learn maths. Two methods which arent going to work are

1. reading your notes and hoping it will all sink in
2. learning off a few key examples, solutions, etc.

By far and away the best way to learn maths is by doing exercises, and there are two main reasons for this. The best way to learn a mathematical fact/ theorem/ etc. is by using it in an exercise. Also the doing of maths is a skill as much as anything and requires practise.
There will be no shortage of exercises for you to try. The notes will contain exercises and the webpage may contain a link to a set of additional exercises. Past exam papers are fair game. Also during lectures there will be some things that will be *left as an exercise*. How much time you can or should devote to doing exercises is a matter of personal taste but be certain that effort is rewarded in maths.
I have not decided on what the format of the tutorials will be. The best format in my opinion is that ye ask questions which I then answer. However I understand that ye have a big workload so it might be better to give you time in class to complete exercises. In this format, I will be available in the class for 1-1 attention. No secrets will be divulged at tutorials and they are primarily for students who have questions about exercises. More general questions on course material shall be answered also.

Reading

Your primary study material shall be the material presented in the lectures; i.e. the lecture notes. Exercises done in tutorials may comprise further worked examples. While the lectures will present everything you need to know about MATH6014, they will not detail all there is to know. Further references are to be found in the library in or about section 510.2454. Good references include:

- M. Crockett & G. Dogett, 2003, Mathematics for Chemists, Vol.1 Ed., Royal Society of Chemistry.
- P. Tebbutt, 1998, Basic Mathematics, John Wiley & Sons.
- J.O.Bird, 2005, Basic Engineering Mathematics, 4th Edition Ed., Newnes.
- K.A. Stroud, 2007, Engineering Mathematics, 6th Edition Ed., Macmillan.

The webpage will contain supplementary material, and contains links and pieces about topics that are at or beyond the scope of the course. Finally the internet provides yet another resource. Even Wikipedia isnt too bad for this area of mathematics! You are encouraged to exploit these resources; they will also be useful for for further maths modules.

Exam

The exam format will roughly follow last year's. Acceding to the maxim that learning off a few key examples, solutions, etc. is bad and doing exercises is good, solutions to past papers shall not be made available (by me at least). Only by trying to do the exam papers yourself can you guarantee proficiency. If you are still stuck at this stage feel free to ask the question come tutorial time.

0.2 Motivation: Eight Applications of Mathematics in Industry

Figure 1: **Cryptography**: Internet security depends on *algebra and number theory*

Figure 2: **Aircraft and automobile design**: The behaviour of air flow and turbulence is modelled by *fluid dynamics* - a mathematical theory of fluids

Figure 3: **Finance**: Financial analysts attempt to predict trends in financial markets using *differential equations*.

Figure 4: **Scheduling Problems**: What is the best way to schedule a number of tasks? Who should a delivery company visit first, second, third,..? The answers to these questions and more like them are to be found using an area of maths called *discrete mathematics*.

Figure 5: **Card Shuffling**: If the casino doesn't shuffle the cards properly in Blackjack, tuned in players can gain a significant advantage against the house. Various branches of mathematics including *algebra*, *geometry* and *probability* can help answer the question: *how many shuffles to mix up a deck of cards?*

Figure 6: **Market Research**: Quite sophisticated mathematics underlies *statistics* - which are the basis on which many companies make very important decisions.

Figure 7: **Structural Design**: Materials technology is littered with equations, etc that then feeds into structural design. Flying buttresses were designed to balance these equations and hence the building itself.

Figure 8: **Medicine**: A load of modern medical equipment measures a patient indirectly by-way of measuring some signal that is emitted by the body. For example, a PET scan can tell a doctor how a sugar is going around your body: the patient takes ingests a mildly radioactive chemical that the body sees as sugar. As this chemical moves around the body it emits radiation which can be detected by the PET scanner. The mathematically-laden theory of *signal analysis* can be used to draw up a representation of what's going in the body.

Chapter 1

Basic Mathematics

1.0.1 Outline of Chapter

- Indices
- Logarithms
- Scientific Notation
- Units
- Equations and Basic Algebra
- Transposition and Evaluation of Formulae

1.1 Indices/ Powers

Introduction

Consider the natural counting numbers 1,2,3,4,... we first encountered these as a young child. Eventually when we got more proficient with counting we discovered that you could *add two numbers together to get another number.* For example, $2 + 3 = 5$. In school we learnt off our addition tables and were able to add small numbers together. A little further on we were introduced to *multiplication.* It is so easy to forget but multiplication is nothing but *repeated addition*, e.g.:

Once we got a handle on multiplication - i.e. after learning off another round of tables - we simply forgot about repeated addition and might have seen multiplication as quite removed from addition. In the realm of natural numbers at least, multiplication is nothing but a *notation* or *simplified way of writing* repeated addition. I ask who in their right mind, once we have written down and declared what $\times$ means, would continue to write

instead of the far more simple 13×19.

Often in mathematics *repeated multiplication* crops up. For example, the area of a rectangle:

In the case of a square, the length and width are equal and so has area

A similar story for the *volume* of a square box or *cube*:

Indices or *powers*, like multiplication, are nothing but a *notation* or *simplified way of writing* repeated multiplication. The difference this time is that there are no tables of indices to learn-off and we must not forget that indices or powers are nothing but repeated multiplication. For example the number:

is a number which we denote by 3^7. The terminology is that 3 is the *base* and 7 is the *exponent* or *power.*

In this course we will try to keep things as straightforward as possible however mathematics is a science of precision in which future results depend on past results. In this vein we will define precisely as many terms as need be:

Definitions

A *natural number* is an ordinary counting number 1,2,3,.... Here the dots signify that this list goes on forever.
The *set* or collection of all natural numbers $\{1, 2, 3, \dots\}$ we call $\mathbb{N}$ for short:

We will use the following notation to signify the sentence "*n is a natural number*":

said "*n in* $\mathbb{N}$" or "*n is an element of* $\mathbb{N}$". The symbol '$\in$' means 'in' or 'is an element of'.

The set of *integers* or *whole numbers* is the set of natural numbers together with 0 and all the negative numbers: $\dots, -3, -2, -1, 0, 1, 2, 3, \dots$. Once again the dots signify that this list continues indefinitely in both directions. We denote the set of "*integerZ*"" by:

The set of *fractions* (or *Quotients*) is the set of all ratios of the kind

Examples include $1/2, -4/3, 211/24$. We will see in Section 1.5 why we don't let $n = 0$ here (it corresponds to division by zero, which is contradictory). A shorthand for "*n is not zero*" is $n \neq 0$.
Like $\mathbb{N}$ and $\mathbb{Z}$ this is also an infinite set which we denote by:

A *real number* is *any* number that can be written as a decimal. Examples:

- $1 = 1.0$
- $3 = 3.0$
- $-5 = -5.0$
- $1/2 = 0.5$
- $2/3 = 0.6666...$
- $\sqrt{2} = 1.41421356...$
- $\pi = 3.14159265...$

In the context of this module a real number is just any number at all be it a natural number, negative number, fraction, square root, etc. For those of you looking to jump the gun *complex numbers* are not real numbers...
Again we write $x \in \mathbb{R}$ to signify "x is a real number".

Now that we have that sorted we can carefully define indices:

Definition

Let $a \in \mathbb{R}$ and $n \in \mathbb{N}$. Then

$$a^n = \underbrace{a \times a \times \cdots \times a}_{n\ multiplicands} \tag{1.1}$$

We can refer to a^n as "*a to the power of n*" and say a^n is "*a power of a*".
Finally a^2 is a *square* (*a-squared*), a^3 is a cube (*a-cubed*) and a^n is an *n-th power*.
For example, 3^7 is an example of a power of 3.

Basic Properties of Powers

Now that we have a shorthand way of writing repeated multiplication we must investigate how they combine together? What happens when we multiply together two powers? If we divide one power by another? What about a power of a power? What if we took two numbers, multiplied them together and took a power of that? We will see that answering these questions will raise more questions.

Before we start our investigation I want you to be aware of the following:

Definition

Let $a \in \mathbb{R}$ and $m, n \in \mathbb{N}$ (both m and n are natural numbers). Then a^m and a^n are *like powers*. Now nobody apart from me in a second calls this the Golden Rule of Powers, but it'll become clear very soon why it is indeed the case:

(The Golden Rule of Powers)

Multiplication of Like Powers

Consider 7^2 and 7^3. What is really happening when we multiply them together?

There is nothing special about 7^2 and 7^3 here. Indeed any real number a and natural numbers m, n could replace the roles of 7, 2 and 3:

Proof:

Try to get any kind of similar rule for unlike powers and you'll see what I mean by the Golden Rule.

Example: *Write* $4 \times 4 \times 16$ *as a power of* 4.

Division of Like Powers

Consider 10^5 and 10^2. What is really happening when we divide, say, 10^5 by 10^2:

Again there is nothing special about 10^5 and 10^2 here. Again let $a \in \mathbb{R}$ and $m, n \in \mathbb{N}$:

Proof

Example: *Write*

$$\frac{10^2 \times 10^5}{100}$$

in the form 10^n *for* $n \in \mathbb{N}$.

Repeated Powers?

What is $(9^3)^4$?

Again this is a general result. Let $a \in \mathbb{R}$ and $m, n \in \mathbb{N}$:

Proof

Example: *Write* $(4^3)^5$ *in the form* 2^n *for* $n \in \mathbb{N}$.

Powers of a Product

What about $(2 \times 5)^3$?

Again this is a general result. Let $a \in \mathbb{R}$ and $m, n \in \mathbb{N}$:

Proof

Example: *Let $x \in \mathbb{R}$. Write $4x^2$ as a square.*

Another similar case is that of a power of a fraction. Just to keep things as streamlined as possible I will introduce a small algebraic fact and combine some algebra and the the rule for a power of a fraction to explain the rest. Let $a, b \in \mathbb{R}$ with $b \neq 0$. Then

$$\frac{a}{b} \equiv a \div b \equiv a \times \frac{1}{b} \tag{1.2}$$

Now let $n \in \mathbb{N}$:

Zero Powers, Negative Powers & Fractional Powers

You may have noticed that our definition for indices/ powers has only defined powers when the exponent is a natural number. What about the following:

(P1) 2^0?

(P2) 3^{-8}??

(P3) $4^{1/2}$???

(P4) $5^{\sqrt{2}}$?????

Note that every natural number is an integer and that every integer is a fraction (e.g. $-11 = -11/1$). What we will do is construct definitions for zero, negative numbers and fractions that *extend the definition for natural numbers* and *make the laws of indices consistent.* In other words we will choose a definition for fractional powers[1] such that for all $a \in \mathbb{R}$, $m, n \in \mathbb{Q}$ the following make sense:

[1]will be for $a > 0$ only

If it makes sense for fractions it will also make sense for integers and natural numbers as these are fractions also. Note at this point we will not define all real powers. To define, for example, $5^{\sqrt{2}}$, in such a way that the definition extends that of natural numbers and the laws of indices still hold, is a bit trickier. The first odd thing about this is that $\sqrt{2} \notin \mathbb{Q}$ - that is $\sqrt{2}$ *cannot be written as a ratio of whole numbers.* The two ways that I know how to define $5^{\sqrt{2}}$ properly involve quite a bit of calculus or quite a bit of sequences and series. We don't worry too much about this - we know it can be done - you can work away with any real exponent as long as you follow the laws of indices!

Zero Powers

In Section 1.5 we will see that a crucial 'fact' about all non-zero numbers is that:

Now consider $2^5/2^5$, using the second law of indices:

Once again this is a result independent of 2 (but note $0^n/0^n = 0/0$ which doesn't make sense... we don't define 0^0 - it won't come up anywhere either). Let $a \in \mathbb{R}$, $a \neq 0$:

Example: *Suppose that $a, b \in \mathbb{R}$ such that $a = b^3$. Simplify*

$$\frac{a^2}{(b^2)^3}$$

Negative Powers

What is 3^{-8}? Using the first and second law of indices:

We know the story by now. Let $a \in \mathbb{R}$, $a \neq 0$. Let $n \in \mathbb{N}$:

Example: *Evaluate* $10^{-1} \times 100$. *What does this tell you about* x^{-1}*?*

Fractional Powers

What is $4^{1/2}$? Now using the third law of indices:

So we see that $4^{1/2}$ is the number that when squared gives 4... Now to keep things easy for us, we will only consider fractional powers of *positive real numbers* (i.e. numbers bigger than zero. We write $a > 0$ for "*a is bigger than zero*"). Try and figure out what $(-1)^{1/2}$ if you want to see what I mean. Hence let $a \in \mathbb{R}$, $a > 0$:

What about $5^{1/6}$? Similar story, using the third law of indices:

So we see that $5^{1/6}$ is the number when brought to the power of 6 gives you 5. This is known as the *sixth root* of 5 which we denote by $\sqrt[6]{5}$, and extends naturally from "*the*" square root (e.g. Let $x \in \mathbb{R}$, $x > 0$. Then $\sqrt{x} = \sqrt[2]{x}$). Again, let $a \in \mathbb{R}$, $a > 0$ and $n \in \mathbb{N}$:

What about $6^{3/5}$? If we agree that whatever it is, it has to agree with, say the third law of indices, so that:

A subtlety of this is that we have $3 \times (1/5) = (1/5) \times 3$:

Finally, let $a \in \mathbb{R}$, $a > 0$ and $m/n \in \mathbb{Q}$:

Example: *Putting our skills together!! Evaluate*

$$\sqrt{\left(\frac{9^2 9^3}{3^4}\right)^5 3^{-6}}$$

Exercises

1. *Evaluate:*

$$(i)\, 36^{1/2} \quad (ii)\, 125^{1/3} \quad (iii)\, 16^{1/4} \quad (iv)\, 1000^{1/3} \quad (v)\, 1000^{2/3}$$
$$(vi)\, 2^{-5} \quad (vii)\, 5^{-2} \quad (viii)\, 8^{2/3} \quad (ix)\, 4^{-1/2} \quad (x)\, 4^{-1}$$

2. *Write these as a/b, where $a, b \in \mathbb{Z}$:*

$$(i)\, 2^{-2} \;\; (ii)\, (1/4)^{1/2} \;\; (iii)\, 32^{-3/5} \;\; (iv)\, 16^{-1/4} \;\; (v)\, 27^{-2/3}$$

3. *Which of each pair is greater?:*

   (i) 2^5 or 5^2

   (ii) $4^{1/2}$ or $(1/2)^4$

   (iii) $2^{-1/2}$ or $(-1/2)^2$

   (iv) $(1/2)^6$ or $(1/2)^7$

   (v) $7^2 \times 7^3$ or $(7^2)^3$

4. *Solve for k:*

$$(i)\, 2^k = 4 \qquad (ii)\, 4^k = 64$$
$$(iii)\, 8^k = 64 \qquad (iv)\, 2^k = 128$$
$$(v)\, 4^k = 2 \qquad (vi)\, 25^k = 5$$
$$(vii)\, 8^k = 4 \qquad (viii)\, 1000^k = 100$$
$$(ix)\, 32^k = 16 \qquad (x)\, 8^k = 1/2$$

5. *Write each of these in the form a^p, where $p \in \mathbb{Q}$:*

$$(i)\, a^7 \div a^2 \qquad (ii)\, a^7 \times a^2$$
$$(iii)\, (a^7)^2 \qquad (iv)\, \sqrt{a}$$
$$(v)\, \sqrt[3]{a} \qquad (vi)\, \sqrt{a^7}$$
$$(vii)\, 1/a^3 \qquad (viii)\, 1/\sqrt{a}$$
$$(ix)\, (\sqrt{a})^3 \qquad (x)\, 1/(a\sqrt{a})$$

6. *Find the value(s) of k in each case.*
   (a) $2^{3k}.2^k = 16$
   (b) $2^{2k+1} = 8^2$
   (c) $16^4.8 = 2^k$
   (d) $5^{2k+1} = 125$
   (e) $3^{k+1} = 9^{k-1}$
   (f) $10^k.10^{2k} = 1,000,000$
   (g) $2^{(k^2)}.2^k = 64$
   (h) $10^{(k^2)} \div 10^{2k} = 1,000$
   (i) $5^k = 1/125$
   (j) $2^k = 1/(4\sqrt{2})$
7. *Write these as 10^p:*

$$(i)\,(10^9)^2 \;\; (ii)\,10^9 \times 10^2 \;\; (iii)\,(10^5.10^3)^2 \;\; (iv)\,(\sqrt{10})^{100}$$

8. *State if these are true or false:*

$$(i)\,2^3.5^3 = 10^3 \;\; (x^6)^7 = x^{13} \;\; (iii)\,(3\sqrt{3})^3 = 3^{4.5}$$
$$(iv)\,(10\sqrt{10})^4 = 10^6 \;\; (v)\,2^7 \times 3^7 = 5^7 \;\; (vi)\,(-3)^4 = -3^4$$

9. *Write $\sqrt[3]{\sqrt{2}}$ as 2^p for $p \in \mathbb{Q}$.*
10. *If*

$$\frac{((2\sqrt{2})^2)^3}{(2\sqrt{2})^2(2\sqrt{2})^3} = 2^n,$$

find the value of n.

1.2 Logarithms

Functions and Graphs

A *function* is an object that takes real numbers as an input and gives a (unique) real number output:

Most if not all of the functions that we will encounter will be of the form:

What this means is that whatever number we feed into the function, x, the expression on the right gives the recipe for calculating $f(x)$.

Examples

1. $f(x) = 3x + 2$.
2. $f(x) = x^2$.
3. $f(x) = 1/(3x + 5)$.
4. $f(x) = \sqrt{x^2 + 1}$.
5. $f(x) = \sin x$.
6. $f(x) = 2^x$.

In general, if we want to say that f is a function we write:

Occasionally we will only take some of the real numbers to be our input. For example:

defines a function on the degree-measure of the acute angles in a right-angled triangle. Now consider the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 10^x$. We can tabulate some of the values that $f(x)$ takes:

Input	-4	-3	-2	-1	$-1/2$	0	$1/2$	1	$4/3$
Output					0.316...		3.162...		21.554...

A *graph* of a function is a picture of a function constructed as follows. Take a pair of axes labeled x and $f(x)$ (or x and y). Now x is the input while $f(x)$ is the output:

Figure 1.1: The graph of a function $f(x)$.

We could plot the points of $f(x) = 10^x$ on a graph:

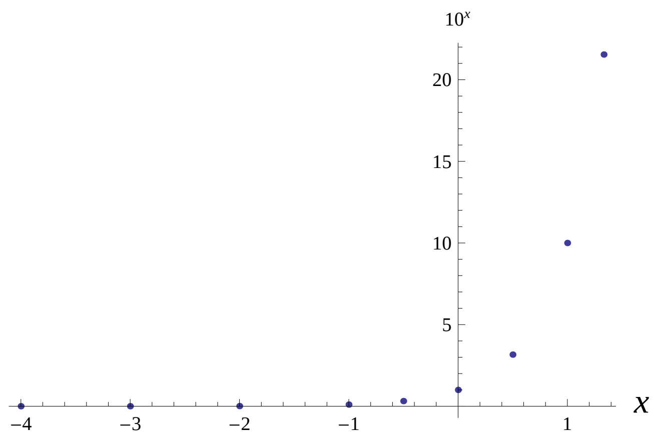


Figure 1.2: The graph of $f(x) = 10^x$.

Now a feature of the graph of 10^x is that it is always increasing and always positive (this will be true for any a^x as long as $a > 1$):

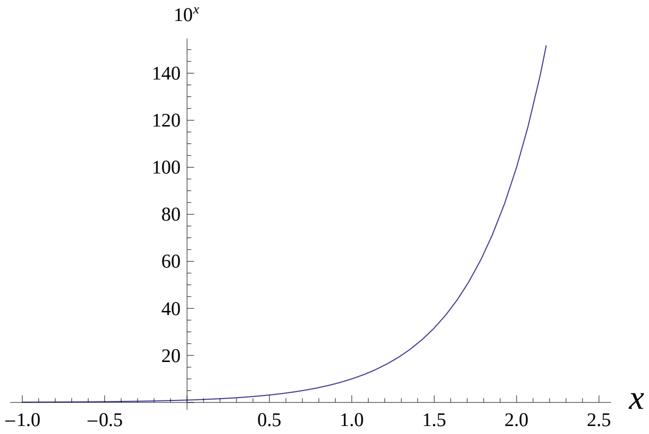


Figure 1.3: To every output we can associate a unique input.

This suggests that we could define a function going in the opposite direction, where "y" would be the input. Now this other function, which we call 'g' only takes *positive* numbers as input. So

where $\mathbb{R}^+$ is the set of positive real numbers. We define g as:

The terminology here is that g is the *inverse* of $f(x) = 10^x$. Again as we did in the last chapter, we can generalise this discussion - i.e there is nothing special here about 10.

Definition

Let $a > 1$ and let $f(x) = a^x$. The *logarithm to base a* is a function $\log_a : \mathbb{R}^+ \to \mathbb{R}$ that is the inverse function of $f(x)$ so that

In other words, $\log_a y$ asks the question:

Examples

Evaluate

1. $\log_2 8$

2. $\log_3 81$

3. $\log_{25} 5$

4. $\log_{10} 0.001$

Properties of Logarithms

Now that we have defined what a logarithm is we must investigate common properties of them. What happens when we take the log of a product? If we take the log of a fraction? What about the log of a power?

The Log of a Product?

What is the log of 8×16 to the base 2:

$$\log_2(8 \times 16) = ?$$

As we've seen time and again, a process like this can be abstracted to prove a general rule for *all* numbers:

Proof: Let $L = \log_a(xy)$:

Now there exist numbers $p, q \in \mathbb{R}$ such that $a^p = x$ and $a^q = y$...:

Well $p = \log_a x$ and $q = \log_a y$, so

Example

Evaluate $\log_4 2 + \log_4 32$.

The Log of a Power

What is $\log_3 8^4$?

So...

(Note we've only stated this for $n \in \mathbb{N}$ but in fact this is true for all $n \in \mathbb{R}$.)
Proof:

Example

Find $\log_5 25^{60}$

The Log of a Fraction

What about $\log_{12}(34/56)$?

So

Proof:

Example

Evaluate $\log_2 80 - \log_2 5$.

The Change of Base Rule - Why Can't I Find $\log_3 12$ on My Calculator?

What is $\log_3 12$ actually equal to as a decimal...

So if we want to change from base a to base b:

Proof:

Example

Evaluate $\log_8 5$ *correct to 5 decimal places:*

Prominent Results

What is $\log_3 3^x$?

What is $4^{\log_4 69}$?

What is $\log_a 1$?

What is $\log_a \sqrt{x}$?

What is $\log_a(1/x)$?

Example

Express as a single logarithm:

$$\log_a x + \frac{1}{2}\log_a y$$

Two Distinguished Bases

For good reason the two bases that 'turn-up' most frequently are 10 and the special real number $e \approx 2.718$. As we will see in the next chapter, any real number at all may be expressed in the form:

Here $a \in [1, 10)$ means "*a is a (real) number between* 1 *and* 10*, possibly equal to* 1 *but not equal to* 10*.* For example,

Now taking the log to the base 10 of both sides:

Hence to find the base-10 logarithm of any number at all we must simply know all the logarithms (to base 10) of all the numbers between 1 and 10... When we write log with no base (cf $\log_8$) then we usually mean $\log_{10}$.

The importance of the base e comes from following observation:

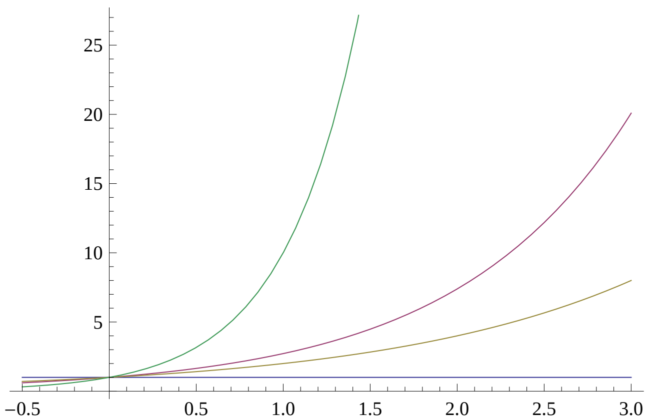


Figure 1.4: The slope of the curve of a^x is similar to a^x itself.

As it turns out, e^x is the unique function of the form $f(x) = a^x$ for which the slope of the curve of a^x is equal to a^x. We will write $\ln x$ for $\log_e x$, where 'ln' is somehow meant to stand for natural logarithm.

Examples

If $4^n = 2000$*, find* n *correct to 4 significant figures*

Expand:

$$\log_a \sqrt{\frac{ab}{c}}$$

Exercises

1. *Solve for x:*

$$(i)\ \log_2 8 = x \qquad (ii)\ \log_{10} x = 2 \qquad (iii)\ \log_3 81 = x$$

$$(iv)\ \log_5 x = 3 \qquad (v)\ \log_2 x = 10 \qquad (vi)\ \log_x 64 = 3$$

$$(vii)\ \log_{10} 10 = x \quad (viii)\ \log_{10} 1 = x \quad (ix)\ \log_{25} x = 1/2$$

2. *Evaluate* $\log_7 35 - \log_7 5$.

3. *Solve for n (to 2 decimal places):* $2^n = 20$.

4. *Use a calculator to evaluate to two decimal places:*

$$(i)\ \log_1 03\ (ii)\ \log_{10} 7\ (iii)\ \log_3 7\ (iv)\ \log_7 3$$

5. *Using the fact that* 3^n *is* increasing, *find the least value of* $n \in \mathbb{N}$ *such that* $3^n > 1,000,000$.

6. *If* $p = \log_3 10$ *and* $q = \log_3 7$*, write* $\log_3 700$ *in terms of* p *and* q.

7. (a) *Find the least value of* $n \in \mathbb{N}$ *such that* $1.03^n > 2$.
   (b) *Suppose that the population of a town increases by 3% every year. Using the previous result, write down how many years it takes to double.*

8. *If* $p = \log_6 5$ *and* $q = \log_6 10$*, write* $\log_6(2\sqrt{6})$ *in terms of* p *and* q.

1.3 Scientific Notation

Powers of Ten

Consider the number 3,487. What *is* this?

Three thousand, four hundred, and eighty seven

How about 0.123?

As claimed before, any real number[2] at all may be written in what's called *scientific notation.* This notation handles extremely large and small numbers and negates the need for too many decimal points. For science and engineering at least, scientific notation greatly eases calculations and data can be presented in a much nicer manner. In scientific notation, real numbers are written in the form:

with $a \in [1, 10)$ and $n \in \mathbb{Z}$.

Examples

1. $300 = 3 \times 10^2$.
2. $4,000 = 4 \times 10^4$.
3. $5,720,000,000 = 5.72 \times 10^9$.
4. $0.0000000061 = 61 \times 10^{-9}$.

Scientific notation allows us to get a rough order on the magnitude of very small and very large numbers.

[2] except 0

How do we write a decimal x in scientific notation? We'll break it up into three cases.

Numbers Between 1 and 10

Write 4.345 in scientific notation.

Abstraction: If $1 \leq x < 10$:

Numbers Bigger than 10

What about 3187.2?

So we have

$$3,187.2 \to 3.1872 \times 10^3 \tag{1.3}$$

In general,

$$a_{n+1}a_na_{n-1}\ldots a_2a_1d_1d_2\cdots \to a_{n+1}.a_n\ldots a_1d_1\cdots \times 10^n \tag{1.4}$$

Examples

1. 4342.43

2. 35.76

3. 1423344.243

Numbers Less than 1

What about 0.022?

So we have

$$0.022 \to 0.022 \times 10^2 \tag{1.5}$$

In general,

$$0.\underbrace{00\ldots0}_{n-1\text{ zeros}}d_1d_2\cdots \to d_1.d_2d_3\cdots \times 10^{-n} \tag{1.6}$$

Moving the decimal spot one to the right is equivalent to multiplying by ten — for every spot you move to the right you must multiply by ten — and if you multiply by 10 n times then you must divide by 10 n times: 10^{-nn}.

Examples

1. 0.00243

2. 0.0076

3. 0.000000243

Examples

1. An electron's mass is about 0.00000000000000000000000000000091093822 kg. In scientific notation, this is

2. The Earth's mass is about 5973600000000000000000000 kg. In scientific notation, this is

3. The Earth's circumference is approximately 40000000 m. In scientific notation, this is

Order of magnitude

Scientific notation also enables simpler order-of-magnitude comparisons. A proton's mass is 0.0000000000000000000000000016726 kg. If this is written as

it is easier to compare this mass with that of the electron, given above. The order of magnitude of the ratio of the masses can be obtained by comparing the exponents instead of the more error-prone task of counting the leading zeros. In this case, -27 is larger than -31 and therefore the proton is roughly four orders of magnitude (about 10000 times) more massive than the electron.
Scientific notation also avoids misunderstandings due to regional differences in certain quantifiers, such as billion, which might indicate either 10^9 or 10^{12}.

Manipulation of Scientific Notation

Multiplication

What about $(9.2 \times 10^8) \times (4.3 \times 10^{10})$:

Hence

$$(a \times 10^n) \times (b \times 10^m) = ab \times 10^{mn} \tag{1.7}$$

Division

What about

$$\frac{1.256 \times 10^{-13}}{8.976 \times 10^{-14}}$$

Hence

$$\frac{a \times 10^n}{b \times 10^m} = \frac{a}{b} \times 10^{n-m} \tag{1.8}$$

Powers

What about $(4.55 \times 10^{-13})^5$?

Hence

$$(a \times 10^n)^m = a^m 10^{nm} \tag{1.9}$$

Exercises

1. *By converting each element to scientific notation, calculate each of the following and write your answer in the form* $a \times 10^n$*, where* $1 \leq a < 10$ *and* $n \in \mathbb{Z}$:

   (a) $400,000 \times 2,000$.

   (b) $6,000 \times 1,400$.

   (c) $25,000 \times 0.0018$.

   (d) $4,500 \times 1.5 \times 10^{-4}$.

   (e) $6,000,100/(3 \times 10^5)$.

   (f) $8,888 \times 10^{-4}/0.000432$.

   (g) $(10,000)^4$.

   (h) $(1.8 \times 10^{-6})^{-2}$.

   (i) $(2.4 \times 10^{-31}) \times (4.123 \times 10^3)/(10^{-18})^3$.

2. *Multiply* $3,700$ *by* 0.2 *and express your answer in the form* $a \times 10^n$*, where* $1 \leq a < 10$ *and* $n \in \mathbb{Z}$.

3. *Write* 2.8×10^3 *as a natural number.*

4. *Express* $(10^5) \times (1.8 \times 10^{-4})^{-4}$ *in scientific notation.*

1.4 Units

SI Units

The central concept of a physics theory is a physical quantity. A physical quantity is any property of matter, time and space that can be measured. Everyday physical quantities include:

time	length	area
volume	distance	speed
acceleration	mass	temperature

When a physical quantity is measured, it is compared with a standard amount, or *unit*, of the same quantity. For example, to say that a length of wire is twelve metres means that the piece of wire is twelve times longer than the metre. The metre is the unit of length. The result of a measurement is always a multiple of a unit. To say that a length is twelve is meaningless.

There are many different units of length: there is the centimetre, the inch, the foot, the yard, the metre, the furlong, the kilometre, the mile, the parsec, etc. In 1960 physicists agreed to use a particular system of units; the *Systéme Internationale*, or the SI.

To relate physical quantities easily, the SI defines *seven* basic quantities and associated units. To ease presentation of laws, for example, a symbol or letter is assigned to stand for physical quantities. Furthermore, the associated units have associated letter symbols. For example, the physical quantity is length, denoted by l (or s often); and the associated SI unit is the metre, denoted by m. Hence the physical quantity twelve metres may be denoted $l = 12$ m. The five basic quantities, their symbol, associated unit and unit symbol are as follows:

Basic Quantity	**Symbol**	**SI Unit**	**Symbol**
length			
time			
mass			
electric current			
temperature			

The unit of every other quantity is called a *derived unit* because it can be expressed as the product or quotient of one or more of the basic units. For example, density is kilogram per cubic metre. The symbol for density is ρ (the Greek letter *rho*). The unit is the kilogram per cubic metre: kg m^{-3}. A few examples of derived units are:

Physical Quantity	**Symbol**	**SI Unit**	**Symbol**
area			
volume			
speed			

Sometimes a derived unit can become quite complex. For example the unit of energy, E, in basic units is kg m^2 s^{-2}. This is given the special name to simplify matters, namely the *joule*, J. Here are a number of common quantities with derived units with their own name:

Physical Quantity	**Symbol**	**SI Unit**	**Derived Unit**	**Symbol**
force				
pressure	P	kg m^{-1} s^{-2}		
power	P	kg m^2 s^{-3}		W
frequency				
magnetic flux density	B		tesla	T
magnetic flux	Φ		weber	Wb
activity of a radioactive source	A		becquerel	Bq

Conventions

- When writing a unit in terms of basic units, a space is left between the symbol of each basic unit. For example, the unit of density is abbreviated kg m^{-3} not kgm^{-3}.
- Sometimes the standard SI units are too large or too small to be used easily. Therefore multiples of the standard units are often used. The most common multiples used are:

Multiple	**Prefix**	**Symbol**
10^9	giga-	G
10^6		
10^3		
10^{-2}		
10^{-3}		
10^{-6}		μ
10^{-9}	nano-	n
10^{-12}	pico-	p

- The name of the unit has the prefix written before it; e.g. 10 kilometres, 5 millinewtons.
- There is no space between the prefix and the symbol for the unit: e.g. 60 millimetres is written as 60 mm; 20 kilowatts is written as 20 kW.
- The kilogram *is* the SI unit of mass - not the gram.

Converting Units

What is 100 kilometres an hour in metres per second? The way to do this is to not worry about the '100' but just change the kilometres and the hour.

So now:

To convert from one unit to another, substitute each unit on the left-hand-side into terms of another on the right-hand side.

Example

Express 100 kilometres in furlongs per fortnight!

Dimensional Analysis

What is the unit of the constant G in Newton's Law of Gravitation? We have that

$$F = \frac{Gm_1m_2}{r^2} \tag{1.10}$$

Example

Analyse the following equation using dimensional analysis. The mass of a body m is given by:

$$m = PA \tag{1.11}$$

where P is air pressure and A is the surface area.

Exercises

1. *Show that the SI unit of area is the square metre.*
2. *If a body of mass m kilograms is moving with a velocity of v metres per second its momentum P is defined by the equation $P = mv$. Find the SI unit of momentum in terms of basic units.*
3. *When the velocity of a body changes its average acceleration a is given by:*

   $$a = \frac{\textit{change in velocity}}{\textit{time taken for change}} \tag{1.12}$$

   Find the SI unit of acceleration.
4. *What is the SI unit of density? Prove that your answer is correct using dimensional analysis.*
5. *Pressure P is defined as force per unit area, i.e. $P = F/A$. If the SI unit of force is the newton (N), find the SI unit of pressure in terms of the newton and the metre.*
6. *How many square centimetres in one square metre?*
   How many cubic centimetres in one cubic metre?
   How many grams in a kilgram?
7. *Convert each of the following to standard SI units:*
   (i) 5 cm^2 (ii) 40 cm^2 (iii) 1 cm^3 (iv) 456 cm^3 (v) 1,000,000,000 cm^3

8. *Convert each of the following to standard SI units. Please write your final answer in scientific notation:*
   (i) 105 km (ii) 57 mm (iii) 6.67×10^{-11} *cm (iv)* 6×10^{27} *grams (v) 9 grams per cubic centimetre (vi) 100 km* h^{-1} *(vii) 5 nN (viii) 10* μ*W (ix) 5 Gm*

9. *The unit of acceleration* a *is the metre per second squared (m* s^{-2}*). Force is equal to mass by acceleration, i.e.* $F = ma$*. The unit of force is the newton. Express the newton in terms of basic units.*

10. *By definition work* W *is equal to force* F *multiplied by distance* s *travelled, i.e.* $W = Fs$*. The unit of work is the joule (J). Express the Joule in terms of the newton.*

11. *Use the results of the previous two questions to write the unit of work, the joule, in terms of basic units.*

12. *By definition power* P *is equal to work done divided by the time taken, i.e.* $P = W/t$*. The unit of power is called the watt (W). Express the watt in terms of the joule and the second. Then express the watt in basic units.*

1.5 Basic Algebra

Five facts about numbers[3]:

1. **Subtraction:** For all real numbers x:

   Example: $5 - 5 = 0$.

2. **Identity:** For all real numbers x:

   Example: $\pi \times 1 = \pi$

3. **Distributivity** For any three real numbers x, y, z:

   This is called the *distributive law*. Note that this works both ways and explains how to group terms and factorise. One way says how to write a product as a sum; the other how to write a sum as a product.

4. **Division:** For all $x \neq 0$,

   Example: $-123/-123 = 1$.

5. **No Zero Divisors:** Suppose that $a, b \in \mathbb{R}$ such that $a \times b = 0$:

   Example: *Which number multiplied by* 5 *gives you* 0*?*

[3]four axioms and one theorem

Why Bother?

Consider the following two problems:

1. *What real number, when added to its square, equals* 0*?*
2. $5^2 - 4^2 = (5-4)(5+4)$. *Is this formula correct for all pairs of numbers?*

Here there are two types of problem. One is a puzzle — what is x? Another is asking us to *prove* that an identity holds.

Example 1

What real number, when added to its square, equals 0*?*
Now first off, can anyone guess... How do you that there are no more solutions??? This is how we do algebra:

Now we use some of our facts in a logical way. Well $x^2 = x \times x$ and $x = x \times 1$:

Now using distributivity we can write this sum as a product:

Now using 'No Zero Divisors':

Now considering $x + 1 = 0$ separately. I'm happy enough that $x = -1$ but if we still weren't convinced we could add -1 to both sides:

We set up x as the fall guy — and by a series of logical deductions, show what x must be.

Example 2

We want to investigate whether or not the statement:

The difference of two real numbers is the product of their sum and their difference.

is true or false. Now let x, y stand for any two numbers at all. Now an equivalent statement is:

for all $x, y \in \mathbb{R}$. How can we show this? Well by distributivity (twice):

Now $xy = yx$...

The statement is true for any pair $x, y \in \mathbb{R}$.

Some More Facts (!)

Multiplication by Zero

For all $x \in \mathbb{R}$, $0 \times x = 0$.

Proof

$0 = 0 + 0$. Therefore $x \times 0 = x \times (0 + 0)$. By distributivity, $x \times (0 + 0) = x \times 0 + x \times 0$. Also $x \times 0$ has an additive inverse $-(x \times 0)$:

Division by Zero is Contradictory

There does not equal a real number equal to $1/0$. (Division by any number y may is realised as multiplication by $1/y$.)

Proof

Assume there exists a real number $1/0$. Now by the previous theorem:

However, by division:

Hence if $1/0$ exists $1 = 0$. This contradicts the assumption that $1 \neq 0$. Hence $1/0$ does not exist. (i.e. division by zero is not allowed) $\bullet$

Canceling above and below

Suppose $a, b, c \in \mathbb{R}$, $b, c \neq 0$. Then

$$\frac{ac}{bc} = \frac{a}{b} \tag{1.13}$$

Proof

Now $c \neq 0$ so from division there exists a number $1/c$ such that $c \times 1/c = 1$.

Remark

Careful!! Let $a, b, c, d \in \mathbb{R}$, $b, c \neq 0$. The following move is nonsense (unless $d = 0$):

$$\frac{ac + d}{bc} = \frac{a\not{c} + d}{b\not{c}} = \frac{a + d}{b}$$

Nowhere does it say that this should make sense. The above theorem is very precise — it only allows cancelations as above.

Four More:

1. For all numbers $-1 \times x = -x$.
2. $-1 \times -1 = +1$.
3. $(-a) \times (-b) = ab$
4. $ax + bx = (a + b)x$

Examples

Unless stated otherwise, all letters are real numbers.

1. *Multiply out:*

$$(x + 1)(3x^2 + x + 1)$$

2. *If*

$$x = \frac{2t}{1+t^2} \text{ , and } y = \frac{1-t^2}{1+t^2},$$

write

$$\frac{x}{1-y}$$

in terms of t.

3. *Express as a single fraction:*

$$\frac{x+2}{3} + \frac{x+5}{4}$$

4. *Simplify*

$$\frac{10ab}{2b}$$

5. *Factorise (i)* $5ab + 10ac$ *(ii)* $6xy - 3y^2$ *(iii)* $pq - pr + p$

6. *Factorise (i)* $ax + bx + ay + by$ *(ii)* $pq + pr - q - r$

7. *Simplify*

$$\frac{4x - 8}{x^2 - 4}$$

8. *Simplify*

$$\left[\frac{1+x}{1-x} - 1\right] \div \frac{1}{1-x}$$

9. *Solve for a^x: $a^x - 3 = -2a^{-x}$.*
   Suppose that a^x solves the equation...

10. *Find the value of x:*

$$\log_3(2x+5) = \log_3(x-8) + 1$$

   Suppose x solves the equation...

Exercises

1. *When* $a = 1$, $b = 2$, $c = 3$, $x = 4$, $y = 5$ *and* $z = 6$, *find the value of: (i)* $2a + 3b + 4c$ *(ii)* abc *(iii)* $3xz + 5bx$ *(iv)* $3(2b + a) + 2(3x + 2y)$ *(v)* $(a + b)^b$ *(vi)* $(2abc/z)^c$ *(vii)* $(5(y + a))/(3(a^2 + b^2))$ *(viii)* $2(b^2 + c + bc3)$
   Selected Answers: (iii) 112 **(vi)** 8

2. *Remove the brackets in each of the following: (i)* $2(x + 4)$ *(ii)* $5(2x^2 + 3x + 4)$ *(iii)* $-3(x - 4)$ *(iv)* $-4(x^2 - 3x + 4)$
   Selected Answers: (iii) $12 - 3x$.

3. *Remove the brackets and then simplify each of the following: (i)* $3(2x+4)+2(5x+3)$ *(ii)* $4(2x+3)+2(4x+6)$ *(iii)* $3(2x+4)+7-3(x+5)-2x-4$ *(iv)* $11x-3(6-x)+13-5(2x-1)$ *(v)* $7(6 - x) + 21 + 5(x - 7) - 3(8 - x)$ *(vi)* $3p + 2(4 - p) + 3(p - 5) - 2p + 6$ *(vii)* $5(q+4)-3q-29+3(q+3)$ *(viii)* $5(2x^2-3x+2)-3(3x^2-6x+2)$ *(ix)* $5(1-x+2x^2)-5(1 - x - 2x^2 - 18x^2)$ *(x)* $2(x + y) + 3(2x + 3y)$ *(xi)* $5(2x - y) - 4(x - 3) + 3(y - 5)$ *(xii)* $4x - [4x - 2(2x - 2)]$
   Selected Answers: (iii) x **(vi)** $2p - 1$ **(ix)** $2x^2$ **(xii)** $4x - 4$

4. *Simplify each of the following: (i)* $2x.5x$ *(ii)* $x.x^2$ *(iii)* $4x^2(2x)$ *(iv)* $(5a^2)(-3a)$ *(v)* $2p.2p.2p$ *(vi)* $(-x)(-x)$ *(vii)* $4ab(3a^2b^2)$ *(viii)* $(3xy)(xy)$ *(ix)* $p.3p(-3p^2)$ *(x)* $(8xy)(xy)(4x)$
   Selected Answers: (iii) $8x^3$ **(vi)** x^2 **(ix)** $-9p^4$

5. *If* $6x(4x^2) = kx^3$, *find the value of* k.

6. *Remove the brackets and then simplify each of the following: (i)* $2x(x+5)+3(x+5)$ *(ii)* $2x(x-1)-3(x-1)$ *(iii)* $2x(2x+3)+3(2x+3)$ *(iv)* $2x(x^2+5x+3)+3(x^2+5x+3)$ *(v)* $4x(2x^2-3x-6)+6(2x^2-4x)$ *(vi)* $a(a+1)+2a(a-3)+6a-3a^2$ *(vii)* $5x(2x-y)-2y(2x-y)$ *(viii)* $3x(2x+4y)+6y^2-6y(x+y)$ *(ix)* $a(b+c)-b(c-a)-c(a-b)$ *(x)* $2[a(a+b)+b(b-a)]$
   Selected Answers: (iii) $4x^2 + 12x + 9$ **(vi)** a **(ix)** $2ab$

7. *Multiply* $2x^2 - 2x + 1$ *by* $x + 1$.
   Selected Answers: (iii) $8x^3$ **(vi)** x^2 **(ix)** $-9p^4$

8. *Write as a single fraction:*

$$(i)\ \frac{1}{5}+\frac{3}{4}$$
$$(ii)\ \frac{7}{5}-\frac{9}{10}+\frac{11}{15}$$
$$(iii)\ \frac{x}{5}+\frac{x}{4}$$
$$(iv)\ \frac{x}{2}+\frac{3x}{4}-\frac{5x}{3}$$
$$(v)\ \frac{x+2}{5}+\frac{x+7}{10}$$
$$(vi)\ \frac{2x+3}{7}+\frac{x+1}{3}$$
$$(vii)\ \frac{5x-3}{2}-\frac{3x-4}{3}$$
$$(viii)\ \frac{2x+5}{3}-\frac{4x-3}{2}$$
$$(ix)\ \frac{5x-1}{4}+\frac{x}{3}-\frac{5}{6}$$
$$(x)\ \frac{5x}{3}-\frac{1}{6}+\frac{2-3x}{2}$$
$$(xi)\ \frac{4x-3}{5}-\frac{x}{2}+\frac{1}{10}$$

Selected Answers:

$$(iii)\ \frac{9x}{20} \quad (1.14)$$
$$(vi)\ \frac{13x+16}{21} \quad (1.15)$$
$$(ix)\ \frac{19x-13}{12} \quad (1.16)$$

9. *Simplify each of the following: (i)* $20ab/5b$ *(ii)* $21pq/7q$ *(iii)* $28p/7$ *(iv)* x^2/x *(v)* $3pq/3pq$ *(vi)* $6x^2y/3xy$ *(vii)* $30p^2q/15p^2$
Note: When I write a/b *we always mean:*

$$\frac{a}{b}=a\times\frac{1}{b}=a\div b=ab^{-1}$$

Selected Answers: (iii) $4p$ **(vi)** $2x$

10. *Simplify (i)* $(x-1)^2-(x+1)^2$ *(ii)* $2(x+2)^2-8(x+1)$.

11. *Simplify:*

$$\left(1-\frac{1}{y}\right)\left(\frac{y}{y-1}\right)$$

12. *Factorise (i)* $4a^2-2a$ *(ii)* $2pq-5r-5q+2pr$ *(iii)* $25a^2-16b^2$ *(iv)* $3x^2-12x$ *(v)* $16a^2-81b^4$

13. *By factorising, simplify*

$$\frac{4x^2 - 64}{x - 4}$$

14. *Simplify*

$$\left(\frac{1}{x} - \frac{1}{x+h}\right) \div h$$

15. *Simplify*

$$\frac{(2x+3)^2 - 6(2x+3)}{2x-3}.$$

16. *Simplify*

$$\left(p - \frac{q^2}{p}\right) \div \left(1 + \frac{q}{p}\right)$$

17. *If* $3\log_a y - \log_a(x+1) = \log_a 2$, *write* y *in terms of* x.

18. *The sides of a triangle have lengths* m^2+n^2, m^2-n^2 *and* $2mn$. *Prove that the triangle is right-angled.*

19. *These statements are either true or false. If true, provide a proof. If false provide a counter-example (i.e. an example where the left-hand side is not equal to the right-hand side). Ignore or account for division by zero.*

    (a)
    $$\frac{1}{a+b} = \frac{1}{a} + \frac{1}{b}$$

    (b)
    $$\frac{a+b}{a} = 1 + b$$

    (c)
    $$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$$

    (d)
    $$\frac{a^2+a}{a} = a + 1$$

    (e)
    $$\frac{a^2+2a}{2a} = a^2 + 1$$

    (f)
    $$\frac{a(b+c)+3a}{a} = b + c + 3a$$

    (g)
    $$\frac{a(b+c)+3a}{a} = b + c + 3$$

(h)

$$\frac{m^2 - n^2}{m - n} = m + n$$

(i)

$$\frac{m^2 + n^2}{m + n} = m + n$$

(j)

$$\frac{m^2 - n^2}{n - m} = -n - m$$

1.6 Transportation and Evaluation of Formulae

What is an Equation/ Formula?

Figure 1.5: An equation or formula is like a weighing scales — when balanced.

Famous Examples

$$\pi = \frac{c}{d} \tag{1.17}$$
$$a^2 = b^2 + c^2 \tag{1.18}$$
$$E = mc^2 \tag{1.19}$$
$$F = \frac{Gm_1m_2}{r^2} \tag{1.20}$$
$$A = \pi r^2 \tag{1.21}$$
$$1 + 1 = 2 \tag{1.22}$$
$$y = mx + c \tag{1.23}$$
$$x_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{1.24}$$
$$\cos^2\theta + \sin^2\theta = 1 \tag{1.25}$$
$$e^{i\pi} + 1 = 0 \tag{1.26}$$

Evaluation of Formulae

The following question was asked on an Irish television quiz programm:

Fill in the missing number: $2 + 4 \times \_ = 30$.

A contestant said the answer was 5. "Correct" said the quizmaster. But they were both wrong. The answer should have been 7.
In Mathematics, by convention alone, multiplications are done before additions, so that

$$2 + 4 \times 5 = 2 + 20 = 22 \text{ , but } 2 + 4 \times 7 = 2 + 28 = 30.$$

So the correct answer is 7.
More generally, the order in which mathematical operations should be done — the *Hierarchy of Mathematics* — is as follows:

1.

2.

3.

4.

Hence if we mean want to say add 2 plus four, then multiply we write:

Example

If $a = 3$*,* $b = 7$ *and* $c = 4$ *evaluate (i)* $a + b\sqrt{a^2 + c^2}$ *and*

$$(ii)\ \frac{a - bc}{3a^2 + bc}$$

(i)

(ii)

Example

If $y = 5.3$, *evaluate, correct to three significant figures:* $\sqrt[4]{2y^3 + 1}$.

Solving Equations

Once we know the basic algebra facts we can solve almost all algebra problems using these in conjunction with combinations of these three *moves*. Given an equation LHS = RHS, any of the following are allowed:

(M1)

(M2)

(M3)

Think of the scale pan...

To solve an equation in x; e.g.

$$x^2 + x = 0,$$

we implicitly suppose that x solves the equation. What we do then is apply a series of moves until we get $x =$ 'something'.

Examples

1. $x - 3 = -10$

2. $x/5 = 1$

3. Suppose $x > 0$. Solve $\sqrt{x} = 2$.

See how we reacted to each equation: we saw something we didn't like, something that was getting in the way of us isolating x on the left-hand side and we chose something from the three moves to get rid of it. What are the obstacles to solving the following equations:

1. $-3x = 12$
2. $2x + 3 = 11$
3. $6x - 7 = 2x + 13$
4. $$\frac{2x}{3} + \frac{x}{4} = \frac{11}{6}$$
5. $x^2 = 35$
6. $4(x - 3) = 4$
7. $3 - \sqrt{x} = -6$
8. $$4 = \frac{5}{x}$$
9. $$\frac{48}{\sqrt{x}} = 3$$
10. $$\frac{1}{x-3} = 17$$
11. Let $x > 0$; $\sqrt{x+4} = 10$
12. $$\sqrt{\frac{x}{4}} = 2$$
13. $$\frac{1}{x-3} + \frac{5}{4} = 1$$
14. Let $x > 0$; $$4 = \sqrt{\frac{x^2}{81}}$$

Again, once we see what is in our way we must apply basic algebra and the four moves to solve the problem. First moves?

1.

2.

3.

4.

5.

6.

7.

8.

9.

10.

11.

12.

13.

14.

Now lets finish each on the board.

Some Longer Examples

1. $3(2x+1) - 3(x+4)$

2. $5 - 4(x - 3) = x - 2(x - 1)$

3. $2 + 5(3x - 1) = 4(2x - 3) + 2(x - 3)$

Equations with More Than One Variable

These are solved exactly the same as above. Remember letters just stand for real numbers — manipulate them as such.

Example

Given $a = b + (c^2 - 1)d$ *write* c *in terms of* a, b *and* d.

A Slight Complication

Sometimes the variable which you are trying to isolate appears twice:

Example

If

$$x = \frac{t-3}{2t-1},$$

write t in terms of x.

A Harder Example

Solve for d:

$$a = \frac{c}{2s}\left(\frac{h^2}{d-h}\right)$$

An Example including Powers

Solve for x:

$$5e^{3x} = 90$$

Using Equations to Solve Problems

Statements in words can be translated to into algebraic expressions. It is common to let x represent the unknown quantity in a problem given by words. For example, if x represents an unknown number then:

- 5 more than the number
- 2 less than the number
- 3 times the number
- 4 times the number, less 1
- a third of the number
- a number subtracted from six
- the difference between two numbers is 8
- two numbers add up to 10

Steps in Constructing an Equation in Solving a Practical Problem

1. Let x equal the unknown number.
2. Write each statement in the problem in terms of x and hence write an equation.
3. Solve the equation.

Examples

Brendan had 15 more t-shirts than Alan. Between them they have 39 t-shirts. How many t-shirts each have Alan and Brendan?

The perimeter of a rectangle is 46 cm. One side is 5 cm longer than the other. Calculate the area of the rectangle.

This year, a woman is four times as old as her son. In five years' time she will be three times as old as him. What age is each of them now.

Exercises

1. *Solve each of the following equations: (i)* $2x = 6$ *(ii)* $6x = 30$ *(iii)* $-2x = -20$ *(iv)* $5x = -15$ *(v)* $5x = 0$ *(vi)* $3x - 7 = 8$ *(vii)* $5x - 2 = -12$ *(viii)* $5x = 12 + 8x$ *(ix)* $7x + 40 = 2x - 10$ *(x)* $2x - 5 = 1 - x$.
   Selected solutions: (iii) $x = 10$ **(vi)** $x = 5$ **(ix)** $x = -10$

2. *Solve each of the following equations: (i)* $3(x+4) = 2(x+8)$ *(ii)* $4(x+4) = 2(x+3)$ *(iii)* $3(x-1)+5(x+1) = 18$ *(iv)* $4(x+5)-2(x+3) = 12$ *(v)* $10(x+4)-3(2x+5)-1 = 0$ *(vi)* $3(2x+1)-3(x+4) = 0$ *(vii)* $7(x-6)+2(x-7) = 5(x-4)$ *(viii)* $5-4(x-3) = x-2(x-1)$ *(ix)* $11 + 4(3x - 1) = 5(2x + 1) + 2(2x - 5)$ *(x)* $2 + 5(3x - 1) = 4(2x - 3) + 2(x - 3)$.
   Selected solutions: (iii) $x = 2$ **(vi)** $x = 3$ **(ix)** $x = 6$

3. *Solve each of the following equations:*

$$(i)\ \frac{x}{3} + \frac{x}{4} = \frac{7}{12}$$

$$(ii)\ \frac{x}{3} - \frac{x}{5} = \frac{27}{20}$$

$$(iii)\ \frac{2x}{5} = \frac{3}{2} + \frac{x}{4}$$

$$(iv)\ \frac{x+4}{3} = \frac{x+1}{4} + \frac{1}{6}$$

$$(v)\ \frac{x+4}{3} - \frac{x+2}{4} = \frac{7}{6}$$

$$(vi)\ \frac{x-5}{3} + \frac{1}{15} = \frac{x-2}{5}$$

$$(vii)\ \frac{3x-1}{2} = \frac{x+8}{3} + \frac{x-1}{6}$$

$$(viii)\ \frac{x-1}{4} - \frac{1}{20} = \frac{2x-3}{5}$$

$$(ix)\ \frac{x-2}{2} = 5 - \frac{x+10}{9}$$

$$(x)\ \frac{x-1}{3} + \frac{x-3}{4} = x - 4$$

   Selected Answers: (iii) $x = 10$ **(vi)** $x = 9$ **(ix)** $x = 8$

   In the following we have "Lhs = Rhs; variable". Solve for "variable".

4. $E = P + k$; k.

5. $F = ma$; m.

6. $y = mx + c$; m.

7. $E = mc^2$; c.

8. $E = V/R$; R.
9. $v^2 = u^2 + 2as$; u.
10. $t = (3 + v)k + c$; v.
11. $s = ut + at^2/2$; a.
12. $z = p/(2s + q)$; s.
13. $s = a/(1 - r)$; r.
14. $A = 4\pi r^2$; r.
15. $x = \sqrt{y + z}$; y.
16. $E = mgh + mv^2/2$; m.
17. $v = \pi h(R^2 + Rr + r^2)/3$; h.
18. $s = n(2a + (n - 1)d)/2$; d.
19. $A = P(1 + r)^3$; r.

Selected answers:

6. $m = \dfrac{y - c}{x}$
9. $u = \sqrt{v^2 - 2as}$
12. $\dfrac{p - q^2}{2z}$
15. $x^2 - z$
18. $\dfrac{2(s - na)}{n(n - 1)}$

Chapter 2

Algebra

2.0.1 Outline of Chapter

- Linear Equations
- Quadratic Equations
- Cubic Equations
- Complex Roots
- Polynomials
- Simultaneous Linear Equations
- Partial Fractions

2.1 Linear Equations

Coordinate Geometry

Just as points on a numberline can be identified with real numbers, points in a plane can be identified with pairs of numbers. We start by drawing two perpendicular coordinate *axes* that intersect at the origin O on each line (where the coordinate in $(0, 0)$). Usually one line is horizontal with positive direction to the right and is called the x-axis; the other line is vertical with positive direction upward and is called the y-axes:

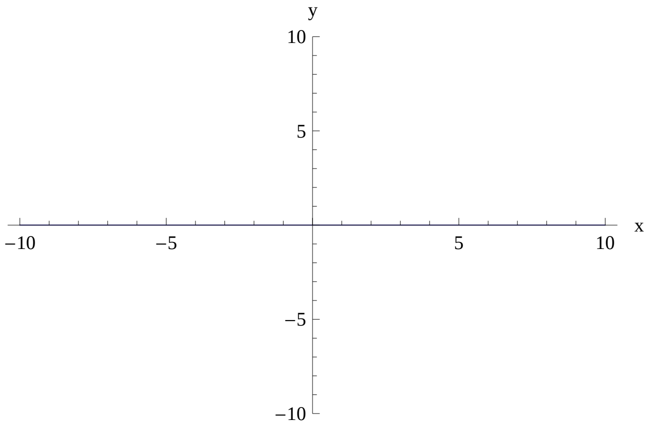


Figure 2.1: When graphing a function, usually the x-axis describes the input variable and the y-axes describes the output variable. To each point P we can associate a pair (a, b) and to each pair (c, d) we can associate a point Q.

This coordinate system is called the *Cartesian coordinate system* in honour of the French mathematician René Descartes.

Lines

We want to find the *equation* of a given line L; such an equation is satisfied by the coordinates of the points on L and by no other point. To find the equation of L we use it's *slope*, which is a measure of the steepness of the line:

Formula

The *slope* of a line that passes through the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is:

The slope of a line is constant — we can take the slope between *any* two points on the line and arrive at the same answer.

Now lets find the equation of the line that passes through a given point, say $P(2, 3)$ with slope 4. Are we happy that this defines a line... We want the equation of the line to be like a membership card:

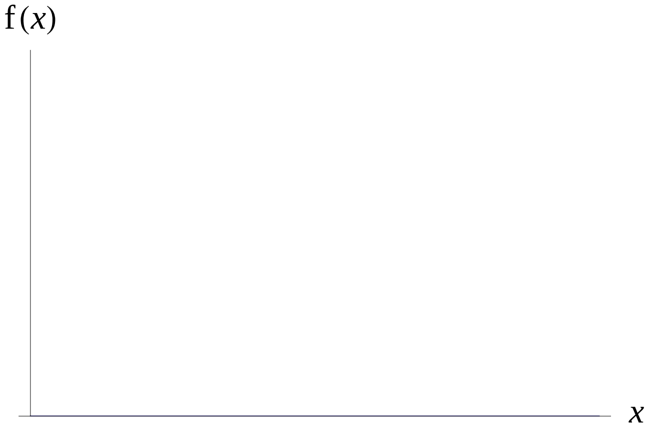


Figure 2.2: The *slope* of a line is the ratio of how much you go up, as you across: *slope* $= |\uparrow|/|\rightarrow|$

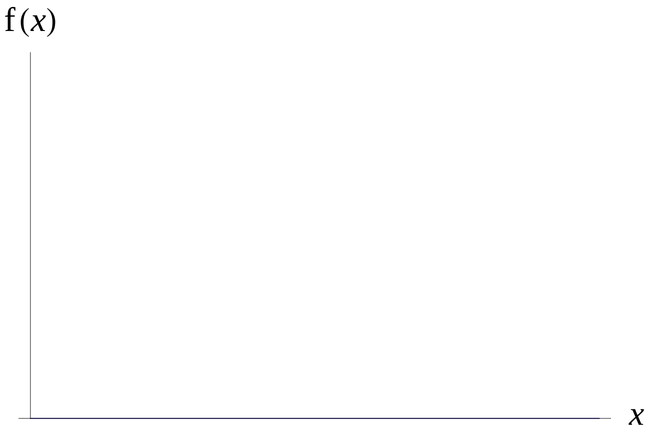


Figure 2.3: Notice that lines with positive slope slant upwards to the right, whereas lines with negative slope slant downwards to the right. Notice also that the steepest lines are those whose slope is big — be it in the positive or negative sense.

Now a point $Q(x, y) \neq P(2, 3)$ is on this line if and only if:

That is,

Now, once again there is nothing special about $(2, 3)$ nor 4 and we could abstract this process to a line defined by a point $P(x_1, y_1)$ and slope m.

Formula

The *equation of the line* passing through the point $P(x_1, y_1)$ and having slope m is

Examples

Find the equation of the line through $(1, -7)$ *with slope* $-1/2$.

Find the equation of the line through the points $(-1, 2)$ *and* $(3, -4)$.

Consider again the line defined by the point $(2, 3)$ and slope 4:

$$y - 3 = 4(x - 2).$$

This shows that any equation of the form:

for m, c constants is another way of writing the equation of the line.

What about c?

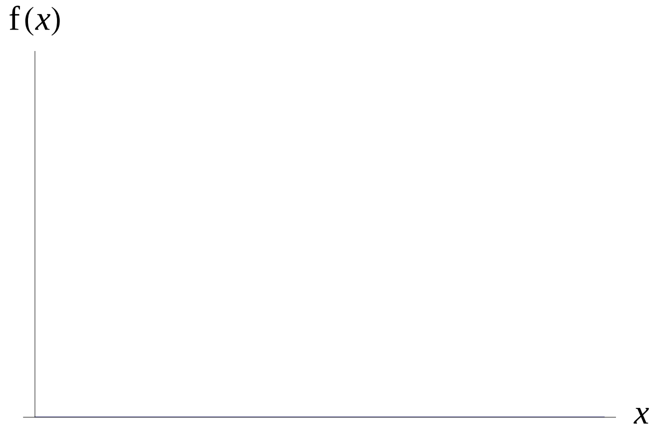


Figure 2.4: When the x-coordinate is zero, the graph of the line cuts the y-axis.

So c is where the line cuts the y-axis.

Formula

The equation of the line with slope m and y-intercept c is given by:

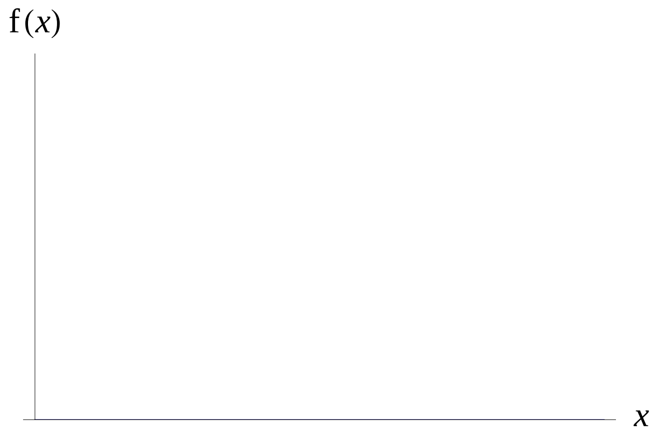


Figure 2.5: If a line is horizontal, then its slope is 0, so its equation is $y = c$, where c is the y-intercept. A vertical line does not have a slope, but we can write its equation as $x = a$, where a is the x-intercept, because the x-intercept of every point on the line is a.

Observe that every equation of the form:

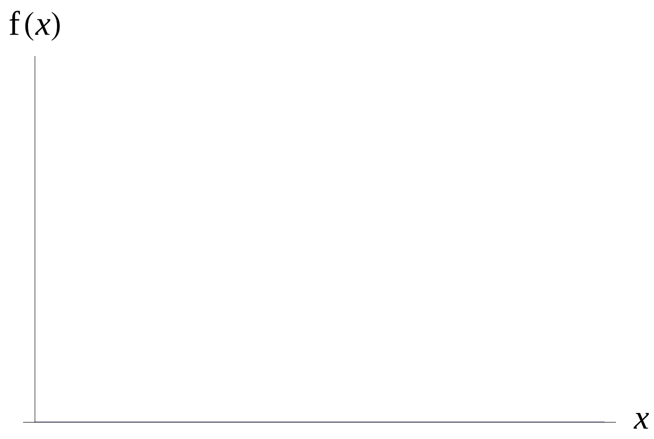
$$AX + BY + C = 0 \tag{2.1}$$

is that of a line:

We call an equation of the form (2.1) a *linear equation*.

Example

Sketch the graph of the line $3x - 5y = 15$.



Parallel and Perpendicular Lines

Slopes can be used to show that lines are parallel or perpendicular:

Theorem

1. *Two lines are parallel if and only if they have the same slope.*
2. *Two lines with slopes* m_1 *and* m_2 *are perpendicular if and only if*

That is, if

Example

Find the equation of the line through the point $(5, 2)$ *that is parallel to the line* $4x+6y+5 = 0$.

Example

Show that the lines $2x + 3y = 1$ *and* $6x - 4y - 1 = 0$ *are perpendicular.*

Linear Models

When we say that an output y is a *linear function* of an input x, we mean that the graph of the function is a line, so we can use the slope-intercept form of the equation of a line to write a formula for the function as:

where m is the slope of the line and c is the y-intercept.
A characteristic feature of linear functions is that they grow at a constant rate. For instance

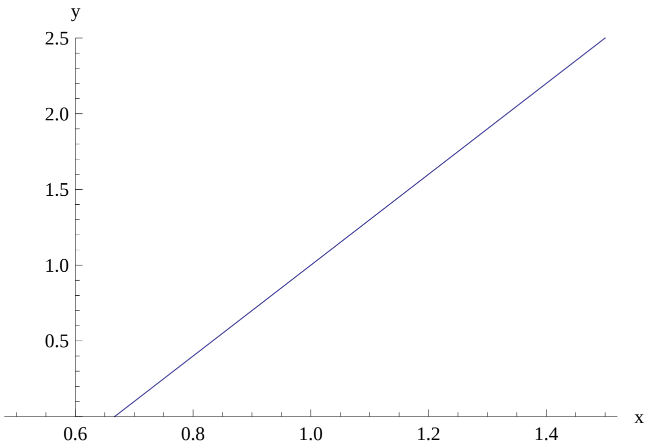


Figure 2.6: A graph of the linear function $y = 3x - 2$. Note that whenever x increases by 0.1, the value of y increase by 0.3. So y increases 3 times as fast as x. Thus, the slope of graph of $y = 3x - 2$, namely 3, can be interpreted as the rate of change of y with respect to x.

Example

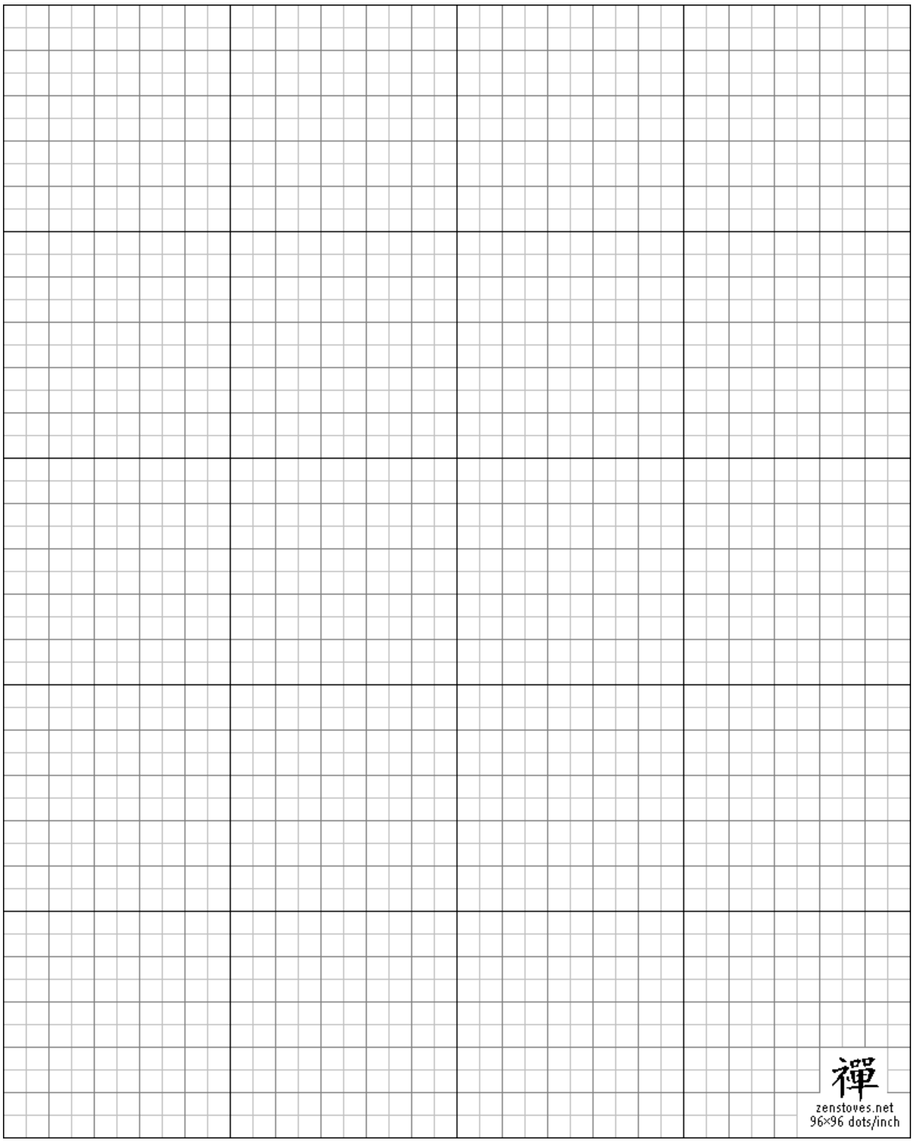
As dry air moves upwards, it expands and cools. If the ground temperature is $20° C$ *and the temperature at a height of 1 km is* $10° C$*, express the temperature* T *(in ° C) as a function of the height* h *(in kilometres), assuming that a linear model is appropriate. Draw the graph of the function. What does the slope represent. What is the temperature at a height of 2.5 km.*

If there is no physical law or principle to help us formulate a model, we construct an *empirical model*, which is based entirely on collected data. We seek a curve that "fits" the data in the sense that it captures the basic trend of the date points.

Example

The below table lists the average carbon dioxide level in the atmosphere, measured in parts per million at Mauna Loa Observatory from 1980 to 2000. Use the data to find a model for the carbon dioxide level. Use the linear model to estimate the average CO_2 level for 1987 and to predict the level for 2010. According to this model, when will the CO_2 level exceed 400 parts per million?

Year	CO_2 level (in ppm)
1980	338.7
1982	341.1
1984	344.4
1986	347.2
1988	351.5
1990	354.2
1992	356.4
1994	358.9
1996	362.6
1998	366.6
2000	369.4


禪
zenstoves.net
96×96 dots/inch

Exercises

1. *Find the slope of the line through* $P(-3, 3)$ *and* $Q(-1, -6)$.

2. *Use slopes to show that the triangle formed by the points* $A(6, -7)$, $B(11, -3)$, $C(2, -2)$ *is a right-angled triangle.*

3. *Use slopes to show that the points* $A(-1, 3)$, $B(3, 11)$ *and* $C(5, 15)$ *lie on the same line.*

4. *Sketch the graph of the equation of (i)* $x = 3$ *(ii)* $y = -2$.

5. *Find the graph of the equation that satisfies the given conditions: (i) Through* $(2, -3)$, *slope* 6. *(ii) Through* $(-3, -5)$, *slope* $-7/2$. *(iii) Slope* 3, y*-intercept* -2.
   Selected Answers: *(iii)* $y = 3x - 2$.

2.2 Quadratic Equations

Introduction

Equations which can be written in the form:

where $a, b, c \in \mathbb{R}$ are real constants with $a \neq 0$ are called *quadratic equations*. Examples:

Take the equation $x^2 - 5x + 6 = 0$. When $x = 2$:

When $x = 3$:

When $x = 2, 3$ both sides of the equation are zero. When this happens we say that $x = 2, 3$ are *roots* of the equation. Solving a quadratic equation involves finding the values of x which satisfy the equation. An analogous problem is finding the *roots* of the quadratic function:

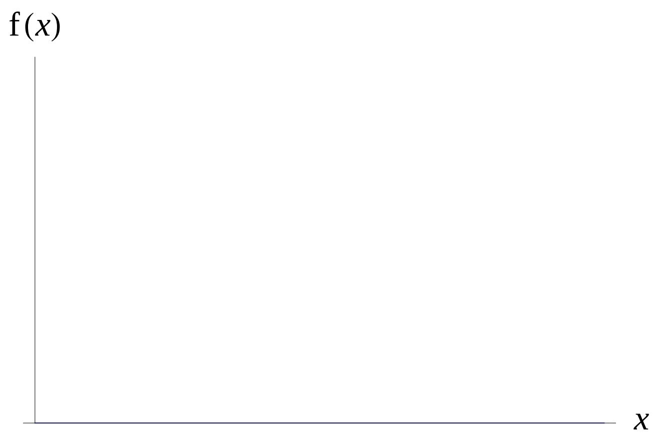


Figure 2.7: The *roots* of $f(x)$ occur where the graph cuts the x-axis.

Finding the Roots

The following is *the* crucial theorem in this section:

No Zero Divisors

Suppose a and b are real numbers and

$$a.b = 0.$$

Hence what we can do is *factorise the quadratic* — i.e. write it as a product. The simplest recipe for factorisation uses the following theorem:

Theorem

Let $f(x) = x^2 + bx + c$ be a quadratic function. Suppose $f(x)$ can be written in the form

Then

If this is the case, then α and β are the roots of $f(x)$.

Proof:

Remark

This means that a quadratic has form

Hence suppose given a quadratic f:

If numbers α and β can be found such that

then

which has roots α and β according to the No Zero Divisors Theorem.

Examples

Factorise the following quadratic functions and hence find the roots:

1. $x^2 - 7x + 10$.

2. $x^2 - 5x - 14$.

3. $x^2 + 12x + 32$.

Remark

The first complication here is when the coefficient of x^2 is not 1. That is

where $a \neq 1$. Recall the aim of our endeavours is to find roots. The roots of $f(x)$ are solutions to the equation

If we multiply both sides of an equation, the solution to the equation is unchanged. Multiply both sides by $1/a$ ((M2): equivalently divide by a):

Therefore the solutions of this equation are the roots of f. Solve this in the above way.

Examples

Factorise the quadratic function $f(x) = 3x^2 - 15x - 42$. *Hence find the roots.*

Factorise the quadratic function $f(x) = 2x^2 - x - 6$. *Hence find the roots.*

Remark

An equivalent (and in my opinion far superior — this method handles cases with $a \neq 1$ much better) method for finding factors of a quadratic f is as follows. Find a re-writing of

$$f(x) = ax^2 + bx + c$$

as

such that $b = k + m$ (so as not to change the quadratic) and $km = ac$. Then

and the roots of $f(x)$ are apparent. In practise this method is known as *doing the cross*.

Examples

1. *Solve the equation* $3x^2 + 10x = 8$.

2. *Solve the equation* $12x^2 - 4x - 5$.

3. *Solve the equation*

$$\frac{6}{x} - \frac{5}{2x-1} = 1.$$

Example

Factorise $5x^2 + 7x - 3$...

Sometimes the polynomial will not have simple roots and then the two techniques for finding the roots will not suffice. There is a formula for these cases. Of course the formula may always be used to extract the two roots but the above methods are preferable.

Theorem

A polynomial

has roots

Example

Find the roots of $f(x) = 5x^2 + 7x - 3$ *correct to two decimal places.*

Only Two Roots???

In each case we have seen that there are two roots — is this always going to be the case?

Proposition

Suppose $f(x) = ax^2 + bx + c$*. Then* $f(x)$ *may be re-written as:*

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b}{4a}\right). \tag{2.2}$$

Remark

It can be shown that because of this, the graph of $f(x)$ is similar to the graph of $f(x) = x^2$ (it is got by translating and stretching the axes.):

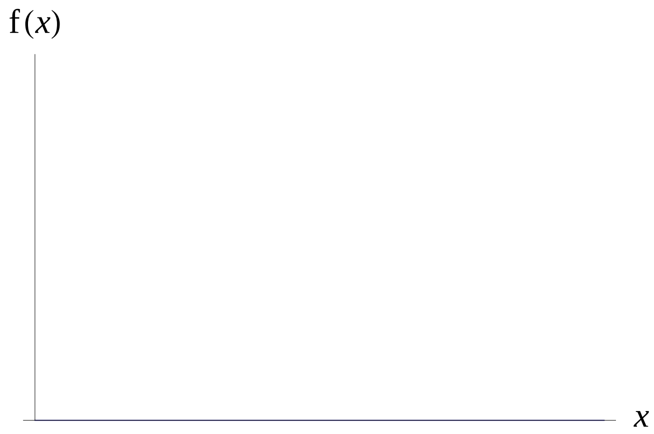


Figure 2.8: Quadratic functions look like x^2 — hence they can only cut the x-axis at most once.

Remark

Examining (*), note that if $b^2 - 4ac < 0$ then there is no (real) number equal to $\sqrt{b^2 - 4ac}$ as a real number squared is always positive. The roots are said to be *unreal* and later it will be seen that they are *complex*. In this case the graph of f does not cut the x-axis at any point:

Figure 2.9: Quadratic functions with $b^2 - 4ac < 0$ have no real roots.

If $b^2 - 4ac = 0$ then the roots are real and equal,

In this case the graph has as a tangent the x-axis:

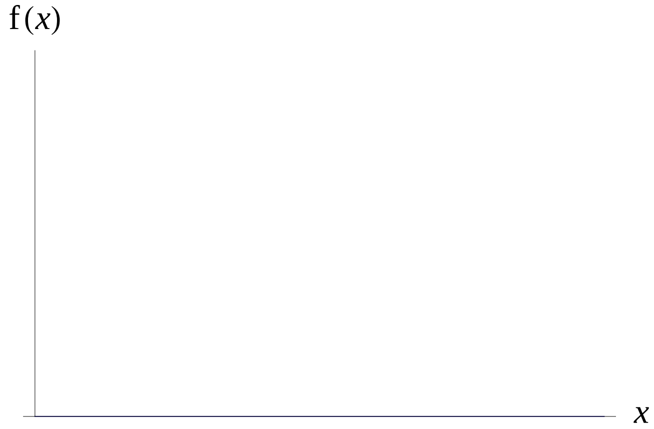


Figure 2.10: Quadratic functions with $b^2 - 4ac = 0$ have two real, repeated roots.

It is only in the case that $b^2 - 4ac > 0$ do we have two distinct, real roots.

Some Harder Examples

Solve the equation

$$(y-2)^2 - 7(y-2) + 10 = 0.$$

Solve the equation

$$6x^4 - 11x^2 - 10.$$

Solve the equation

$$\left(t - \frac{6}{t}\right)^2 - 6\left(t - \frac{6}{t}\right) + 5 = 0.$$

Solve the equation

$$1 + \sqrt{x+1} = x.$$

Exercises

To check your answer, substitute in for x. Please use the α-β method or the cross method. You can use your own method in tests & exams. ***Beware: typos possible!***

1. *Find the roots of the following functions: (i)* $(2x-5)(x+3)$ *(ii)* $2x(3x-1)$ *(iii)* $x^2-8x+12$ *(iv)* $x^2-7x-18$ *(v)* $x^2-9x-10$ *(vi)* $x^2+5x-36$ *(vii)* $2x^2-x-6$ *(viii)* $3x^2+x-10$ *(ix)* $3x^2+19x-14$ *(x)* $5x^2-13x+6$

2. *Factor and solve for x (i)* $3x^2-4x=0$ *(ii)* $3x^2-7x=0$ *(iii)* $4x^2-1=0$ *(iv)* $4x^2-49=0$

3. *Solve for x:*
$$(2x-5)(x-2)=15.$$

4. *Solve the equation:*
$$(x+3)^2=(x+1)(2x+3).$$

5. *Solve the equation*
$$x^2-13x+42=0.$$
Hence solve the equation
$$(2t-3)^2=13(2t-3)+42=0.$$

6. *Solve the equation*
$$2x^2+9x-5=0.$$
Hence solve the equation
$$2\left(x-\frac{1}{2}\right)^2+9\left(x-\frac{1}{2}\right)-5=0.$$

7. *Solve the equation*
$$6x^2-11x-10=0.$$
Hence solve the equation
$$6(t-1)^2-11(t-1)-10=0.$$

8. *Solve the equation*
$$x^2-19x+70=0.$$
Hence solve the equation
$$y^4-19y^2+70=0.$$

9. *Solve each of the following equations*
$$\begin{array}{ll} (i) & \frac{x+7}{3}+\frac{2}{x}=4 \\ (ii) & \frac{1}{x+1}+\frac{x}{5}=1 \\ (iii) & \frac{1}{x+1}-\frac{1}{x+2}=\frac{1}{2} \\ (iv) & \frac{2}{x-1}-\frac{2}{x}=\frac{1}{3} \\ (v) & \frac{3}{x-1}-\frac{2}{x+1}=1 \\ (vi) & \frac{5}{2x+3}-\frac{2}{4x-3}=\frac{1}{3} \end{array}$$

10. *Solve the equation* $x^2 + 5x - 14 = 0$*. Hence find the four values of* t *that satisfy the equation*
$$\left(t - \frac{8}{t}\right)^2 + 5\left(t - \frac{8}{t}\right) - 14 = 0.$$

11. *Solve the equation* $x^2 - 12x + 27 = 0$*. Hence find the four roots of the function*
$$g(t) = \left(2t - \frac{5}{t}\right)^2 - 12\left(2t - \frac{5}{t}\right) + 27.$$

12. *Find the roots of the following functions, correct to two decimal places: (i)* $x^2 + 2x - 5$ *(ii)* $3x^2 - x - 1$ *(iii)* $5x^2 - 4x - 2$

13. *Solve for* x*: (i)* $3x^2 + 7x = 2$ *(ii)* $3x^2 + 5x = 3$

14. *Express the following equations in the form* $ax^2 + bx + c$ *and hence use the quadratic formula to solve for* x *correct to two decimal places:*
$$(i)\ \frac{7}{x} = 3 + 2x.$$
$$(ii)\ \frac{1}{x+1} + \frac{2}{x-3} = 4.$$

15. *Find the roots of:*
$$3x^2 - 2x - 2.$$
Hence find the roots, correct to one decimal place, of the function
$$g(z) = 3(2z - 1)^2 - 2(2z - 1) - 2.$$

16. *Given that* $x^2 + y^2 + z^2 = 29$*, calculate the two possible values of* y *when* $x = 2$ *and* $z = -3$.

17. *If* 2 *is a root of* $f(x) = 3x^2 - 4x + c = 0$*, find the value of* c.

18. *Find the two values of* y *which satisfies the equation*
$$y^3 + \frac{27}{y^3} = 28.$$

19. *A real number is such that the sum of the number and its square is 10. Find the two real numbers which have this property.*

20. *Two consecutive numbers have product 42. Write down an equation and hence find two solutions (e.g. 5 and 6 AND 11 and 12).*

21. *Find the values of* x *for which: (i)* $\sqrt{x+3} = x - 3$ *(i)* $x + \sqrt{x} = 2$.

22. * *Show that for all values of* $k \in \mathbb{R}$, $f(x) = kx^2 - 3kx + k$ *has real roots.*

23. * *Show that the equation* $(x-1)(x+7) = k(x+2)$ *always has two distinct solutions, where* $k \in \mathbb{R}$.

24. * *By plugging in*

$$x_+ = \frac{-b + \sqrt{b^2 - 4ac}}{2a},$$

into $f(x) = ax^2 + bx + c$, *show that* x_+ *is a root of* $f(x)$.

25. * *By multiplying out, prove that:*

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b}{4a}\right). \tag{2.3}$$

where $f(x) = ax^2 + bx + c$.
Now solve for x, *thus deriving the formula for the roots of a quadratic function.*

2.3 Cubic Equations

Introduction

Consider the function defined by:

$$f(x) = (x - 1)(x - 2)(x - 3) \quad (^*).$$

That is, any function of the form $g(x) = (x - \alpha)(x - \beta)(x - \gamma)$ can be written in the form:

with $a, b, c, d \in \mathbb{R}$, $a \neq 0$. Such a function is called a *cubic* function or a *cubic polynomial*.

What are the roots of $f(x)$?

Hence we have a mini-theorem:

Theorem

Suppose $f:\mathbb{R}\to\mathbb{R}$ *is a cubic function with a factor* $(x-k)$ *such that*

$$f(x)=(x-k)q(x),$$

for some quadratic function $q:\mathbb{R}\to\mathbb{R}$. *Then* k *is a root of* $f(x)$ *(*$f(k)=0$*).*

Or in short:

$(x-k)$ a factor $\Rightarrow k$ a root.

Proof

Now the[1] *converse* statement would read:

"Theorem"

Suppose $f:\mathbb{R}\to\mathbb{R}$ *is a cubic function with a root* k. *Then* $(x-k)$ *is a factor of* $f(x)$ *(*$f(x)=(x-k)q(x)$ *for some quadratic function* $q(x)$*.).*

Or in short

k *a root* $\Rightarrow (x-k)$ *a factor.*

Remark

If this "theorem" were true we would have a method for finding the roots of a cubic $f(x)$:

STEP 1: Find *a* root k.

STEP 2: By the "theorem", there is a quadratic $q(x)$ such that $f(x)=(x-k)q(x)$. Find this quadratic.

STEP 3: Find the roots of the quadratic α and β.

STEP 4: The roots of $f(x)$ are k, α and β.

[1]fancy word for oppposite

Example

Show that 2 is a root of

$$p(x) = 2x^3 + x^2 - 8x - 4.$$

Show that $(x - 2)$ is a factor of $p(x)$. Hence find the roots of $p(x)$.

We want to find $a, b, c \in \mathbb{R}$ such that:

$$2x^3 + x^2 - 8x - 4 = (x - 2)(ax^2 + bx + c)$$

Therefore

$$p(x) = (x - 2)(2x^2 + 5x + 2).$$

Now when is $p(x) = 0$?

The Factor Theorem

Is this "theorem" true? The answer is yes:

The Factor Theorem for Cubics

Or, in short,

$$k \text{ a root } \Leftrightarrow (x-k) \text{ a factor,}$$

that is, k is a root is equivalent to $(x-k)$ being a factor.

Proof

How to Find that First Root?

In an exam situation there are two situations:

- you are given the root, or,
- you are just told to solve the cubic equation. In this case there will *always* be a root in $\{-3,-2,-1,0,1,2,3\}$ so to find the root you have to compute $\{f(-3),f(-2),\dots,f(2),f(3)\}$ until you find it (don't forget — if $f(k)=0$ then k is a root).
  Three hints!
  One: if there is a constant term at the end (e.g. $x^3-3x^2+2\underline{-6}$, 0 will *never be a root*.
  Two, the root will have to be a factor of the constant term (e.g. if the last term is -8 the only possibilities are $\{\pm 1,\pm 2\}$)).
  Three, you are better off computing in this order: $\{f(1),f(2),f(-2),f(-2),f(3),f(-3)\}$.
  If the root is not given to you it is likely to be smaller rather than bigger.

Examples

Show that -5 *is a root of* $x^3 + 5x^2 - 4x - 20$.

Find a root of $x^3 + 4x^2 + x - 6$.

How to Find the Quadratic?

Suppose $f(x) = ax^3 + bx^2 + cx + d$ is a cubic polynomial ($a, b, c, d \in \mathbb{R}$, $a \neq 0$). Once we have found a root, we need to know how to find the quadratic. Here we present two methods — please choose your favourite.

Method 1: Compare the Coefficients

If we find a root k ($f(k) = 0$), we know that $(x - k)$ is a factor such that:

$$f(x) = (x - k)q(x),$$

for some quadratic $q(x)$.

Step 1: Let $q(x) = ex^2 + fx + g$.

Step 2: Write $f(x) = (x - k)(ex^2 + fx + g)$ and multiply out:

Step 3: We want this to be equal to $f(x)$ so we compare coefficients and solve:

These coefficients e, f, g are the coefficients of the quadratic.

Example

Given that 7 is a root of $f(x) = x^3 - 2x^2 - 29x - 42$, write $f(x)$ as a product of a linear function and a quadratic function.

Method 2: Polynomial Long Division

It's pretty difficult to explain this method in generality but roughly, given that k is a root of some cubic $f(x)$, if we can write $f(x) = (x - k)q(x)$, then

$$q(x) = \frac{f(x)}{(x-k)} = f(x) \div (x - k).$$

Can we even do this?? The answer is yes and after a bit of practise it works nicely.

The Factor Theorem for Cubics guarantees that the remainder will always be zero.

Examples

1. *If* $(x-3)$ *is a factor of* $f(x) = x^3 + kx^2 - 4x + 12$, *find* $k \in \mathbb{R}$, *find the value of* k.

2. *Find the roots of* $p(x) = 2x^3 - x^2 - 2x + 1 = 0$.

3. *Compute* $4x^3 - 11x + 3 \div 2x - 3$.

4. *Find the roots of* $q(x) = x^3 + x^2 - 5x + 3 = 0$.

Three Roots?

Using differential calculus we can show the following:

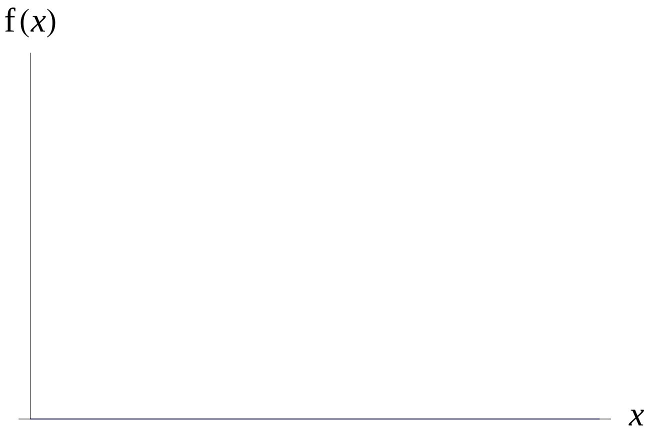


Figure 2.11: A cubic function $p(x) = ax^3 + bx^2 + cx + d$ with $a > 0$ and three real distinct roots.

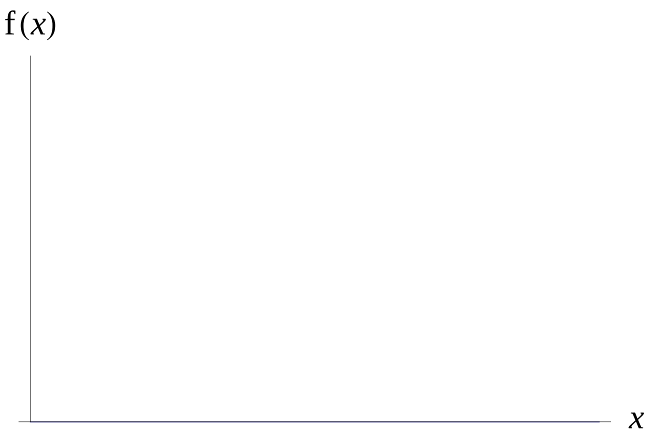


Figure 2.12: A cubic function $p(x) = ax^3 + bx^2 + cx + d$ with $a < 0$ and three real distinct roots.

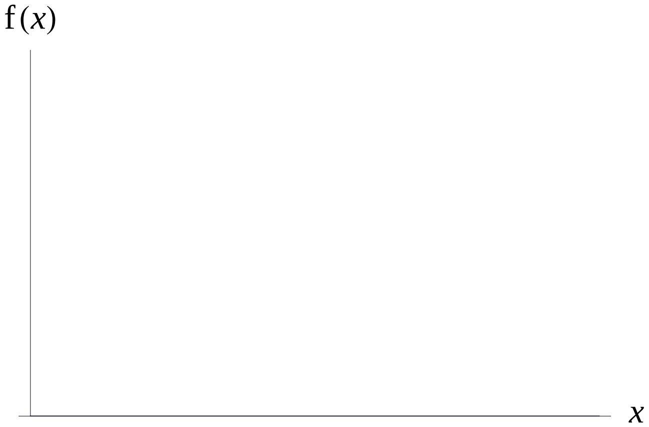


Figure 2.13: A cubic function $p(x) = ax^3 + bx^2 + cx + d$ with $a > 0$ and three real roots — two of which are repeated.

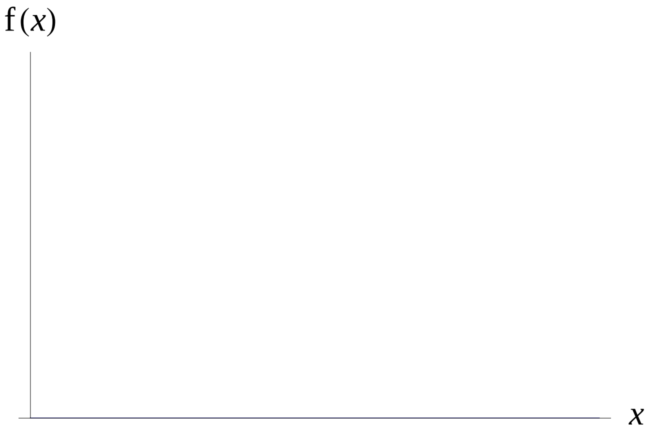


Figure 2.14: The cubic function $f(x) = x^3$ has three real, repeated roots. Namely $f(x) = (x - 0)(x - 0)(x - 0)$ — 0 is the root.

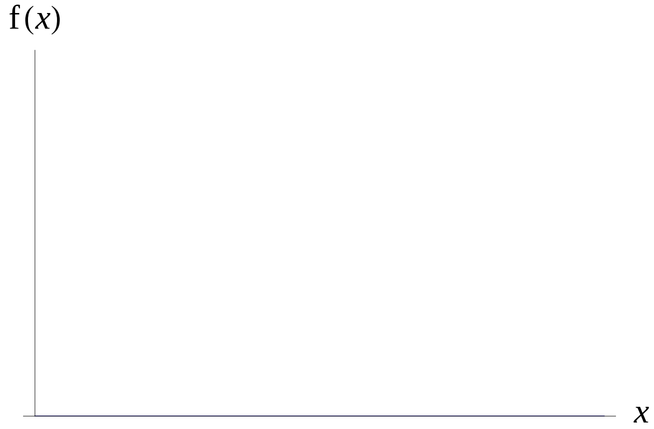


Figure 2.15: A cubic function $p(x) = ax^3 + bx^2 + cx + d$ with $a > 0$ and one real roots and two complex roots.

Exercises

1. *Divide each of the following:*

$$\begin{array}{rc} (i) & x^3 + 3x^2 - 10x - 24 \div x - 3. \\ (ii) & 2x^3 - 3x^2 - 12x + 20 \div 2x + 5. \\ (iii) & 6x^3 + 7x^2 - 10x - 6 \div 2x + 1. \\ (iv) & 2x^3 + x^2 - 16x - 15 \div 2x + 5. \\ (v) & 6x^3 - 13x^2 + 4 \div 2x + 1. \\ (vi) & 2x^3 + 13x^2 - 36 \div 2x - 3. \end{array}$$

   Selected Answers: (iii) $3x^2 + 2x - 6$ **(vi)** $x^2 + 8x + 12$**.**

2. *If* $(x - 2)$ *is a factor of* $x^3 + 2x^2 - 5x + k$*, find the value of the real number* k *(HINT: Use the Factor Theorem).*

3. *If* $(x + 2)$ *is a factor of* $x^3 + tx^2 + 4x - 8$*, where* $t \in \mathbb{R}$*, find the value of* t*.*

4. *Find the roots of these functions:*

$$\begin{array}{rc} (i) & x^3 + 4x^2 + x - 6. \\ (ii) & x^3 + 3x^2 - 4x - 12. \\ (iii) & 3x^3 - 11x^2 + x + 15. \end{array}$$

   Selected Answer: (iii) $\{3, -1, 5/3\}$**.**

5. *Show that* -1 *is a root of* $x^3 - 7x - 6$*. Find the other two roots.*

6. *Write down a cubic polynomial with roots* -1*,* 2 *and* 5*.*

7. *Find the quadratic equation with roots* $3 + \sqrt{5}$ *and* $3 - \sqrt{5}$*. Hence, or otherwise, find the cubic equation with roots* $3 + \sqrt{5}, 3 - \sqrt{5}$ *and* 4*.*

8. *Show that* $3/5$ *is a root of the function* $f(x) = 5x^3 + 7x^2 - 11x + 3$. *Hence find the other two roots in the form* $n \pm \sqrt{m}$, *where* $m, n \in \mathbb{N}$.

9. * $q(x) = x^2 + bx + c$ *has roots* $3 + \sqrt{7}$ *and* $3 - \sqrt{7}$. *Find the values of* b *and* c. $p(x) = x^3 + dx^2 + ex + f$ *has* 1 *as a root — and* $3 \pm \sqrt{7}$ *are also roots. Find the values of* d, e *and* f.

2.4 Polynomials*

Apart from the end of the previous week's lecture, when we said that a cubic $p(x)$ had only one real root in the case when:

this section is for your own information. However, some of this material may help you understand some of the last two sections.

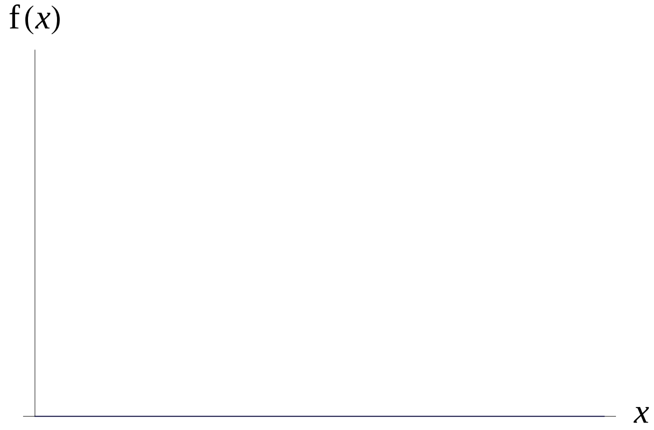


Figure 2.16: A cubic function with only one root.

What is a polynomial?

Definition

We call the biggest power of x in the polynomial it's *degree*. We have already seen three types of polynomials (degrees 1,2,3):

What would a degree 0 polynomial look like? We have also looked at how to find the roots of these polynomials. How do we find the roots of a more general polynomial? What do the graphs of these polynomials look like? Will a degree 100 polynomial really have 100 roots like a degree 3 polynomial has 3?

A Formula for all Polynomials

We started with a linear polynomial:

No problem finding the roots of this. A quadratic polynomial:

Now a nice way to solve this is by factorisation — but there is also a formula:

We have seen how to solve a cubic polynomial:

We used the Factor Theorem. Is there a formula?? What about the quartic (or degree 4)? Can we use the factor theorem?? Is there a formula? What about degree 6,7,8,...

An amazing bit of mathematics was done by a Norwegian mathematician called Abel. He proved that something called the *Abel Impossibility Theorem*:

Theorem [Abel, 1824]

A French mathematician called Galois developed a beautiful theory which explained why this was the case. So formulae are out unfortunately.

The Factor Theorem

Is the factor theorem true for polynomials of arbitrary degree??

The answer is yes, and I have a proof at http://irishjip.wordpress.com/2010/09/08/an-inductive-proof-of-the-factor-theorem/. In general, this gives us a method of solving polynomials of arbitrary degree.

Example

How to solve a quartic equation?

Complex Numbers

The original reason that people started to use complex numbers is that they allowed them to solve more equations! Consider the equation:

What number when added to it's square equals 0? Of course this question is equivalent to asking about the roots of the quadratic $x^2 + 1$:

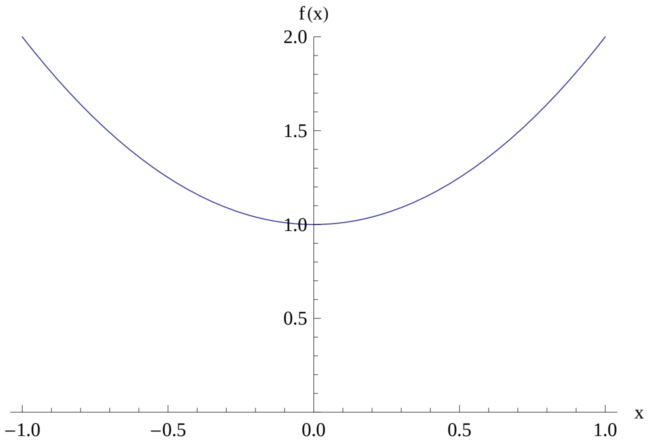


Figure 2.17: The quadratic $q(x) = x^2 + 1$ has no real roots. In fact any quadratic of the form $p(x) = x^2 + a$ has no real roots for $a > 0$.

Also, quite rightly you should note that this is equivalent to:

The square of what number is -1?

So what we do is say, well let "i" be the object who's square is -1:

So the roots of $x^2 + 1$ are $\pm i$!
A number of the form $a + ib$, with $a, b \in \mathbb{R}$ is a *complex number.* A mathematician called Gauss, while still a student (albeit a doctoral student), showed that for normal equations such as:

the complex numbers were all we needed — mathematics wouldn't need another new type of number to solve some equation like this.

This result can be extended — along with the factor theorem — to show that every polynomial of degree n can be written in the form:

Thus result is called the fundamental theorem of algebra.

The Geometry of the Graph of a Polynomial: Turning Points & The Conjugate Root Theorem

We know a number of polynomials that have complex roots; well we'll draw them:

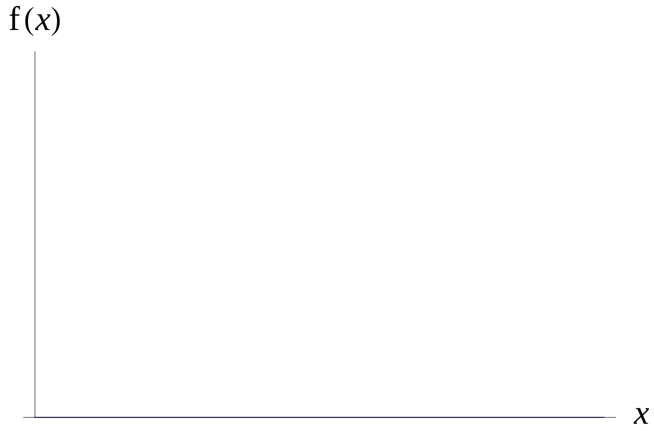


Figure 2.18: Is it true that there is always an even number of complex roots?

This is a fact called the *Conjugate Root Theorem*:

Theorem

In conjunction with a fact from differential calculus, we can use this theorem to guess what polynomials of arbitrary degree look like:

Differential Calculus Fact

In general, an n degree polynomial has a horizontal tangent at $n-1$ locations — this means, in general, $n-1$ turning points.

For the rest of this lecture we'll draw pictures on the whiteboard...

2.5 Simultaneous Linear Equations

The Intersection of Two Lines

Suppose we are given two lines $L_1 \equiv y = x - 1$ and $L_2 \equiv y = -x + 8$. How do we find their intersection?

What does it take to be on L_1?

What does it take to be on L_2

What does it take to be in the intersection, which we could write $L_1 \cap L_2$ (*said* L_1 *intersection* L_2.)?

Hence we are looking for a solution (x, y) to the *simultaneous equations:*

Now the geometry should inform us that there is a *unique* solution[2]. One way to look at this is to say well all the points on L_1 have coordinates $(x, x-1)$ and the lines on L_2 have coordinates $(x, -x+8)$:

Is there a point where these general points coincide? Well we will need their y coordinates to coincide:

So at the point where $x = 3$ we know that $(x, x-1) = (x, -x+8)$. Hence the intersection is $(3, 2)$. This method, while nicely geometric, is not the standard method. Also this method only seems to work for lines: what about equations of the form:

Well to be honest these two equations *do* represent lines but we might not always know this. Here we present two methods (the second of which we'll generalise to work well in three variables!)

The Substitution Method

Given two *simultaneous equations*, for example:

$$\begin{aligned} 3x - 4y &= 4 \\ x - y &= 6 \end{aligned}$$

Step 1: Take the first equation and write y (or x if you so please) in terms of x:

[2] unless the lines are parallel — or equal.

Step 2: Now replace any instance of y (respectively x) in the second equation with y in terms of x — i.e. eliminate y from the second equation so it is an equation in x alone:

Step 3: Now solve this for x:

Step 4: Now substitute back into y in terms of x to find y:

Hence the solution: $(20, 14)$.

Example

Solve the simultaneous equations:

$$3x - 9y = -6$$
$$x - y = 2$$

Solution:

The Row Method

First some facts (we wouldn't normally place these facts here but when we generalise this method to three variables x, y, z we'll need them). Suppose we have the following simultaneous equations:

Fact 1: *Multiplying both sides of an equation by a constant will leave the solution unchanged.*
Fact 2: *Changing the order of the equations leaves the solution unchanged* — i.e. writing (B) above (A) doesn't change the solutions.
Fact 3: *We can add the equations together as follows and the solution is unchanged:*

Why? Because we added the same things to both sides! How does this work in practise? What we do is use these facts to eliminate a variable — either x or y — from both sets of equations. This is the method thought in secondary schools — I much prefer the above method; however this method works far better with three variables.

Examples:
Solve the equations:

$$x - 3y = -6 \ \ (A)$$
$$3x + y = 2 \ \ (B)$$

Solution: I want to balance up the coefficients of y:

Now if I add (B) to (A) the ys cancel:

Now I can simply back substitute to find y:

So my solution is $(0, 2)$.

Solve the equations:

$$x + y = 1 \;\; (A)$$
$$3x + 4y = 2 \;\; (B)$$

Solution: I want to balance up the coefficients of y:

Now if I add (B) to (A) the ys cancel:

Now I can simply back substitute to find y:

So my solution is $(2, -1)$.

The Intersection of Three Planes

Planes could be considered to be three dimensional versions of lines. Think of a plane as an infinite flat sheet in space. We are not going to prove the following:

Proposition

Let $a, b, c, d \in \mathbb{R}$.

is the equation of a plane.

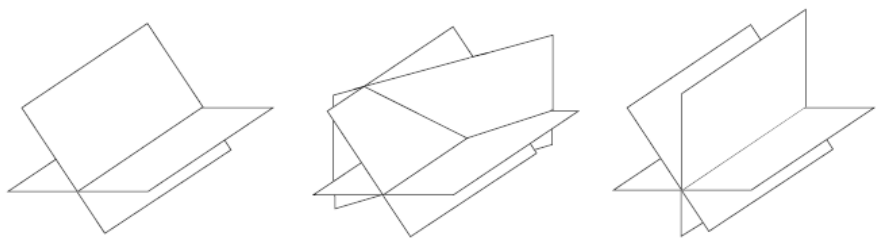


Figure 2.19: How planes intersect.

Luckily for MATH6014, case 2 in the diagram will be the case (any set of three simultaneous linear equations they give you will have a *unique* solution). Think about a geometric method of finding the intersection of three planes, say:

$$\begin{aligned} 2x - 3y + z &= -2 \ \ (A) \\ 5x - y + 2z &= 3 \ \ (B) \\ 3x + 2y - z &= 11 \ \ (C) \end{aligned}$$

Now looking at the first picture, two planes intersect along a line. Hence if we could find the lines of intersection along planes (A) & (B) and maybe (B) & (C). One way to do this is to solve each of them separately for z:

Now if a point (x, y, z) is going to be in the intersection of (A) and (B) then the zs will certainly have to agree:

Hence $x + 5y = 7$ could be considered[3] the equation of the line of intersection between (A) and (B).

Now looking at the intersection between planes (A) and (C). For plane (C),

$$z = 3x + 2y - 11.$$

Hence if a point is going to be in the intersection of (A) and (C) then the zs will have to agree:

$$\begin{aligned} -2 - 2x + 3y &\overset{!}{=} 3x + 2y - 11 \\ \Rightarrow 5x - y &= 9 \end{aligned}$$

That is $5x - y = 9$ is the equation of the line of intersection of planes (A) and (C).

[3] as z runs over all the real numbers

Now to find the intersection of the three planes, we find the intersection of these two lines. In other words we solve the simultaneous equations:

$$x + 5y = 7 \tag{2.4}$$
$$5x - y = 9 \tag{2.5}$$

Using the substitution method, we could, from the first equation, write $x = 7 - 5y$, and substitute it into the second:

$$\begin{aligned} 5(7-5y) - y &= -1 \\ \Rightarrow 35 - 25y - y &= 9 \\ \Rightarrow -26y &= -26 \\ \Rightarrow y &= 1 \end{aligned}$$

Now to find the corresponding x-coordinate substitute into (2.4) or (2.5) (here I use (2.4)):

$$\begin{aligned} x + 5(1) &= 7 \\ \Rightarrow x &= 2 \end{aligned}$$

Now to find the z-coordinate, substitute $(x, y) = (2, 1)$ into (A), (B) or (C) (here I use (A)):

$$\begin{aligned} 2(2) - 3(1) + z &= -2 \\ \Rightarrow z &= -2 - 1 = -3. \end{aligned}$$

Hence the solution is $(x, y, z) = (2, 1, -3)$.

Examples

Solve the simultaneous equations:

$$\begin{aligned} x + y + z &= 6 \\ 3x - 2y - 3z &= -10 \\ 2x + y - 3z &= -5 \end{aligned}$$

Solve the simultaneous equations:

$$\begin{aligned} a - b + c &= 4 \\ a + b - c &= 2 \\ a + b + c &= 8 \end{aligned}$$

Reduced Row Method*

There is another (equivalent) way of solving these three variable simultaneous equations (this method also works for the two variable case — and indeed the 4+ variable case). First note that the variable names are essentially irrelevant; the equations:

have the same solutions. All we have done is relabel $x \leftrightarrow m$ and $y \leftrightarrow n$. Consequently all we are actually interested in are the coefficients; which can be recorded in a *matrix*. For example,

$$\begin{aligned} x + y + z &= 6 \\ 3x - 2y - 3z &= -10 \\ 2x + y - 3z &= -5 \end{aligned}$$

converts to

Conversely given such a matrix we can recover the corresponding equations. For example,

$$\left[\begin{array}{ccc|c} 2 & -3 & 1 & -2 \\ 5 & -1 & 2 & 3 \\ 3 & 2 & -1 & 11 \end{array}\right]$$

converts to

Now remember the three facts we wrote about simultaneous equations:

1. we can multiply both sides of any equation
2. we can swap the position of any equation
3. we can add any equation to another

In the matrix picture these become:

These are known as the *elementary row operations.* Remember — applying elementary row operations is *not going to change the solution.* If we can use elementary row operations to transform our matrix form into:

Then we can use *back substitution* to solve the system. This is because the above matrix corresponds to:

Examples

Use the reduced row method to solve

$$\begin{aligned} x + y + z &= 1 \\ 2x + 3y + z &= 4 \\ 4x + 9y + z &= 16 \end{aligned}$$

Use the reduced row method to solve

$$\begin{aligned} 5x - 4y + z &= 3 \\ 3x + y - 2z &= 31 \\ x + 4y &= 21 \end{aligned}$$

Exercises

1. *Solve the following simultaneous equations:*

   (i)

$$\begin{aligned} 5x + 7y &= 0 \\ -3x + 4y &= 2 \end{aligned}$$

   (ii)

$$\begin{aligned} 5a + 7b &= 0 \\ 6a - 8b &= -4 \end{aligned}$$

   (iii)

$$\begin{aligned} 5x_1 + 7x_2 &= 0 \\ 7x_1 + 18x_2 &= 2 \end{aligned}$$

   Remark on the solutions to (i) and (ii).

2. *Solve*

$$\begin{aligned} 2x + y + z + 7 &= 0 \\ x + 2y + z + 8 &= 0 \\ x + y + 2z + 9 &= 0 \end{aligned}$$

3. *Solve*

$$\begin{aligned} x + y &= z \\ 3x + 2y - 4z &= -1 \\ x - 3y + 3z &= 2 \end{aligned}$$

4. *Solve*

$$x + y = 2; \;\; x + z = 1; \;\; y + z = 7$$

5. *Solve*

$$x + 3y - 8 = y - 2z - 5 = z - x = 0.$$

6. *An apple, a banana and a cucumber cost 48 c. Two apples and three bananas cost 71 c. Three apples, a banana and three cucumbers cost 1.20. Write down three equations and hence find the price of each item.*

7. *Solve*

$$\begin{aligned} x + y - 2z &= 316 \\ 2x + y - 3z &= 654 \\ 3x + 4y - 6z &= 279 \end{aligned}$$

Hence solve

$$\begin{aligned} 3a + b - 2(c-1) &= 316 \\ 6a + b - 3(c-1) &= 654 \\ 9a + 4b - 6(c-1) &= 279 \end{aligned}$$

[HINT: Let $3a = x$, $b = y$ and $c - 1 = z$.]

8. *Solve*

$$\begin{aligned} x + y + z &= 6 \\ x - 2y - z &= -2 \\ 3x - 5y + z &= 0 \end{aligned}$$

and

$$\begin{aligned} (2a+1) + (b-1) + (c-3)^3 &= 6 \\ (2a+1) - 2(b-1) - (c-3)^3 &= -2 \\ 3(2a+1) - 5(b-1) + (c-3)^3 &= 0 \end{aligned}$$

Selected Solutions: Check answers by substituting in your values.

2.6 Partial Fractions

This chapter will give us a an algebraic technique that allows us to write a 'fraction' as a sum of (supposedly) simpler 'fractions'.

Adding Fractions

Let $a, b, c, d \in \mathbb{R}$ such that $b \neq 0$, $d \neq 0$. Now

$$\frac{a}{b} + \frac{c}{d} =$$

So we can see that we can write the sum any fractions with denominators b,d as a single fraction with denominator bd. Now I ask the reverse question:

Given a fraction a/b can I write a/b as a sum of two fractions?

Example

Write $1/12$ *as a sum of two simpler fractions.*

Rational Functions

Definition

Any function of the form:

for $a_i \in \mathbb{R}$, $n \in \mathbb{N}$ is a *polynomial.* If $a_n \neq 0$ then p is said to be *of degree n.*

Examples

Suppose that all $a_i \in \mathbb{R}$ with *leading term* non-zero:

1. $p(x) = a_1 x + a_0$ is a line or a linear polynomial.
2. $q(x) = a_2 x^2 + a_1 x + a_0$ is a quadratic or a quadratic polynomial.
3. $r(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0$ is a cubic or a cubic polynomial.

Definition

Suppose that $p(x)$ and $q(x)$ are polynomials. Any function of the form:

is called a *rational function*.

Examples

1. $$\frac{x+5}{x^2+x-2}$$
2. $$\frac{x^3+x}{x-1}$$
3. $$\frac{x^2+2x-1}{2x^3+3x^2-2x}$$

The remainder of this section will be concerned with writing rational functions as a sum of simpler 'fractions' called *partial fractions*. To mirror the addition of a/b and c/d from earlier on, consider:

$$\frac{2}{x-1} - \frac{1}{x+2}$$

That is example 1 above has partial fraction expansion:

$$\frac{x+5}{x^2+x-2} = \frac{2}{x-1} - \frac{1}{x+2}$$

To see how the method of partial fractions works in general, let's consider a rational function f:

where p and q are polynomials. We will see that it will be possible to write f as a sum of simpler fractions provided the *degree of p is less than the degree of q*. If it isn't, we must first divide q into p using long division (same method as when we did the factor theorem example). *In MATH6014 the degree of the top will always be less than the degree of the bottom.*

General Method for Partial Fractions

Let $f(x) = p(x)/q(x)$ be a rational function with $q(x)$ a quadratic polynomial.

1. Factorise $q(x)$ into $(ax + c)(bx + d)$.

2. To each factor of $q(x)$ we associate a term in the partial fraction decomposition via the following rule:

   To each linear factor of the form $(ax + c)$ there corresponds a partial fraction term of the form:

   Example: *Suppose $f(x) = p(x)/q(x)$ and $q(x) = (x - 1)(2x - 1)$. What is the partial fraction expansion of $f(x)$?*

3. Write the partial fraction expansion as a single fraction "$f(x)$", and set it equal to $f(x)$. Compare the numerators of $f(x)$, $u(x)$; and the numerator of "$f(x)$", $v(x)$; by setting them equal to each other:

   Find the coefficients in the partial expansion using the following method:
   The coefficients of $u(x)$ must equal those of $v(x)$. Solve the resulting simultaneous equations.

Example: *Let*

$$f(x) = \frac{7}{(x-1)(x-2)}$$

Hence $f(x)$ has partial expansion

Evaluate A and B.

Examples

1. *Autumn 2008*:
   Express the following as the sum of two partial fractions:

$$\frac{3x+7}{(x-1)(x-2)}.$$

Finishing this off:

2. *Find the partial fraction expansion of*

$$\frac{2x}{x^2 - 16}.$$

3. *Find the partial fraction expansion of*

$$\frac{7}{2x^2 + 5x - 12}$$

Exercises

1. *Factorise the following polynomials: (i)* $x^2 - 4x - 5$ *(ii)* $x^2 - 2x$ *(iii)* $15x^2 + x - 6$

2. *Write each as a single fraction:*

$$\frac{4x-3}{5} + \frac{x-3}{3}$$

$$\frac{1}{x-1} - \frac{2}{2x+3}$$

$$\frac{x}{x-1} + \frac{2}{x}$$

$$\frac{1}{x+1} - \frac{3}{2x-1}$$

3. *Find the Partial Fraction Expansions of (note that A, B won't necessarily be whole numbers):*

$$\frac{4x+3}{x(x+1)}.$$

$$\frac{3x+4}{(x+1)(x+2)}.$$

$$\frac{4-16x}{x(x-1)}.$$

$$\frac{2x-5}{(x-2)(x-3)}.$$

$$\frac{1}{x^2-4}.$$

$$\frac{4x}{x^2-x-6}.$$

$$\frac{6x-13}{2x^2+3x-2}.$$

Chapter 3

Linear Graphs

3.0.1 Outline of Chapter

- Manipulation of Data and Plotting of Graphs
- Evaluation and Interpretation of Constants
- Reduction of a Non-linear Relationship to a Linear Form

3.1 Reduction of a Non-linear Relationship to a Linear Form

Review of Logs

We defined the *logarithm to base a* as the inverse function of $f(x) = a^x$. Another way of looking at logarithms is to consider them as a way of converting multiplication to addition; division to subtraction; and powers to multiplication — by way of the *laws of logs*:

Examples

1. $\log(5x)$

2. $\log((3x-4)/y)$:

3. $\log\sqrt{10x^2-11}$.

Non-Linear Relationships

We saw earlier that many mathematical relationships take the form of a linear relationship:

In particular, if we know that some process is a linear one, but we don't know the *parameters* m and c, then we can make measurements of Y and X, and plot a straight-line graph of Y vs X:

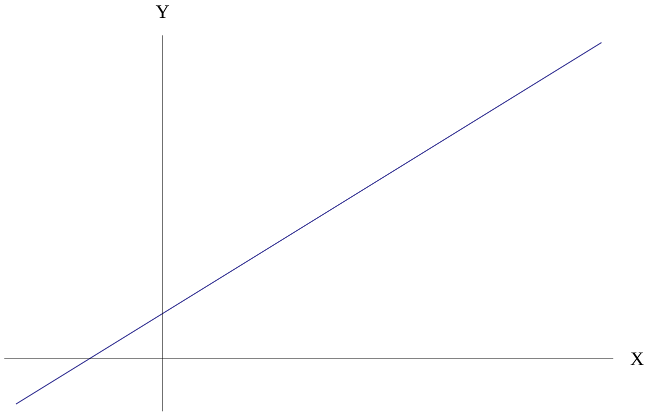


Figure 3.1: If Y and X are related via $Y = mX + c$, then the graph is a straight-line and we know that m is the slope and c is the y-intercept.

Example: Radioactive Decay

However not all mathematical relationships take such a tractable form. For example, it can be shown that the decay of a radioactive material is modeled by:

In this case $N(t)$ is the number of particles at a time t, N_0 is the initial number of particles and k is a constant. If we were to take measurements of $N(t_i)$ at various times t_i, and plotted them, we would get something like:

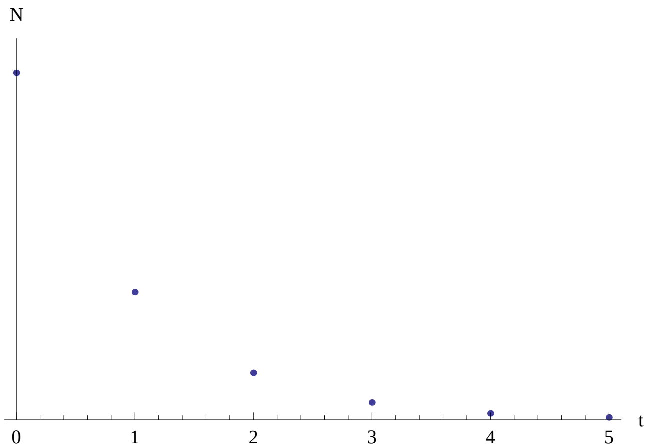


Figure 3.2: In Radioactive Decay, the number of particles reduced exponentially. It is very difficult to extract the parameters N_0 and k from such a plot.

However — and this will be our tool in this section — we can convert the multiplication and powers in the decay equation by taking logs:

Now what we can do is, as before, measure $N(t)$ and t at various t, but this time we will plot $\log(N(t))$ vs t — and the graph will be a straight line. Furthermore, the parameters $-k$ and $\log N_0$ — which we want to calculate — will be the given by:

We will review plotting on Friday, but for today we just want to focus on writing non-linear relationships in linear form.

Examples

In all of the following a and b are *constants*, e is the natural base, and the other two letters are *variables*. If the variables are x and y the aim is to write the following in the form:

for some functions f and g and constants m and c dependent on a and b alone. Please identify m and c.

1. *Write in linear form:*

$$F = ab^T.$$

2. *Write in linear form:*

$$v = ah^b.$$

3. *Write in linear form:*

$$v = ae^{bt}.$$

What if it's not all Multiplication and Powers?

The distance traveled by an object in t seconds, with initial speed u and constant acceleration a is given by:

This is not in linear form; can we put it in linear form. The answer is yes, but we have to utilise a tricky method known as *completing the square*. We must outline the general method for completing the square.

Completing the Square

Let $q(x)$ be a quadratic (we'll write q for short):

What I am aiming to do is find real numbers h and k such that I could write q in the form:

Multiplying out[1]:

$$q = a(x^2 + 2hx + h^2) + k.$$

Now what does this compare with what we have:

$$q = a\left(x^2 + \frac{b}{a}x\right) + c$$

Now we will use a trick — we will add and subtract ah^2 as follows:

$$a\left(x^2 + \frac{b}{a}x \underbrace{+h^2}_{\star}\right) + c \underbrace{-ah^2}_{\star\star}.$$

Now to finish this off we need to compare the two:

$$q = a(x^2 + 2hx + h^2) + k = a\left(x^2 + \frac{b}{a}x + h^2\right) + c - ah^2$$

We therefore need:

This we *can* do. As ever, examples shows off the theory.

[1] $(x + y)^2 = (x + y)(x + y) = x^2 + xy + xy + y^2 = x^2 + 2xy + y^2$

Examples

Complete the Square:

1. $x^2 - 4x + 29$.

2. $2x^2 + 6x + 3$.

Hence, looking at

$$ut + \frac{1}{2}at^2$$

Exercises

1. *For the following, a and b are the constants. Please write in linear form* $Y = mX + c$*; identifying* Y, m, X, c*:*

$$P = ae^{bt}.$$
$$R = aV^{b}.$$
$$y = ae^{bx}.$$
$$y = ax^{b}.$$
$$v = ae^{tb}.$$
$$T = aL^{b}.$$
$$p = at^{b}.$$

2. *By completing the square, write in linear form:*

$$q = x^2 + 6x + 13$$
$$q = x^2 + 2x + 5$$
$$q = x^2 - 4x + 29$$
$$q = x^2 - 3x + 10$$
$$q = 4x^2 + 12x + 13$$
$$y = ax^2 + bx$$
$$G = ax^2 + bx$$
$$F = \frac{1}{3}at^2 + bt$$

3.2 Manipulation of Data and Plotting of Graphs

Introduction

Consider the spread of a disease *Armpitus shmellyus* in a population. Suppose a medical team recorded the number of people infected over a series of days and compiled the data:

Day, t	8	12	18	23	28
Infectants, I	36	59	96	129	164

Suppose the relationship is thought to be of the form:

$$I(t) = at^b, \tag{3.1}$$

for some constants $a, b \in \mathbb{R}$. This would be an example of *polynomial growth.* Today we will look at a method of testing whether this data satisfies this law.

The Method

1. Write the equation in linear form. This means that starting with $y = f(x)$ we end up with:

   where $Y = g_1(x, y)$ and $X = g_2(x, y)$ (more usually each depend on y or x); and m and c are independent of x and y.

2. Take your data for x and y and manipulate the data to come up with a table:

x	y	X	Y
⋮	⋮	⋮	⋮

3. Plot the points X vs Y on graph paper.

4. If there is a reasonable straight-line fit we say that the law is verified. Draw the line if this is the case.

Example

Show that Armpitus shmellyus *data is a good fit to $I = at^b$.*

1. Taking the log of both sides (which base??):

   That is, we have $Y = \log I$ and $X = \log t$.

2. We now manipulate the data to fit the linear model:

t	I	$X = \log t$	$Y = \log I$
8	36		
12	59		
18	96		
23	129		
28	164		

3. Now we plot X vs Y — morryah $\log t$ vs $\log I$:

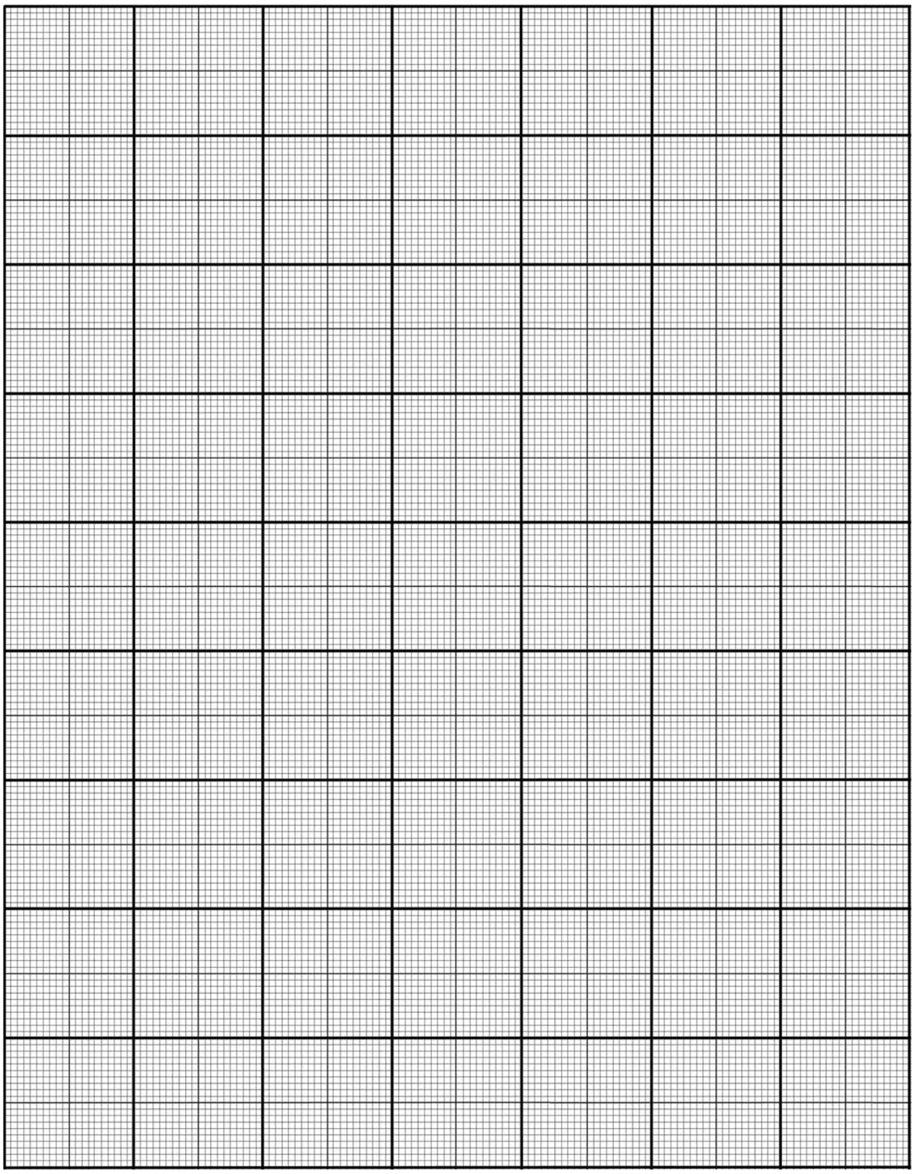


Figure 3.3: If Y and X are related via $Y = mX + c$, then the graph is a straight-line.

4. Hence the law is verified.

Another Example: Summer 2007

Pressure P and velocity v are believed to be connected by the law $v = ah^b$, where a and b are constants.

h	*10.6*	*13.4*	*17.2*	*24.6*	*29.3*
v	*9.77*	*11.0*	*12.44*	*14.88*	*16.24*

Write the equation in linear form. Plot a suitable graph to verify that the law is true.

1. Taking the log of both sides:

   That is, we have $Y = \log v$ and $X = \log h$.

2. We now manipulate the data to fit the linear model:

h	v	$X = \log h$	$Y = \log v$
10.6	9.77		
13.4	11.0		
17.2	12.44		
24.6	14.88		
29.3	16.24		

3. Now we plot X vs Y — morryah $\log h$ vs $\log v$:

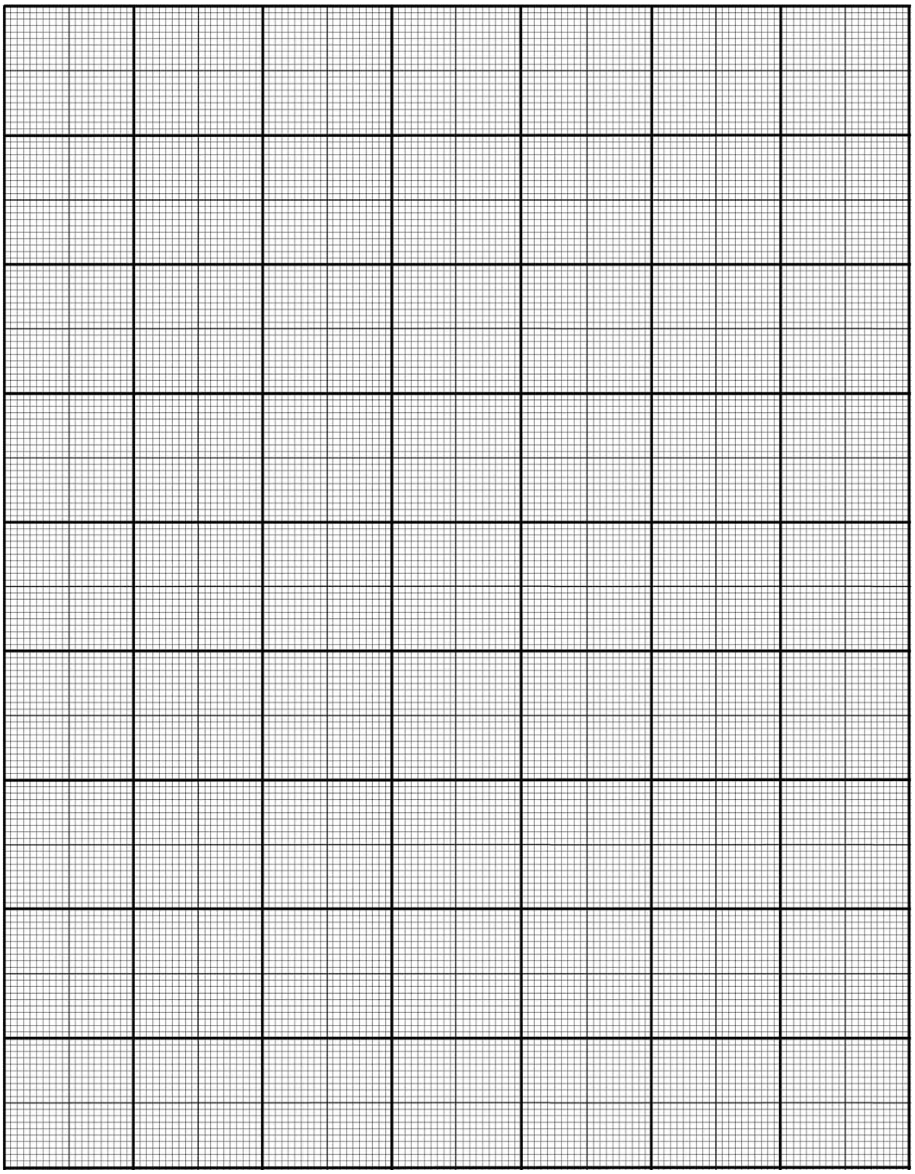


Figure 3.4: If Y and X are related via $Y = mX + c$, then the graph is a straight-line.

4. Hence the law is verified.

Exercises

1. *v is believed to be related to t according to the law $v = ae^{bt}$.*

t	27.3	37.5	42.5	47.8	56.3
v	700	190	100	50	17

   Show by plotting a graph of $\log v$ against t that the law relating these quantities is as stated.
   [HINT: The linear form is $\log v = bt + \log a$.]

2. *In an experiment on moments, a bar was loaded with a load W, at a distance x from the fulcrum. The results of the experiment were:*

x	10	20	30	40	50	60
W	55	27.5	18.33	13.75	11	9.17

   Verify that a law of the form $W = ax^b$ is true where a and b are constants by plotting $logW$ against $logx$.

3. *It is believed that x and y are related by a law of the form $y = ar^{kx}$ where a and k are constants. Values of x and y were measured and the results are as shown:*

x	0.25	0.9	2.1	2.8	3.7	4.8
y	6.0	10.0	25.0	42.5	85.0	198.0

   Show by plotting $\log y$ against x that the law is true.

Chapter 4

Trigonometry

4.0.1 Outline of Chapter

- Trigonometric Ratios
- Sine, Cosine and Other Rules
- The Unit Circle
- Trigonometric Graphs
- Trigonometric Equations

4.1 Trigonometric Ratios

In a right-angled triangle, special ratios exist between the angles and the lengths of the sides. We look at three of these ratios. Consider the right-angled triangle below:

Triangle Facts

1. The angles in a triangle add up to 180°.
2. If the triangle is right-angled we have Pythagoras Theorem:

Example

Three Special Triangles

4.1.1 The Inverse Trigonometric Functions

Given a value of $\sin A$, $\cos A$ or $\tan A$, we can find the value of θ using the $\sin^{-1}$, $\cos^{-1}$ or $\tan^{-1}$ button on our calculators.

Example

Find the angle A:

4.1.2 Calculating the Length of a Side in a R.A.T.

We can use a trigonometric ratio to calculate the length of a side of a triangle in a R.A.T. if we know the length of one side and one angle:

1. Draw the triangle filling in all the information we have.
2. Choose the trigonometric ratio that links the required side with the known angle and known side.
3. Write down this equation and solve.

Examples

Suppose Δ *is a right-angled triangle with an acute angle* $A = 32°$ *and a hypotenuse of length 8 cm. How long is the side opposite* $32°$*?*

In Δabc*,* $|\langle bca| = 90°$*,* $|\langle abc| = 34°$*, and* $|ac| = 20$ *m. Calculate* $|bc|$*, correct to two decimal places.*

Calculate the length of the hypotenuse of a triangle with an acute angle 31° with adjacent side of length 10.

Practical Applications

Many practical problems in navigation, surveying, engineering and geography involve solving a triangle. Mark on your triangle the angles and lengths you know, and label what you need to calculate, using the correct ratio to link the angle or length required with the known angle or length. Angles of elevation occur quite often in problems that can be solved with trigonometry.

Examples

A ladder, of length 5 m, rests against a vertical wall so that the base of the ladder is 1.5 m from the foot wall. Calculate the angle between the ladder and the ground, to the nearest degree.

When the angle of elevation of the sun is 28°, an upright flagpole casts a shadow of length 6 m. Calculate the height of the pole, correct to one decimal place.

Exercises

1. *Write down the value of the three trigonometric ratios of the angle A:*

2. *Use Pythagoras Theorem to find x, the length of the missing side, keeping roots where necessary. Then find the three trigonometric ratios of the angle A.*

3. *Using a 30-60 triangle, calculate* $\cos 60^\circ$.

4. *Show that* $\tan 80 \neq 2 \tan 40$ *and that* $\cos 50 \neq 2 \cos 25$.

5. *Find each angle: (i)* $\tan\theta = \sqrt{3}$ *(ii)* $\sin\theta = 1/\sqrt{2}$.

6. *Calculate, to the nearest degree, the angles marked by X:*

7. *In* Δxyz, $|\langle xyz| = 90^\circ$, $|xz| = 200$ *cm, and* $|yz| = 150$ *cm. Write down the ratio* $\sin\langle yxz$, *in it's simplest form. Calculate, to the nearest degree* $|\langle yxz|$ *and* $|\langle yzx|$.

8. *Calculate — to two decimal places — the length of the sides marked by A*

4.2 The Unit Circle: Extending the Definition of the Trigonometric Ratios

The Unit Circle

The familiar trigonometric functions sin, cos & tan were first encountered in the context of a right angled triangle. Where $\theta \neq 90^\circ$ in the triangle:

Here the concern is primarily with angles $0 < \theta < 90^\circ$. The following construction extends sin, cos & tan to *any* angle.

We call the different sectors of the unit circle *quadrants* as follows:

In the first quadrant, the sin, cos and tan are given by:

Therefore extend the definitions of sin, cos and tan so that:

Definition

Let $0^\circ \leq \theta < 360^\circ$ (*θ is an angle of degree greater than or equal to* 0 *and less than* 360). Suppose θ is the angle made with the positive x-axis and a line segment to a point $p = (x, y)$ on the unit circle. Then

Therefore the points on the unit circle are the points $(x, y) = (\cos\theta, \sin\theta)$ where θ is the angle between the line segment from $(0, 0)$ to $p(x, y)$ and the positive x-axis. This allows us to prove a little theorem about sin and cos.

Theorem

For all angles θ:

Proof:

•

Where are the Trigonometric Ratios Plus or Minus?

In the first quadrant, the sin, cos and tan functions are all positive (i.e. for angles between 0 and 90). However this is not necessarily the case for angles outside this quadrant. In the second quadrant:

In the third quadrant:

Finally, in the fourth:

Putting this all together:

People use the mnemonics *All Small Tin Cans* or *CAST* to remember this.

Final Extension

Angles Bigger than $360°$

Finally the sin, cos and tan functions are extended to the entire real number line by the principle that $360° \equiv 0°$. Therefore the angle $400°$ is equivalent to the angle $360 + 40 = 40°$; and we define sin, cos and tan to be periodic in the sense that every 360 they will begin to repeat themselves.

Negative Angles

We say that angles have a positive and negative orientation:

Therefore $-\theta \equiv 360^\circ - \theta$. This agrees with the above as:

How to Find the Sin, Cos and Tan of Angles in Quadrants 2, 3 and 4

Quadrant 2:

By (a)symmetry, $\cos\theta = -\cos\alpha$, $\sin\theta = \sin\alpha$ and $\tan\theta$:

So to find the sin, cos and tan of angles in the second quadrant, you draw the reference angle α — and change signs according to CAST:

Quadrant 3:

By (a)symmetry, $\cos\theta = -\cos\alpha$, $\sin\theta = -\sin\alpha$ and $\tan\theta$:

So to find the sin, cos and tan of angles in the third quadrant, you draw the reference angle α — and change signs according to CAST:

Quadrant 4:

By (a)symmetry, $\cos\theta = \cos\alpha$, $\sin\theta = -\sin\alpha$ and $\tan\theta$:

So to find the sin, cos and tan of angles in the fourth quadrant, you draw the reference angle α — and change signs according to CAST:

Note this calculation gives a little theorem.

Theorem

For $0 \leq \theta < 360$*, we have that*

Example

Express $\sin 1000°$ *as the sine of an angle* θ *with* $0 \leq \theta < 360°$.

Examples

1. *Find* $\cos 270$, $\sin 270$ *and* $\tan 270$.

2. *Find* $\sin 240$.

3. *Find* $\cos 150 + \sin 150$.

4. *Find all angles* $0 \leq \theta \leq 360$ *such that* $\cos\theta = 1/2$.

Exercises

Note all angles are measured in degrees (i.e. in Q.1, $225 = 225°$).

1. *Use the special triangles to find (please don't write roots/ surds in decimal form — keep them as roots).*
   (a) $\sin 225$.
   (b) $\sin 120$.
   (c) $\tan 30 + \tan 60$.

2. *In each case, find two values of θ, where $0 \le \theta \le 360$.*
   (a) $\sin\theta = -\sqrt{3}/2$.
   (b) $\sin\theta = -1/2$.
   (c) $\tan\theta = \sqrt{3}$.

3. *Find the values of θ such that $\sin\theta = 0$; $0 \le \theta \le 360$.*

4. *Find the value of A if $\cos A + 1 = 0$, $0 \le A \le 360$.*

5. *Using a calculator where necessary, find, correct to two decimal places, where $0 < B < 360$: (i) $5\sin B = 2$ (ii) $2\tan B + 7 = 0$.*

6. *If $\alpha = 30$, investigate if $\tan(2\alpha) = 2\tan\alpha$.*

7. *If $x = 10$ and $y = 23$, investigate if $\sin(x + y) = \sin x + \sin y$.*

8. *There is only one angle between 0 and 360 whose sin is $-\sqrt{3}/2$ and whose cos is $-1/2$. Find this angle.*

4.3 Trigonometric Equations

Radians

When trigonometrical functions are studied in more depth it becomes apparent that degrees are not a satisfactory nor natural unit of angle. The natural unit is that of a *radian*. We can use the unit circle to define radian measure:

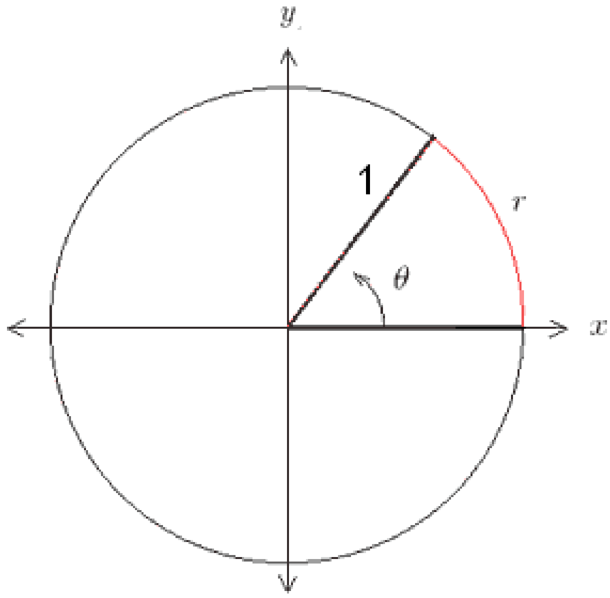


Figure 4.1: The angle θ, measured in radians, is the length of the arc subtended by θ. This arc *always* extends from the point $(1, 0)$ (East) to the point $(\cos\theta, \sin\theta)$.

For MATH6014 we will largely work in degrees but then convert to radians. What would be a good conversion? Well what is 180° in radians? Well the arc for 180° comprises the upper part of the unit circle. Now the circumference of a circle is given by $2\pi r$ where r is the radius. For a unit circle the radius is 1 so that the circumference of the circle is 2π — the upper part then, half the full circle, must have length $2\pi/2 = \pi$ — and hence radian measure π (also $360° \equiv 2\pi$).

Radian Conversion

$$\pi \text{ radians } = 180° \tag{4.1}$$

We can either work off this naturally (e.g. 20° is one ninth of 180° and hence $\pi/9$ radians); or else use the following:

$$\begin{aligned} 180° &= \pi \\ \Rightarrow 1° &= \frac{\pi}{180} \end{aligned} \tag{4.2}$$

That is you can replace every instance of "°" with $\pi/180$; e.g.

$$\begin{aligned} 20^\circ &= 20\left(\frac{\pi}{180}\right) \\ &= \frac{20}{180}\pi = \frac{1}{9}\pi = \frac{\pi}{9}. \end{aligned}$$

Trigonometric Equations

Trigonometric equations are equations involving the trigonometric ratios of some angle. Our aim is to find the angle which satisfies the the equation. For example,

$$\sqrt{2}\sin A\cos A - \cos A = 0\,,\ 0^\circ \leq A \leq 360^\circ \tag{4.3}$$

$$2\sin^2 x - \cos x = 1\,,\ 0^\circ \leq x \leq 360^\circ \tag{4.4}$$

$$\cos 5\theta + \cos\theta = 0\,,\ 0^\circ \leq \theta \leq 90^\circ \tag{4.5}$$

$$\frac{12}{5}\sin A - \frac{3}{2}\sin A = \frac{1}{2}\,,\ 0^\circ \leq A \leq 360^\circ \tag{4.6}$$

We will be asked to solve equations within a very specific range. In general, if the range is given in degrees, answer in degrees; if the range is given in radians (i.e. with πs), use radians. We will be answering the examples below and I will answer in both degrees and radians — first I will use the $^\circ = \pi/180$ but afterwards just by looking at the fraction that degree angle has of 180°.

The main principle use we will use to solve these equations is the *no zero-divisors theorem*. That is if $ab = 0$ then $a = 0$ or $b = 0$. It is difficult to say when a sum is zero — but it is easy to say when a product is zero — whenever any of the factors are zero:

WRITE THE SUM AS A PRODUCT.

Example: Equation (4.3)

First off write the sum as a product — i.e. factorise:

$$\cos A(\sqrt{2}\sin A - 1) = 0,$$

therefore we either have $\cos A = 0$ or $(\sqrt{2}\sin A - 1) = 0$.

Now for angles between 0° and 360° we look for when the x-coordinate in the unit circle is 1:

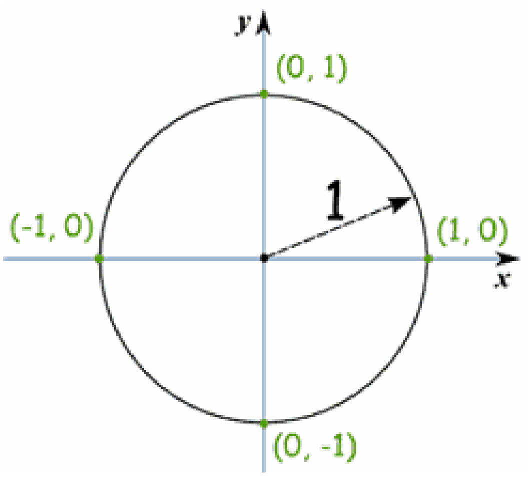


Figure 4.2: cos is equal to 0 at 90° and 270°.

Now we also need to find where $\sin A = 1/\sqrt{2}$. The $1/\sqrt{2}$ is related to the 45° triangle:

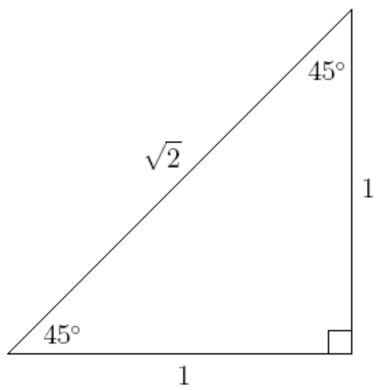
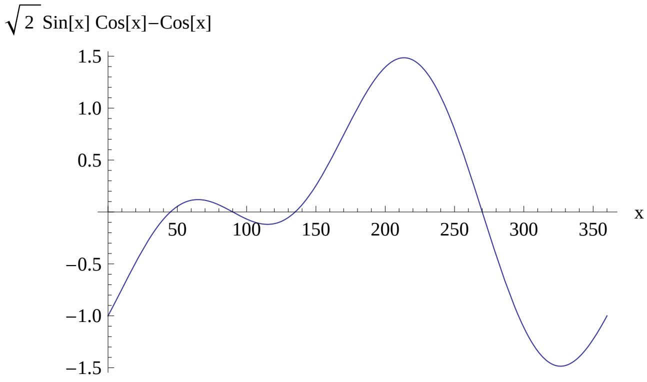


Figure 4.3: $\sin 45 = 1/\sqrt{2}$.

We also know that sin > 0 in the second quadrant so there will be an angle in the second quadrant with reference angle 30°. This corresponds to $180° - 30° = 150°$.
Ans: $\{30, 90, 150, 270\}$.



In Radians:

$$
\begin{aligned}
30^\circ &= 30\frac{\pi}{180} = \frac{\pi}{6}\\
90^\circ &= 90\frac{\pi}{180} = \frac{\pi}{2}\\
150^\circ &= 150\frac{\pi}{180} = \frac{5\pi}{6}\\
270^\circ &= 270\frac{\pi}{180} = \frac{3}{2}\pi
\end{aligned}
$$

Example: Equation (4.4)

The key here is to note that

$$\sin^2 A + \cos^2 A = 1,$$

so there is a relationship between, for example, $\sin^2$ and $\cos^2$. In this particular example we have:

$$
\begin{aligned}
2(1-\cos^2 x) - \cos x &= 1,\\
\Rightarrow 2 - 2\cos^2 x - \cos x - 1 &= 0,\\
\Rightarrow 2\cos^2 x + \cos x - 1 &= 0,\\
\Rightarrow 2(\cos x)^2 + (\cos x) - 1 &= 0.
\end{aligned}
$$

This is a quadratic equation in $\cos x$. Let $y = \cos x$;

$$
\begin{aligned}
2y^2 + y - 1 &= 0,\\
\Rightarrow 2y^2 + 2y - y - 1 &= 0,\\
\Rightarrow 2y(y+1) - 1(y+1) &= 0,\\
\Rightarrow (y+1)(2y-1) &= 0.\\
\Rightarrow y = -1 \text{ or } y &= 1/2
\end{aligned}
$$

That is we are looking for angles which have a cos of -1 or $1/2$.

Cosines equal to ± 1 or 0 are related to the unit circle:

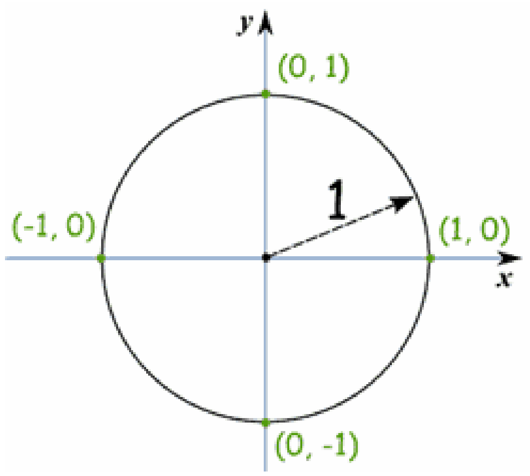


Figure 4.4: 180 is the only angle with $\cos 180° = -1$.

1/2 is related to the 30-60 triangle:

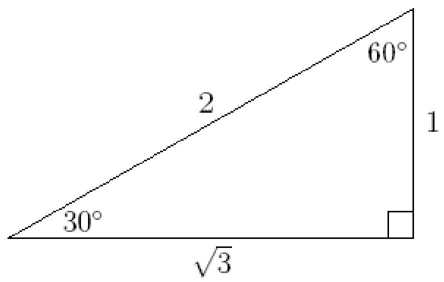
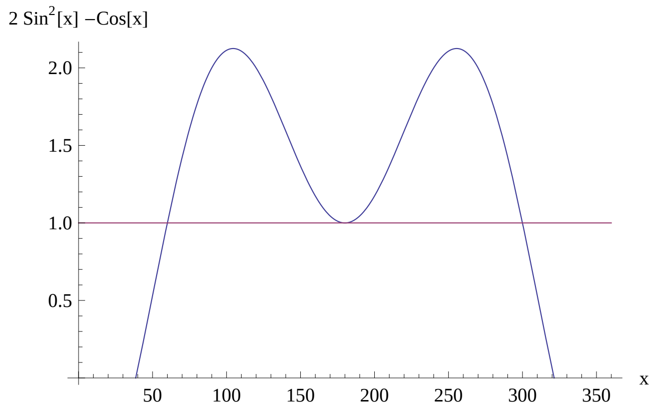


Figure 4.5: 60 is the angle with $\cos 60° = 1/2$.

We also know that cos is positive in the fourth quadrant. Using the reference angle 60 we have $360 - 60 = 300°$ is another solution.
Ans: $\{60, 180, 300\}$.

In Radians: Now 60 is one third of 180 so $60° = \pi/3$. Now $300° = 5 \times 60°$ so $300° = 5\pi/3$.



Example: Equation (4.5)

At first glance there doesn't seem to be any obvious way to employ our principle — write sums as products:

$$\cos 5\theta + \cos\theta.$$

However there are ways of handling these:

Compound Angle Formulae

We have a number of formulae in the tables — the proofs of which do not concern us:

$$\cos(A - B) = \cos A\cos B + \sin A\sin B \tag{4.7}$$
$$\cos(A + B) = \cos A\cos B - \sin A\sin B \tag{4.8}$$
$$\sin(A + B) = \sin A\cos B + \sin B\cos A \tag{4.9}$$
$$\sin(A - B) = \sin A\cos B - \sin B\cos A \tag{4.10}$$

We can mess around with these to generate the *sum-to-product* formulae — also to be found in the tables:

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \tag{4.11}$$
$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right) \tag{4.12}$$
$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \tag{4.13}$$
$$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right) \tag{4.14}$$

These tell us how to deal with equations such as this one. We use equation (4.11) here to re-write $\cos 5\theta + \cos\theta$ as a product:

$$\begin{aligned}\cos 5\theta + \cos\theta &= 2\cos\left(\frac{5\theta+\theta}{2}\right)\cos\left(\frac{5\theta-\theta}{2}\right)\\ &= 2\cos 3\theta\cos 2\theta.\end{aligned}$$

Now looking a the original equation:

$$\cos 5\theta + \cos\theta = 0$$
$$\Rightarrow 2\cos 3\theta \cos 2\theta = 0$$

Now we use the no-zero divisors theorem. The only time this is 0 is when $\cos 3\theta = 0$ or $\cos 2\theta = 0$. We'll look at each separately.

$\cos 3\theta = 0$: First we want to find out where $\cos A = 0$. By looking at the unit circle:

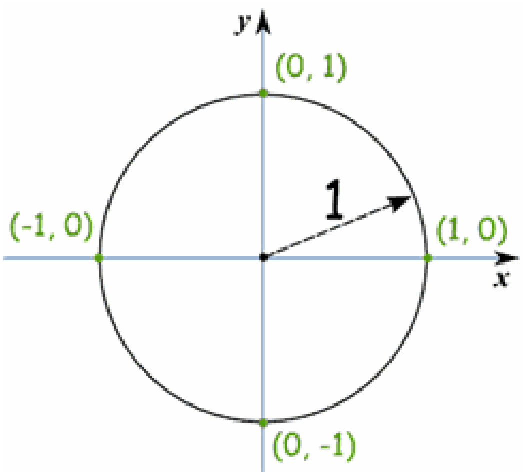


Figure 4.6: 90° and 270° are the angles in the unit circle with a cos of zero. However we must be aware of what range of angles we are interested in. In this example, the question asked for solutions in the range $0° \le \theta \le 90°$.

Now we know that $\cos A = 0$ when $A = 90°, 270°$. However $\cos A$ will have the same value if we add on any multiple of 360°. In fact the full solutions set is

$$\{A : \cos A = 0\} = \{90° + k360°, 270° + k360° : k \in \mathbb{Z}\}.$$

What we do in this situation is take as many solutions[1] as we need and hope we can discard most. My solutions to $\cos A = 0$ are:

$$A = 90°, 270°, 90° + 360°, 270° + 360°, 90° + 720°, 270° + 720°, 90° - 360°, 270° - 360°, \ldots$$
$$= 90°, 270°, 450°, 630°, 810°, 990°, -270°, -90°, \ldots$$

Now if I'm solving the equation $\cos 3\theta = 0$ I hence have the solutions:

$$3\theta = 90°, 270°, 450°, 630°, 810°, 990°, -270°, -90°, \ldots$$

Now it's clear that the negative numbers won't give me solutions $0° \le \theta \le 90°$. So I can discard these. Also I can discard any of the numbers larger than 450° because $450/3 = 150$ which is not in the required range. Hence I'm left with:

$$3\theta = 90°, 270°$$
$$\theta = 90/3, 270°$$
$$= 30°, 90°.$$

[1] at this point — we are going to solve for $\cos 3\theta = 0$. So for example, we know $\cos 90° = 0$ but that means $3\theta = 90° \Rightarrow \theta = 30°$.

This is only the first lot of solutions:

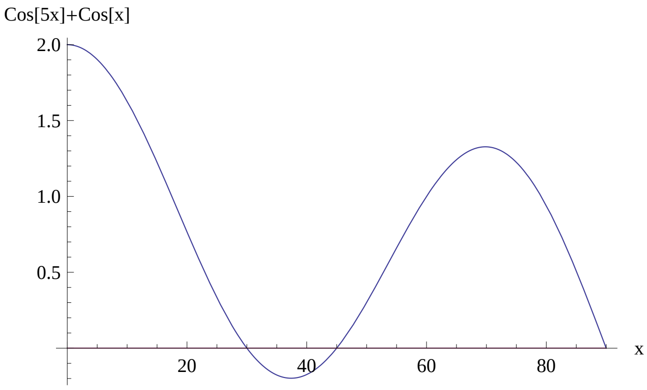
$\cos 2\theta = 0$**:** We can jump forward to:

$$2\theta = 90°, 270°, 450°, 630°, 810°, 990°, -270°, -90°$$

Again we can throw out negative entries and also we can throw out anything bigger than 270° as $270/2 = 135°$ — which is again outside the range. So we are left with:

$$\begin{aligned} 2\theta &= 90° \\ \Rightarrow \theta &= \frac{90}{2} = 45° \end{aligned}$$

Ans: $\{30°, 45°, 90°\}$.



In Radians: 30 is a sixth of 180 so $30° = \pi/6$. 45 is a quarter of 180 so $45° = \pi/4$. $90° = \pi/2$.

Example: Equation (4.6)

Now this is certainly the least straightforward:

$$\frac{12}{5}\sin A - \frac{3}{2}\cos A$$

However one of the beauties of trigonometric functions is that they are all so similar — and in fact if you add a multiple of $\sin x$ to a multiple of $\cos x$ you will get something that looks like both of them — i.e. another waveform. We can show that we can write:

$$a\sin x + b\cos x = R\sin(x+\alpha) \tag{4.15}$$

$$a\sin x - b\cos x = R\sin(x-\alpha) \tag{4.16}$$

for a given R and α related to a and b. Once we do this we can solve the original equation:

$$\begin{aligned} \frac{12}{5}\sin A - \frac{3}{2}\cos A &= \frac{1}{2} \\ \Rightarrow R\sin(A-\alpha) &= \frac{1}{2} \\ \Rightarrow \sin(A-\alpha) &= \frac{1}{2R} \\ \Rightarrow A-\alpha &\simeq \sin^{-1}(1/2R). \end{aligned}$$

So let's get to work on R and α. The trick is to compare $R\sin(x \pm \alpha)$ with $a\sin x \pm b\cos x$. Now, using the compound angle formula (4.9):

$$\begin{aligned} R\sin(x+\alpha) &= R\cos\alpha\sin x + R\sin\alpha\cos x \\ &= (R\cos\alpha)\sin x + (R\sin\alpha)\cos x \end{aligned}$$

Hence to be equal to $a\sin x + b\cos x$ we require:

$$R\cos\alpha = a \tag{4.17}$$
$$R\sin\alpha = b \tag{4.18}$$

Now squaring both sides we get:

$$\begin{aligned} R^2\cos^2\alpha &= a^2 \\ R^2\sin^2\alpha &= b^2 \\ \Rightarrow R^2\cos^2\alpha + R^2\sin^2\alpha &= a^2 + b^2 \\ \Rightarrow R^2\underbrace{(\cos^2\alpha + \sin^2\alpha)}_{=1} &= a^2 + b^2 \\ \Rightarrow R^2 &= a^2 + b^2 \\ \Rightarrow R &= \sqrt{a^2+b^2}. \end{aligned}$$

So we know what R is. What about α? Well if we again look at (4.17) and (4.18), and divide one by the other we have:

$$\begin{aligned} \frac{R\sin\alpha}{R\cos\alpha} &= \frac{b}{a} \\ \Rightarrow \frac{\sin\alpha}{\cos\alpha} &= \frac{b}{a} \\ \Rightarrow \tan\alpha &= \frac{b}{a} \\ \Rightarrow \alpha &= \tan^{-1}\left(\frac{b}{a}\right) \end{aligned}$$

For MATH6014 we don't have to go through all this analysis in class we can just remember it as a theorem

Theorem

We have that

$$a\sin x + b\cos x = R\sin(x \pm \alpha), \tag{4.19}$$

where

$$R = \sqrt{a^2+b^2}\ , \text{ and} \tag{4.20}$$
$$\alpha = \tan^{-1}\left(\frac{b}{a}\right) \tag{4.21}$$

Finally we can get back to the original equation! Rewrite

$$\frac{12}{5}\sin A - \frac{3}{2}\cos A = R\sin(A-\alpha),$$

where:

$$\begin{aligned} R &= \sqrt{\left(\frac{12}{5}\right)^2 + \left(\frac{3}{2}\right)^2} \\ &= \sqrt{\frac{144}{25} + \frac{9}{4}} \\ &= \sqrt{\frac{801}{100}}, \end{aligned}$$

where the last part was done by calculator. Split the square root (some of ye're calculators will do this anyway):

$$R = \sqrt{801}/10.$$

You *may* use decimals here as we're dealing with numbers that are not *nice*[2], but ensure to take maybe three decimal places.

Finally α:

$$\begin{aligned} \alpha &= \tan^{-1}\left(\frac{b}{a}\right) \\ &= \tan^{-1}\left(\frac{3/2}{12/5}\right) \\ &= \tan^{-1}\left(\frac{15}{24}\right) \end{aligned}$$

That is we can write:

$$\frac{12}{5}\sin A - \frac{3}{2}\cos A = \frac{\sqrt{801}}{10}\sin(A - \tan^{-1}(15)/24).$$

Hence solve as above:

$$\begin{aligned} \frac{\sqrt{801}}{10}\sin(A-\tan^{-1}(15)/24) &= \frac{1}{2} \\ \Rightarrow \sin(A-\tan^{-1}(15/24)) &= \frac{10}{2\sqrt{801}} \end{aligned}$$

Now we do inverse sin on the calculator:

$$\sin^{-1}(10/2\sqrt{801}) = 10.1756^\circ$$

[2]I would define nice numbers in the context of trigonometry to include $0, 1, -1, 2, 3, \sqrt{2}, \sqrt{3}$ or other simple numbers. If things are getting messy feel free to use decimals — take as many places as possible though while keeping your work neat — perhaps three places.

However the calculator will only give us the solution in the first quadrant. Sine is also positive in the second quadrant so we must find another solution in there using 10.176° as a *reference angle*:

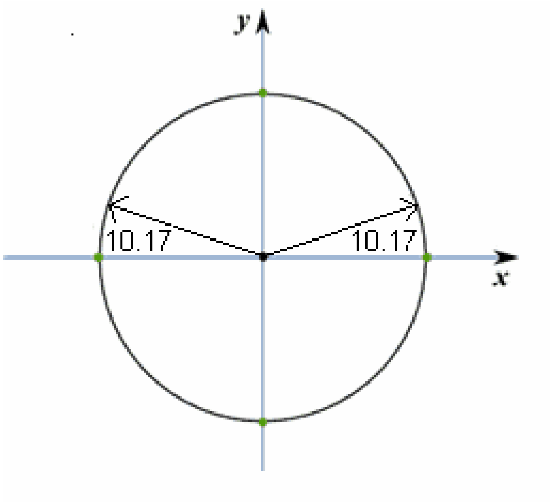


Figure 4.7: Sine is positive in both the first and second quadrants so when we want to know which angle A has a sine of, say $1/2$ — the calculator will only give us the first quadrant solution, $A = 30°$. To find the other solution in the second quadrant we must use 30° as a *reference angle* as shown here. In this case the other solution is $180° - 10.176° = 169.824°$.

So again we have some solutions:

$$A - \tan^{-1}(15/24) = 10.176, 169.824$$

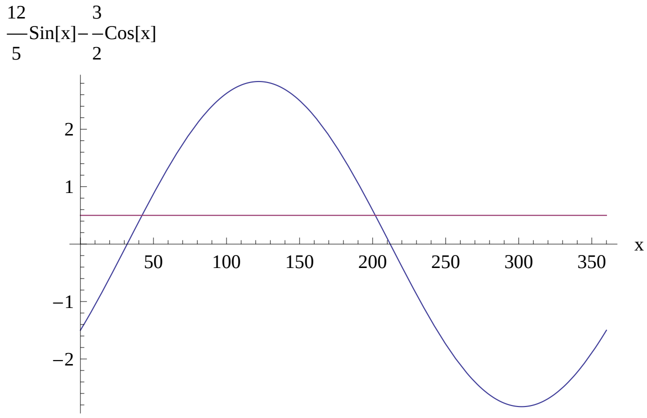
However note that we are asked for solutions in the range $0° \leq A \leq 360°$. Hence we might add on a couple of multiples of 360° — both positive *and* negative. We may as well use the decimal approximation to $\tan^{-1}(15/24)$ also, using a calculator $\tan^{-1}(15/24) = 32.005°$. Luckily we can choose α to be between 0° and 180° so we don't have to worry about other quadrants here (this is an exception — you see if you choose α in the third quadrant what happens is that you get the 'other' side of the cycle. In this context, evaluating α, you can always just use the calculator.):

$$\begin{aligned} A - 32° &= 10.18°, 169.82°, 10.18° + 360°, 169.82° + 360°, 10.18° - 360°, 169.82° - 360°, \ldots \\ &= 10.18°, 169.82°, 370.18°, 529.82°, -349.2°, -190.18°, \ldots \\ \Rightarrow A &= 10.18° + 32°, 169.82° + 32°, 370.18° + 32°, 529.82° + 32°, -349.2° + 32°, -190.18° + 32°, \ldots \end{aligned}$$

All but two of these solutions are not in the required range so can be discarded:

$$\begin{aligned} A &= 10.18° + 32°, 169.82° + 32°, \cancel{370.18° + 32°}, \cancel{529.82° + 32°}, \cancel{-349.2° + 32°}, \cancel{-190.18° + 32°}, \cancel{\ldots} \\ &= 42.8°, 201.82°. \end{aligned}$$

Ans: $\{42.8°, 201.82°\}$



In Radians:

$$\begin{aligned} 42.8^\circ &= 42.8 \cdot \frac{\pi}{180} \\ &= 0.747 \\ 201.82^\circ &= 201.82^\circ \cdot \frac{\pi}{180} \\ &= 3.522 \end{aligned}$$

Exercises

Solve the given equations in the given ranges:

$$\begin{aligned}
\cos 4x = 0\,&;\ \text{all solutions}\\
2\sin 5x - \sqrt{3}\,&;\ 0 \le \pi\\
\tan^2\theta - \tan\theta = 0\,&;\ 0 \le \theta \le 2\pi\\
2\cos^2\alpha + \sin\alpha = 2\,&;\ 0 \le \alpha \le 2\pi\\
\sin 3x + \sin x = 0\,&;\ 0^\circ \le x \le 360^\circ\\
4\sin\theta + 3\cos\theta = 2\,&;\ 0^\circ \le \theta \le 360^\circ\\
\sin\theta - \sqrt{2}\cos\theta = \frac{4}{5}\,&;\ 0^\circ \le \theta \le 360^\circ
\end{aligned}$$

4.4 Trigonometric Graphs

As we have said before, the graphs of sine and cosine are waveforms. That is their graphs look like waves. In general, we will be examining the graphs of functions of the form:

All of the numbers A, ω, α determine aspects of the graph of $y = f(t)$. Note that in *all* examples of graph plotting, the angles are measured in radians. Some examples,

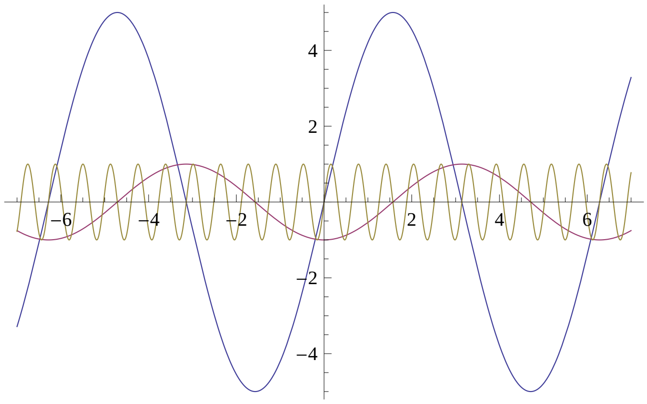


Figure 4.8: The graphs of $5\sin t$, $\sin(t - \pi/2)$ and $\sin 10t$.

The properties of the graph we are interested in are *amplitude, period, frequency, phase shift and maxima.*

Amplitude

Definition

The *amplitude* of a periodic function, A, is it's maximum/ minimum value.

From our unit circle we know that the sine and cosine functions take values between -1 and 1. That is:

Hence, for example, the function $f(t) = 5\sin t$ takes values between $+5$ and -5 and hence $5\sin t$ is the function (above) with the "large" waves.

Theorem

If a trigonometric function is represented by $f(t) = A\sin(\omega t + \alpha)$ *or* $f(t) = A\cos(\omega t + \alpha)$ *then the amplitude of* f *is given by* A.

Examples

What are the amplitudes of $10\sin t$, $20\cos t$ *and* $-3\sin x$*?*

Period

Definition

The *period* of a periodic function is the time taken to make one complete cycle.

Consider the function $f(t) = \sin 10t$. Now a normalised trigonometric function like $g(t) = \cos t$ repeats itself every 2π so the period is ordinarily 2π. However, $f(t) = \sin 10t$ has repeated itself 10 times over the course of the interval $[0, 2\pi]$; i.e. $f(2\pi) = 20\pi$ — ten cycles have already passed. Hence in this case the period is $2\pi/10$ — $f(t)$ repeats itself every $2\pi/10$:

Hence, above, $\sin 10t$ is the functions with a lot of cycles — it repeats itself more often than the others.

Theorem

If a trigonometric function is represented by $f(t) = A\sin(\omega t + \alpha)$ *or* $f(t) = A\cos(\omega t + \alpha)$ *then the period of* f *is given by* $2\pi/\omega$.

Examples

What are the periods of $\sin 5t$, $\cos t/2$ *and* $\sin \pi x$*?*

This leads nicely onto

Frequency

If the period is small and therefore the function repeats itself often we say that the function has high *frequency*. Conversely, if the period is large so that the function does not repeat itself often we say that the function has low frequency.

Definition

The *frequency* of a periodic function is the number of cycles that occur in unit time.

We can show quite simply that:

Theorem

The frequency of a periodic function, f*, is given by:* $1/T$*; i.e.* $f = \omega/2\pi$.

Examples

What are the frequencies of $\sin 0.8t$, $\cos t/4$ *and* $\sin 2\pi x$*?*

Phase Angle or Phase Displacement

Note that $\sin 0 = 0$ and $\cos 0 = 1$. That is both of these functions, $\sin t$ and $\cos t$, either start at 0 or 1 ($-\cos t$ starts at -1):

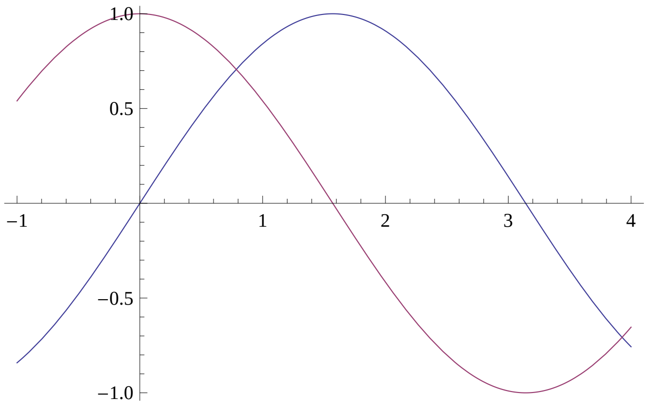


Figure 4.9: The graphs of $\sin t$ and $\cos t$.

Maxima and Minima

4.5 The Sine and Cosine Rules

If a triangle is not right-angled, it is possible to find missing sides or angles using the Sine or Cosine Rules.

The Sine Rule

In the triangle shown,

we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}. \tag{4.22}$$

Proof

Drop a perpendicular onto the side c:

Now we can measure h using two different methods. In the triangle on the left we have that:

On the right we have,

Hence we have

Example

Find $|qs|$ correct to one decimal place.
Solution: Firstly, $|\langle psq| = 180^\circ - 43^\circ - 68^\circ = 69^\circ$. Therefore $|\langle qsr|$:

Cosine Rule

In the triangle shown,

we have

$$a^2 = b^2 + c^2 - 3bc\cos A. \tag{4.23}$$

Example

In the example above, find $|qr|$ correct to one decimal place.
Solution: In this case we do not know an angle and an opposite side so we must use the Cosine Rule:

Example

Find the angle A to the nearest degree.

Exercises

1. *Find, correct to one decimal place, $|bd|$ and $|cd|$.*

2. *Find, to the nearest whole number, $|ac|$ and $|ad|$.*

3. *Find the angle, A.*

4. *The sides of a triangle have lengths 9 cm, 5 cm and 7 cm. Find the measurement of the smallest angle to the nearest degree.*

5. *Let ABC be a triangle such that $|AB| = 10$ cm, $\langle A = 30^\circ$ and $\langle B = 100^\circ$. Solve the triangle.*

6. *Find, correct to 3 significant figures, the values of a, B and C:*

4.6 Area and Volume

1. The floor of a room 3.2 m long by 2.7 m wide is to be covered with tiles. Each tile is 10 cm by 15 cm. Find the number of tiles needed to cover the floor.

2. A closed cylindrical metal can has an external radius of 7 cm and a height of 10 cm. Calculate the total surface area of the can.

3. Calculate the volume of a sphere of radius 4.5 cm. Four of these spheres fit flush into a cylinder of radius 4.5 cm and of height h. Calculate the height, h, of the cylinder, the volume of the cylinder and the fraction of the volume of the cylinder taken up by the four spheres.

4. The curved surface area of a cylinder is 219.8 cm^2 and it's radius is 3.5 cm. Find the height of the cylinder and the volume of the cylinder.

5. A solid lead sphere os radius 6 cm is melted down and recast as a solid cylinder of height 18 cm. Calculate the radius of the sphere.

6. A cylinder has a volume of 500 cm^3. If it's height is 10 cm, calculate it's radius.

7. A picture is 40 cm long by 20 cm wide. If a constant-width, rounded frame of area 0.095 m^2 is put around the picture, calculate the width of the frame.