MATH6037 - Mathematics for Science 2.1 with Maple

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Contents

0.1 Introduction

Lecturer

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This page will comprise the webpage for this module and as such shall be the venue for course announcements including definitive dates for continuous assessments. This page shall also house such resources as a copy of these initial handouts, the exercises, a copy of the course notes, links, as well as supplementary material. Please note that not all items here are relevant to MATH6037; only those in the category 'MATH6037'. Feel free to use the comment function therein as a point of contact.

Module Objective

This module contains further calculus including methods of integration and partial differentiation. An introduction to numerical methods and the theory of Laplace transforms completes the module.

Module Content

Further Calculus

Integration by Parts and Partial Fractions. Functions of two or more variables. Surfaces. Partial Derivatives. Applications to Error Analysis.

Numerical Methods

Solving equations using the Bisection Method and the Newton-Raphson Method. Approximate definite integrals using the Midpoint, Trapezoidal and Simpson's Rules.

Introduction to Laplace Transforms

Definition to transform. Determining the Laplace transform of basic functions. Development of rules. First shift theorem. Transform of a derivative. Inverse transforms. Applications to solving Differential Equations. Applications to include the Damped Harmonic Oscillator.

Assessment

Total Marks 100: End of Year Written Examination 70 marks; Continuous Assessment 30 marks.

Continuous Assessment

The Continuous Assessment will be divided equally between a one hour written exam in Week 6 and your weekly participation in the Maple Lab.
Absence from a test will not be considered accept in truly extraordinary cases. Plenty of notice will be given of the test date. For example, routine medical and dental appointments will not be considered an adequate excuse for missing the test.

Lectures

It will be vital to attend all lectures as although I intend that there will be a copy of the course notes available within the month, many of the examples, proofs, etc. will be completed by us in class.

Maple Labs

Maple Labs will commence next week and are designed both to introduce you to this software and to aid your understanding of the course material.

Exercises

There are many ways to learn maths. Two methods which arent going to work are

1. reading your notes and hoping it will all sink in
2. learning off a few key examples, solutions, etc.

By far and away the best way to learn maths is by doing exercises, and there are two main reasons for this. The best way to learn a mathematical fact/ theorem/ etc. is by using it in an exercise. Also the doing of maths is a skill as much as anything and requires practise.
I will present ye with a set of exercises every week. In this module the "Lecture-Supervised Learning" is comprised of you doing these exercises, giving them to me on a weekly basis, marking them, and returning them. In addition I will provide a set of solutions online. To protect myself from mounting corrections I must warn you that the only work from the previous week shall be corrected. For example, do not expect me to correct work you did in week 2 to be corrected in week 10. Everyone shall have access to the solution sets however. The webpage may contain a link to a set of additional exercises. Past exam papers are fair game. Also during lectures there will be some things that will be *left as an exercise*. How much time you can or should devote to doing exercises is a matter of personal taste but be certain that effort is rewarded in maths.

Reading

Your primary study material shall be the material presented in the lectures; i.e. the lecture notes. Exercises done in tutorials may comprise further worked examples. While the lectures will present everything you need to know about MATH6037, they will not detail all there is to know. Further references are to be found in the library in or about section 510 and 510.2462. Good references include:

- J. Bird, 2006, *Higher Engineering Mathematics*, Fifth Ed., Newnes.
- A. Croft & R. Davison, 2004, *Mathematics for Engineers — A Modern Interactive Approach*, Pearson & Prentice Hall,

The webpage will contain supplementary material, and contains links and pieces about topics that are at or beyond the scope of the course. Finally the internet provides yet another resource. Even Wikipedia isnt too bad for this area of mathematics! You are encouraged to exploit these resources; they will also be useful for for further maths modules.

Exam

The exam format will roughly follow last year's. Acceding to the maxim that learning off a few key examples, solutions, etc. is bad and doing exercises is good, solutions to past papers shall not be made available (by me at least). Only by trying to do the exam papers yourself can you guarantee proficiency. If you are still stuck at this stage feel free to ask the question come tutorial time.

0.2 Motivation: What makes a good Door Closer?

Figure 1: A good door closer should close automatically, close in a gentle manner and close as fast as possible.

One possible design would be to put a mass on the door and attach a spring to it (just for ease of explanation we'll only worry about one dimension).

Assuming that the door is swinging freely the only force closing the door is the force of the spring. Now *Hooke's Law* states that the force of a spring is directly proportion to it's distance from the equilibrium position. If the door is designed so that the equilibrium position of the spring corresponds to when the door is closed flush, then if $x(t)$ is the position of the door t seconds after release, then the force of the spring at time t is given by:

where $k \in \mathbb{R}$ is known as the spring constant.

We will see later on that this system *does* close the door automatically but the balance between closing the door gently and and closing the door quickly is lost. Indeed if the door is released from rest at $t = 0$, then the speed of the door will have the following behaviour:

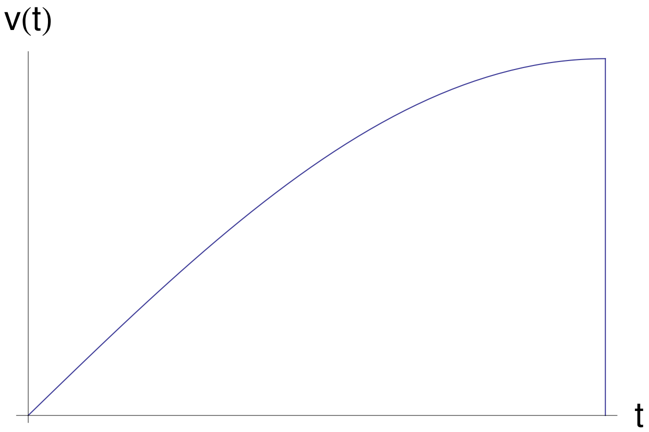


Figure 2: With a spring system alone, the door will quickly pick up speed and slam into the door-frame at maximum speed.

Clearly we need to slow down the door as it approaches the door-frame. A simple model uses a *hydraulic damper*:

Figure 3: A hydraulic damper increases its resistance to motion in direct proportion to speed.

With the force due to the hydraulic damper proportional to speed, the force of the hydraulic damper at time t will be:

for some $\lambda \in \mathbb{R}$. Now by Newton's Second Law:

and the fact that speed is the first derivative of distance, and in turn acceleration is the first derivative of speed, means that the *equation of motion* is given by:

We will see much later on that suitably chosen k and λ will provide us with a system that closes automatically, closes in a gentle manner and closes as fast as possible.
Equations of this form turn up in many branches of physics and engineering. For example, the oscillations of an electric circuit containing an inductance L, resistance R and capacitance C in series are described by

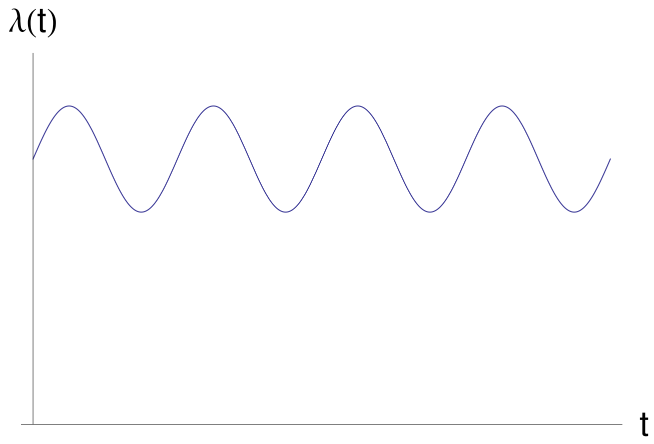
$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0, \tag{1}$$

in which the variable $q(t)$ represents the charge on one plate of the capactitor. These class of equations, *linear differential equations*,

may be solved in various different ways. In this module we will explore one such method — that of *Laplace Transforms*.

Figure 4: Top Gear dropped a VW Beetle from a height of 1 mile and it spun in the air as it fell.

If we are trying to formulate a model for the fall of this car we would have to try and account for the way the roll of the car means that the coefficient of the drag term $(\lambda v(t))$ varies between its maximum and minimum in a wave-like way:



A function with this behaviour is:

$$\lambda(t) = \frac{1}{2}(M+m) + \frac{1}{2}(M-m)\sin\omega t \tag{2}$$

where M and m are the maximum and minimum of $\lambda(t)$ and ω is a constant related to the angular frequency. Then the equation of motion is of the form:

Neither the method of using Laplace Transforms nor any other method I know of solves this differential equation.

Unfortunately this is typical, and for many systems for which a differential equation may be drawn, it may be impossible to solve the equations. There are a number of numerical techniques which can give approximate answers. However if we are participating in some industrial project with millions spent on it we don't want to be chancing our arms on any old estimate or guess. *Approximation Theory* aims to control these errors as follows. Suppose we have a Differential Equation with solution $y(x)$. An approximate solution $A_y(x)$ to the equation can be found using some numerical method. If the approximation method is sufficiently 'nice' we may be able to come up with a measure of the error:

Here $|\cdot|$ is some measure of the *distance* between $y(x)$ and $A_y(x)$. The most common measure here would be maximum error:

We would call the parameter ε here the *control* or the *acceptable error*. Some classes of problem are even nicer in that with increasing computational power we can develop a sequence of approximate solutions $\{A_y^1(x), A_y^2(x), A_y^3(x), \dots\}$ with decreasing errors $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \dots\}$:

Even nicer still from a mathematical point of view if we can find a sequence of approximations with errors decreasing to zero:

In this case we say that the sequence of approximations *converges*.

In this module we will take a first foray into the approximation theory of numerical methods by estimating the roots of equations and of estimating numerical integrals.

The first chapter will focus on some of the mathematical background needed to look at these areas.

Chapter 1

Further Calculus

1.0.1 Outline of Chapter

- Review of Integration
- Integration by Parts and Partial Fractions
- Functions of two or more variables
- Surfaces
- Partial Derivatives
- Applications to Error Analysis

1.1 Review of Integration

Differentiation

In the figure below, the line from a to b is called a *secant* line.

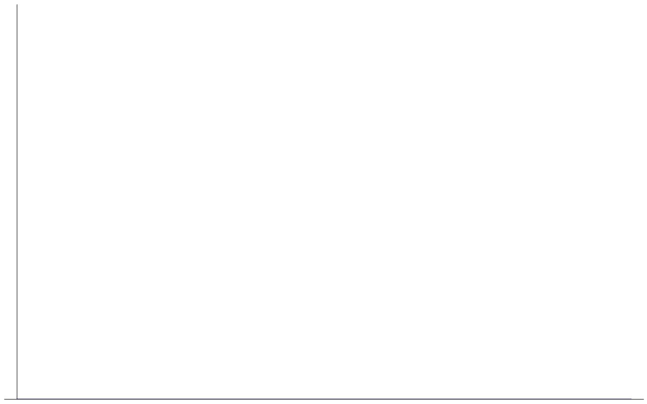


Figure 1.1: Secant Line.

Introduce the idea of slope. The slope of a line is something intuitive. A steep hill has a greater slope than a gentle rolling hill. The slope of the secant line is simply the ratio of how much the line travels vertically as the line travels horizontally. Denote slope by m:

What about the slope of the curve? From a to b it is continuously changing. Maybe at one point its slope is equal to that of the secant but that doesn't tell much. It could be estimated, however, using a ruler the slope at any point. It would be the tangent, as shown:

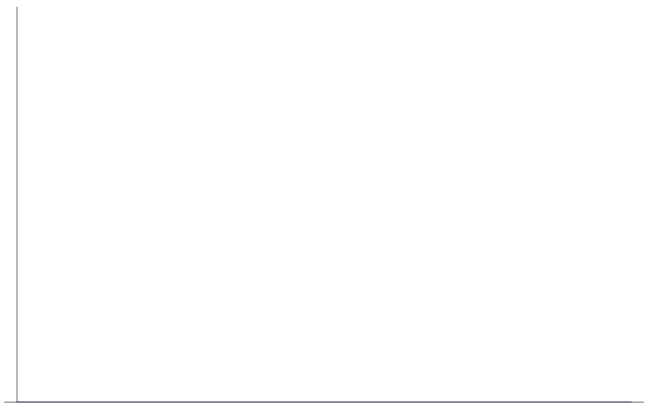


Figure 1.2: Tangent Line

The above line *is* the slope of the curve at x_0.

Construct a secant line:

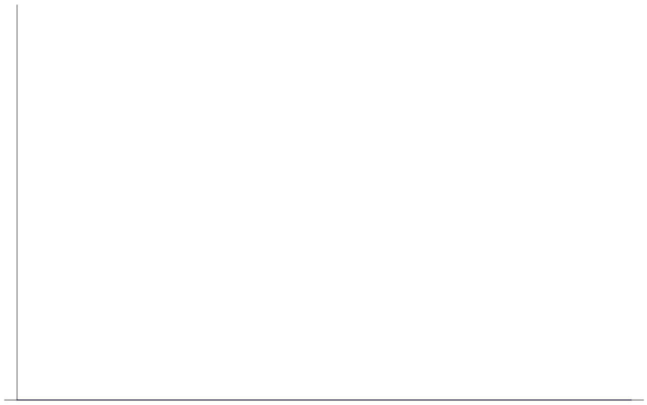


Figure 1.3: Secant line and Tangent line

Now the slope of this secant is given by:

It is apparent that the secant line has a slope that is close, in value, to that of the tangent line. Let h become smaller and smaller:

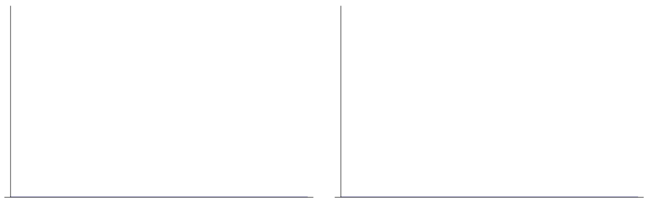


Figure 1.4: Secant line approaching slope of Tangent line

The slope of the secant line is almost identical to that of our tangent. Let $h \to 0$. Of course, if $h = 0$ there is no secant. But if h got *so close to* 0 *as doesn't matter* then there would be a secant and hence a slope:

This $f'(x)$ is the *derivative of f(x).* This gives the slope of the curve at *every* point on the curve.

In the *Leibniz notation*, y is equivalent to $f(x)$. However, the notation for the derivative of y is:

It must be understood that if $y = f(x)$; then

and there is no notion of canceling the ds; it is just a notation. It is an illuminating one because if the second graph of figure 1.4 is magnified about the secant:

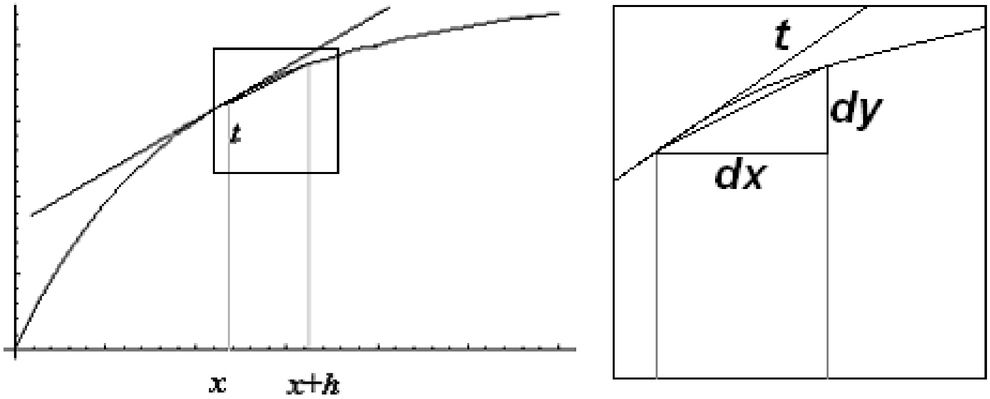
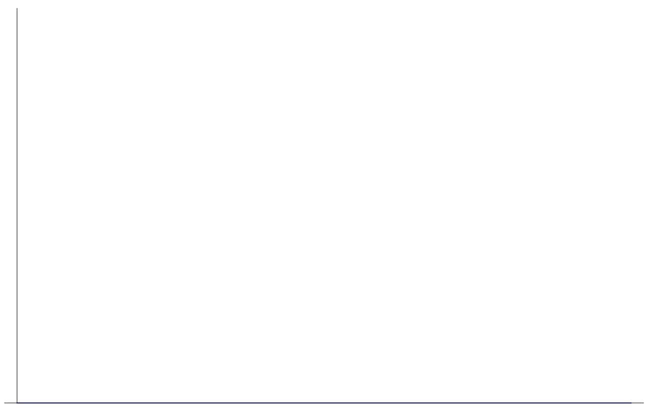


Figure 1.5: Leibniz notation for the derivative

If dy is associated with a small variation in $y \sim f(x+h) - f(x)$; and dx associated with a small variation in $x \sim h$; then dy/dx makes sense.

Integration

What is the area of the shaded region under the curve $f(x)$?



Start by subdividing the region into n strips S_1, S_2, ..., S_n of equal width as Figure 1.6.

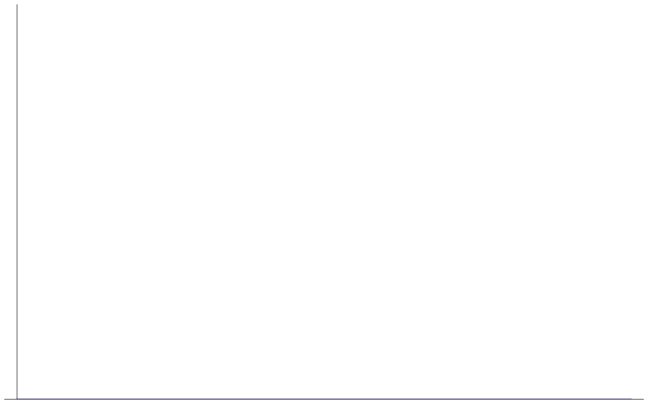
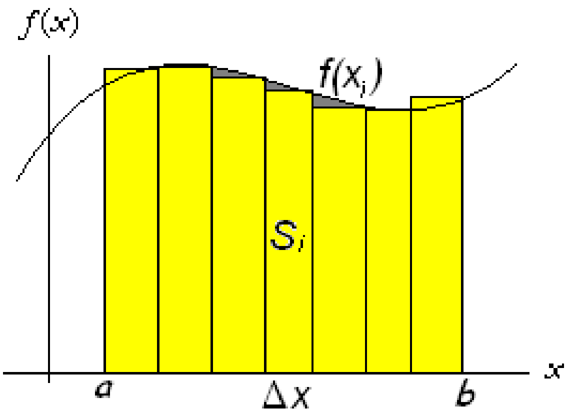


Figure 1.6:

The width of the interval $[a, b]$ is $b - a$ so the width of each of the n strips is

Approximate the ith strip S_i by a rectangle with width Δx and height $f(x_i)$, which is the value of f at the right endpoint. Then the area of the ith rectangle is $f(x_i)\ \Delta x$:



The area of the original shaded region is approximated by the sum of these rectangles:

This approximation becomes better and better as the number of strips increases, that is, as $n \to \infty$. Therefore the area of the shaded region is given by the limit of the sum of the areas of approximating rectangles:

Definition: The Definite Integral

If $f(x)$ is a *continuous* function defined in $[a, b]$ and x_i, Δx are as defined above, then the *definite integral of f from a to b* is

So an integral is an infinite sum. Associate $\int \cdot \, dx \sim \lim_{n\to\infty} \sum_n \cdot \Delta x$.

Fundamental Theorem of Calculus

If f is a function with derivative f' then

Examples

1. *Evaluate*

$$\int_0^2 3x^2 \, dx$$

2. *Evaluate*

$$\int_1^e \frac{1}{x} \, dx$$

3. *Evaluate*

$$\int_0^{\pi} -\sin x \, dx$$

Definition: The Indefinite Integral

If $f(x)$ is a function and it derivative with respect to x is $f'(x)$, then

where c is called the *constant of integration.*

The Indefinite Integral $\int f(x)\, dx$ asks the questions:

Note the constant of integration. It's inclusion is vital because if $f(x)$ is a function with derivative $f'(x)$ then $f(x) + c$ also has derivative $f'(x)$ as:

Geometrically a curve $f(x)$ with slope $f'(x)$ has the same slope as a curve that is shifted upwards; $f(x)+c$. Note that the constant of integration can be disregarded for the indefinite integral. Suppose the integrand is $f'(x)$ and the anti-derivative is $f(x) + c$. Then:

Finding the derivative of a function f at x is finding the slope of the tangent to the curve at x. Integration meanwhile measures the area between two points $x = a$ and $x = b$.

The Fundamental Theorem of Calculus states however that differentiation and integration are intimately related; that is given a function f:

i.e. differentiation and integration are essentially inverse processes.

Examples

Integrate 1-3:

1. $\int 3x^2\,dx$

2. $\int (1/x)\,dx$

3. $\int -\cos x\,dx$

4. *Evaluate:*

$$\int_0^{\pi} 4x^3\,dx$$

Straight Integration

From the Fundamental Theorem of Calculus

$$\int f'(x)\,dx = f(x) + c \tag{1.1}$$

Thus:

$f(x)$	$\int f(x)$
$x^n\ (n \neq -1)$	$\frac{x^{n+1}}{n+1} + c$
$\cos x$	$\sin x + c$
$\sin x$	$-\cos x + c$
e^x	$e^x + c$
$\sec^2 x$	$\tan x + c$
$\frac{1}{x}$	$\ln\lvert x\rvert + c$

Examples

Integrate:

1. $\int \sqrt{x}\,dx$

2. $\int (1/x^2)\, dx$

Let $a \in \mathbb{R}$. Now

$$\frac{d}{dx} e^{ax} = a e^{ax}$$

Example

Evaluate:

$$\int_0^1 e^{-x}\, dx$$

Also because

$$\frac{d}{dx}\sin nx = n\cos nx \text{ , and}$$
$$\frac{d}{dx}\cos nx = -n\sin nx$$

Example

Integrate $\int \cos 2x\, dx$.

Also, let $a > 0$;

$$\frac{d}{dx}\frac{1}{a}\tan^{-1}\frac{x}{a} = \frac{1}{a}\frac{1}{1+x^2/a^2}.\frac{1}{a}$$

Example

Evaluate:

$$\int_0^1 \frac{1}{1+x^2}\, dx$$

Also

$$\frac{d}{dx}\sin^{-1}\frac{x}{a} = \frac{1}{\sqrt{1-x^2/a^2}}\frac{1}{a}$$

Example

Integrate:

$$\int \frac{1}{\sqrt{1-x^2}}\,dx$$

Properties of Integration

Proposition

Let f, g be integrable functions and $k \in \mathbb{R}$:

(a)

$$\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx \tag{1.2}$$

(b)

$$\int k\,f(x)\,dx = k\int f(x)\,dx\ ,\ \textit{where } k \in \mathbb{R} \tag{1.3}$$

The Substitution Method for Evaluating Integrals

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du \tag{1.4}$$

where $u = g(x)$

Examples

Spot the patterns:

$$\int 2x^2\sqrt{x^3+1}\,dx$$

$$\int t(5+3t^2)^8\,dt$$

$$\int x^2 e^{x^3}\,dx$$

$$\int s^2\sqrt[5]{7-4s^3}\,ds$$

$$\int \sqrt{1+\frac{1}{3x}}\,\frac{dx}{x^2}$$

$$\int x^2\sec^2(x^3+1)\,dx$$

$$\int \sin^2 x\cos x\,dx$$

Examples

Evaluate 1-2:

1.
$$\int_0^{\pi/4} e^{\tan x} \sec^2 x \, dx$$

2.
$$\int_0^{\sqrt{3}} \frac{x}{\sqrt{x^2+1}} \, dx$$

3. *Integrate:*

$$\int \frac{dx}{\sqrt{15 + 2x - x^2}}$$

LIATE

If we cannot see a $g(x)$, $g'(x)$ pattern we can use the LIATE rule. Choose u according to the most complicated expression in the following hierarchy:

L

I

A

T

E

In general this works well.

Examples

1. *Integrate:*

$$\int x^2 \sec^2(x^3 + 1)\, dx$$

2. *Evaluate:*

$$\int_0^{\pi^2/4} \frac{\cos\sqrt{x}}{\sqrt{x}}\, dx$$

Exercises

1. *Evaluate*

   (a)
   $$\int_0^1 (2x+5)\,dx$$

   (b)
   $$\int_0^1 \frac{2x+5}{x^2+5x+1}\,dx$$

   (c)
   $$\int_0^1 e^{2x+5}\,dx$$

2. (a) *The following integral could be found by expanding $(1-x^2)^5$. Note however that the derivative of $(1-x^2)$ is $-2x$. By making a substitution, evaluate:*
   $$\int_0^1 x(1-x^2)^5\,dx$$

   (b) *By noting that $a^{-n} = 1/a^n$, evaluate*
   $$\int_1^2 \frac{dx}{e^x}$$
   correct to 3 decimal places.

3. *It can be shown that for $x \geq 1$,*
   $$\frac{1}{x^2} \leq \frac{1}{x} \leq \frac{1}{\sqrt{x}}$$
   By integrating these functions between suitable values, show that
   $$\frac{1}{2} \leq \log x \leq 2\sqrt{2} - 2$$

4. *Evaluate:*

   (a)
   $$\int_1^2 \frac{(x+1)^2}{2x}\,dx$$

   (b)
   $$\int_0^{\sqrt{\pi/2}} x\cos(x^2)\,dx$$

   (c)
   $$\int_1^{5/2} \frac{dx}{\sqrt{(4-x)(2+x)}}$$

5. *Let*
$$f(x) = \frac{e^x + e^{-x}}{2}\ , \text{ and } g(x) = \frac{e^x - e^{-x}}{2}$$
Prove that $f'(x) = g(x)$. *Hence find*
$$\int_0^{\log 1/2} 2f(x)g(x)\,dx$$

6. (a) *Suppose that* $f(x) = ax^2 + bx + c$. *There is a process called* completing the square *where we write:*
$$ax^2 + bx + c = (x+p)^2 + q,$$
for some $p, q \in \mathbb{R}$. *Complete the square of* $x^2 + 4x + 5$. *Now making a substitution of the form* $u = (x + p)$, *integrate*
$$\int \frac{1}{x^2 + 4x + 5}\,dx$$
   (b) *Integrate the following:*
$$\int \frac{2x + 5}{x^2 + 4x + 5}\,dx$$

7. *By manipulating the right-hand side, show that*
$$\frac{1}{(e^x + 1)^2} = 1 - \frac{e^x}{e^x + 1} - \frac{e^x}{(e^x + 1)^2}$$
Hence find
$$\int \frac{dx}{(e^x + 1)^2}$$

8. *Evaluate*
$$\int_0^3 \frac{x}{x^2 + 9}\,dx$$

1.2 Integration by Parts

Introduction

We should at this stage be aware of the sum, product, quotient and chain rules for differentiation:

In all cases here u and v are understood to be $u(x)$ and $v(x)$ — functions of x. In theory, because of the Fundamental Theorem of Calculus;

we should be able to integrate both sides of each of the above rules to generate a new one for integration.

A Sum Rule for Integration

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx} \tag{1.5}$$

i.e. we may integrate term by term.

A Chain Rule for Integration

$$\frac{d}{dx}(g(f(x))) = g'(f(x))f'(x) \tag{1.6}$$

Let $u = f(x)$:

Of course this is more well known as the *Substitution Rule* — but really it's a Chain Rule for Integration.

Integration by Parts — A Product Rule for Integration

What about a Product or Quotient Rule for integration? Well first off a quotient rule wouldn't be much use:

$$\int \frac{vu' - uv'}{v^2}\,dx = \frac{u}{v} \tag{1.7}$$

But what about a Product Rule? Well

$$\frac{d}{dx}(ux) = u\frac{dv}{dx} + v\frac{du}{dx} \tag{1.8}$$

Now instead of doing what we did above, notice that

$$u\frac{dv}{dx} = \frac{d}{dx}(ux) - v\frac{du}{dx}$$

Now integrating with respect to x:

$$\int u\,dv = uv - \int v\,du \tag{1.9}$$

This formula is known as the *Integration by Parts* formula. It will be very prominent in our study of Laplace Transforms. In practise you will be confronted by an integral of the form:

In terms of the notation, if $f(x)$ is split into a product $f(x) = g(x)h(x)$ then:

Now in terms of (1.9):

In general, $f(x)$ will be already 'split' and the only issue will be the choice of u and the choice of dv. Note first of all that once u is chosen, dv is just whatever is left. Note that whatever we choose u to be, we will have no problem differentiating it to find du. To find v, we must integrate dv. However, in general, integration is more difficult that differentiation. Hence a general *heuristic* or strategy is to choose u to be the term that is harder to integrate. Consider the following hierarchy:

L

I

A

T

E

This is a hierarchy of classes of functions in *decreasing difficulty of integration*. Hence therefore, if you choose u to be the first element in this hierarchy to be found in the integrand, then automatically dv will be easier to integrate than du. This is known for obvious reasons as the LIATE Rule.

Examples

1. *Find*

$$I = \int x \sin x \, dx$$

2. *Find*

$$I = \int \ln x \, dx$$

3. *Find*

$$I = \int t^2 e^t \, dx$$

4. *Find*

$$I = \int e^x \sin x \, dx$$

If we combine the formula for integration by parts with the Fundamental Theorem of Calculus:

Example

Calculate

$$\int_0^1 \tan^{-1} x \, dx$$

Exercises

1. *Find* $\int x \cos x \, dx$ *and check your solution.*

2. *Find* $\int xe^{2x} \, dx$ *and check your solution.*

3. *Evaluate*
$$\int_0^{\pi/2} x \cos 2x \, dx$$

4. *Evaluate*
$$\int_1^2 \log x \, dx,$$
giving your answer in the form $\log p + q$ *where* $p, q \in \mathbb{Q}$.

5. *Find* $\int \log 2x \, dx$.

6. *Integrate* $\int \sin^{-1} x \, dx$

7. *Integrate* $\int x \log x \, dx$.

8. *Integrate* $\int \theta \sec^2 \theta \, d\theta$.

9. *Evaluate*
$$\int_0^1 \frac{\ln x}{x^2} \, dx$$

10. *Evaluate*
$$\int_1^4 \sqrt{t} \, \ln t \, dt$$

1.3 Partial Fractions

This chapter will serve two purposes. Firstly it will give us a an algebraic technique that allows us to write a 'fraction' as a sum of (supposedly) simpler 'fractions' and as a corollary it will give us another integration technique.

Adding Fractions

Let $a, b, c, d \in \mathbb{R}$ such that $b \neq 0$, $d \neq 0$. Now

$$\frac{a}{b} + \frac{c}{d} =$$

So we can see that we can write the sum any fractions with denominators b,d as a single fraction with denominator bd. Now I ask the reverse question:

Given a fraction a/b can I write a/b as a sum of two fractions?

Let $\alpha, \beta \in \mathbb{R}$ such that $b = \alpha\beta$. Then:

$$\frac{x}{\alpha} + \frac{y}{\beta}$$

Now compare:

$$\frac{a}{b} = \frac{a}{\alpha\beta} = \frac{\alpha x + \beta y}{\alpha\beta}$$

So not only can be do it we can do it in an infinite number of ways!

Example

Write 1/12 *as a sum of two simpler fractions.*

Rational Functions

Definition

Any function of the form:

for $a_i \in \mathbb{R}$, $n \in \mathbb{N}$ is a *polynomial.* If $a_n \neq 0$ then p is said to be *of degree* n.

Examples

Suppose that all $a_i \in \mathbb{R}$ with *leading term* non-zero:

1. $p(x) = a_1x + a_0$ is a line or a linear polynomial.
2. $q(x) = a_2x^2 + a_1x + a_0$ is a quadratic or a quadratic polynomial.
3. $r(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ is a cubic or a cubic polynomial.
4. $s(x) = x^{100} - 99x^{50} + 2$ is a polynomial of degree 100.

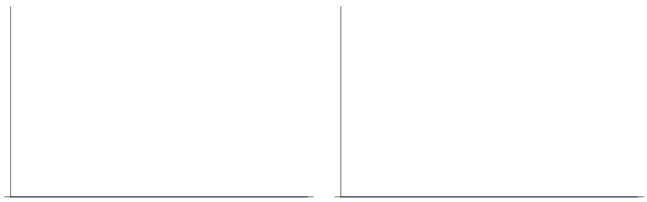


Figure 1.7: Plots of a linear and a quadratic polynomial on the left. A plot of a cubic polynomial on the right.

Take a general polynomial, say $p(x)$. How can the roots of p be found?

p in general is a sum, not a product. However the following theorem gives us a scheme to find the roots of p:

Theorem

Suppose a and b are numbers and

$$a.b = 0.$$

Then either:

To use this however, we need to be able to write p as a product! To write a sum as a product is to factorise. The below theorem gives a clue:

Theorem: Factor Theorem

A number k is a root of a polynomial $p(x)$ if and only if $(x - k)$ is a factor of p.

Proof

See http://irishjip.wordpress.com/2010/09/08/an-inductive-proof-of-the-factor-theorem/ •

Example

Let $p(x) = 6x^3 - 11x^2 + 6x - 1$. *Show that* $p(1) = 0$. *Hence solve*

$$6x^3 - 11x^2 + 6x - 1 = 0$$

For the moment suppress the restriction to real functions ($x \in \mathbb{R}$) and consider functions defined on the *complex numbers.* It is a deep result in algebra and complex analysis that:

Theorem: Fundamental Theorem of Algebra

Every non-constant polynomial p of degree n can be written in the form

for some $c \in \mathbb{R}$, $a_1, a_2, \cdots \in \mathbb{C}$

Remark

The a_i here are the roots of f and this theorem proves that a polynomial of degree n has n roots, some of which may be complex.

Theorem: Fundamental Theorem of Algebra for Real Polynomials

Every non-constant polynomial p of degree n can be written in the form

for some $c \in \mathbb{R}$, $b_1, b_2, \cdots \in \mathbb{R}$, $c_1, c_2, \cdots \in \mathbb{R}$.

Remark

We can break down every polynomial with real coefficients to a product of linear and quadratic terms.

Definition

Suppose that $p(x)$ and $q(x)$ are polynomials. Any function of the form:

is called a *rational function.*

Examples

1.
$$\frac{x+5}{x^2+x-2}$$

2.
$$\frac{x^3+x}{x-1}$$

3.
$$\frac{x^2+2x-1}{2x^3+3x^2-2x}$$

The remainder of this section will be concerned with writing rational functions as a sum of simpler 'fractions' called *partial fractions.* To mirror the addition of a/b and c/d from earlier on, consider:

$$\frac{2}{x-1} - \frac{1}{x+2}$$

That is example 1 above has partial fraction expansion:

$$\frac{x+5}{x^2+x-2} = \frac{2}{x-1} - \frac{1}{x+2}$$

Now why the hell would be do this? The primary reason for this module is for doing *Inverse Laplace Transforms.* Frequently rational functions will arise here and we will need to expand them in order to apply this $\mathfrak{L}^{-1}$ operator. However for now we could consider the integral:

$$\int \frac{x+5}{x^2+x-2}\,dx$$

So to integrate rational fractions it may be useful to express them in a partial fraction form. To see how the method of partial fractions works in general, let's consider a rational function f:

where p and q are polynomials.

We will see that it will be possible to write f as a sum of simpler fractions provided the *degree of* p *is less than the degree of* q. If it isn't, we must first divide q into p using long division (same method as when we did the factor theorem example). When we've done this we will end up with an expression of the form:

$$f = s(x) + \frac{r(x)}{q(x)} \tag{1.10}$$

where $s(x)$ is a polynomial and $\deg(r) < \deg(q)$.

Examples

By dividing $x - 1$ *into* $x^3 + x$, *write*

$$\frac{x^3 + x}{x - 1}$$

in the same form as (1.10).

Write the following in the same form as (1.10):

$$\frac{x^4 + 3x^2 - 2}{x^2 + 1}$$

General Method for Partial Fractions

Let $f(x) = p(x)/q(x)$ be a rational function.

1. Write $f(x)$ in the same form as (1.10).
2. Factor $q(x)$ as far as possible using the Factor Theorem for real polynomials.
3. To each factor of $q(x)$ we associate a term in the partial fraction decomposition via the following rule:

I To each *non-repeated* linear factor of the form $(ax + b)$ (i.e. no other factor of $q(x)$ is a constant multiple of $(ax + b)$) there corresponds a partial fraction term of the form:

Example: *Suppose* $f(x) = p(x)/q(x)$*, with deg*$(q) <$ *deg*(p)*, and* $q(x) = (x - 1)(2x - 1)(-x + 2)$*. What is the partial fraction expansion of* $f(x)$*?*

II To each linear factor of the form $(ax + b)^n$ (i.e. a repeated linear factor of $q(x)$) there corresponds a sum of n partial fraction terms of the form:

Example: *Suppose* $f(x) = p(x)/q(x)$*, with deg*$(q) <$ *deg*(p)*, and* $q(x) = (x - 1)^3(2x - 1)(-x + 2)^2$*. What is the partial fraction expansion of* $f(x)$*?*

III To each non-repeated *quadratic* factor of $q(x)$ of the form $(ax^2 + bx + c)$ (i.e. no other factor of $q(x)$ is a constant multiple of $(ax^2 + bx + c)$) there corresponds a partial fraction term of the form:

Example: *Suppose* $f(x) = p(x)/q(x)$*, with deg*$(q) <$ *deg*(p)*, and* $q(x) = (x - 1)^2(x^2 + x + 1)(2x^2 + 3)$*. What is the partial fraction expansion of* $f(x)$*?*

IV To each quadratic factor of the form $(ax^2 + bx + c)^n$ (i.e. a repeated linear factor of $q(x)$) there corresponds a sum of n partial fraction terms of the form:

Example: *Suppose* $f(x) = p(x)/q(x)$*, with* $deg(q) < deg(p)$*, and* $q(x) = (x-1)^2(2x-1)(2x^2+3)^2$*. What is the partial fraction expansion of* $f(x)$*?*

4. Write the partial fraction expansion as a single fraction "$f(x)$", and set it equal to $f(x)$. Compare the numerators of $f(x)$, $u(x)$; and the numerator of "$f(x)$", $v(x)$; by setting them equal to each other:

   Find the coefficients in the partial expansion using one of two methods:

   (a) The coefficients of $u(x)$ must equal those of $v(x)$. Solve the resulting simultaneous equations.

   (b) If $u(x)$ and $v(x)$ agree on all points then $f(x)$="$f(x)$". Generate m simultaneous equations in m variables by plugging in m different values $x_1, x_2, \ldots, x_m$ and solving the equations:

   Example: *Let*

$$f(x) = \frac{7}{(x-1)(x-2)}$$

Hence $f(x)$ *has partial expansion*

Evaluate A*,* B *using both methods above.*

Examples

1. *Find the partial fraction expansion of*

$$\frac{7}{2x^2 + 5x - 12}$$

2. *Evaluate*

$$\int \frac{6x^2 - 3x_1}{(4x+1)(x^2+1)}\, dx$$

3. *Evaluate*

$$\int \frac{dx}{x^5 - x^2}$$

Exercises

1. *Factorise the following polynomials: (i)* $x^2 - 4x - 5$ *(ii)* $x^2 - 2x$ *(iii)* $15x^2 + x - 6$

2. *Divide each of the following: (i)*$2x^2 - 7x - 4 \div x - 4$ *(ii)* $3x^3 - 2x^2 - 19x - 6 \div 3x + 1$ *(iii)* $2x^3 + x^2 - 16x - 15 \div 2x + 5$ *(iv)* $8x^3 + 27 \div 2x + 3$ *(v)* $2x^3 - 7x^2 - 7x - 10 \div 2x - 5$ *(vi)* $6x^3 - 13x^2 \div 2x + 1$

3. *Search for a root of the following cubics and hence use the Factor Theorem to factorise: (i)* $2x^3 + x^2 - 8x - 4$ *(ii)* $x^3 + 4x^2 + x - 6$ *(iii)* $3x^3 - 11x^2 + x + 15$

4. *Write each as a single fraction:*

$$\frac{4x-3}{5} + \frac{x-3}{3}$$
$$\frac{1}{x-1} - \frac{2}{2x+3}$$
$$\frac{x}{x-1} + \frac{2}{x}$$
$$\frac{1}{x+1} - \frac{3}{2x-1}$$

5. *Write out the partial fraction expansion of the following. Do not evaluate coefficients.*

$$\frac{x^3-1}{x(x-2)^2}$$
$$\frac{x^2+x}{x^3-x^2+x-1}$$
$$\frac{x^2-2x-3}{(x-1)(x^2+2x+2)}$$

6. *Write out the partial fraction expansion of the following. Do evaluate coefficients.*

$$\frac{x-1}{x^3-x^2-2x}$$
$$\frac{1}{x^3+3x^2}$$
$$\frac{1}{(x+2)^2}$$

7. *Evaluate*

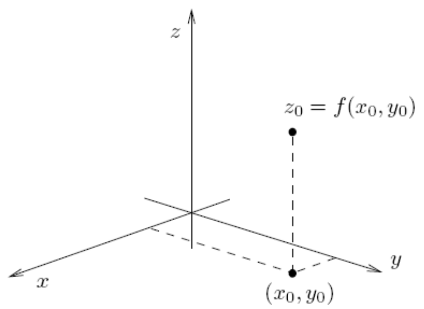
$$\int \frac{1}{x^2-4}\,dx$$
$$\int \frac{1}{(8-x)(6-x)}\,dx$$
$$\int \frac{5x-2}{x^2-4}\,dx$$

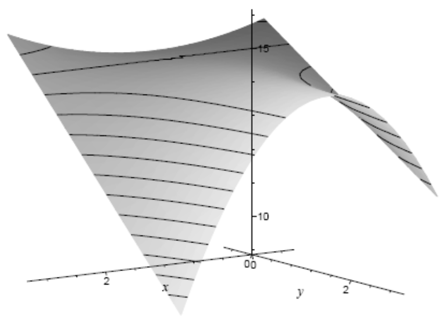
1.4 Multivariable Calculus

Functions of Several Variables: Surfaces

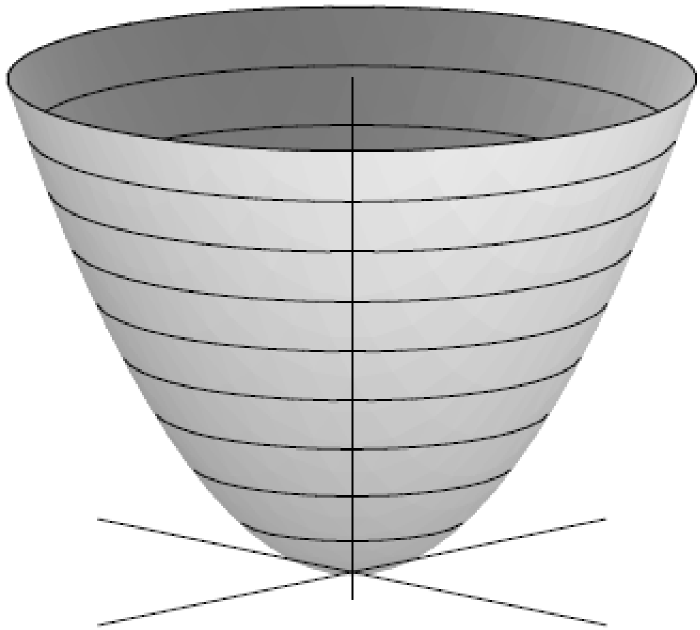
Many equations in engineering, physics and mathematics tie together more than two variables. For example Ohm's Law ($V = IR$) and the equation for an ideal gas, $PV = nRT$, which gives the relationship between pressure (P), volume (V) and temperature (T). If we vary any two of these then the behaviour of the third can be calculated:

How P varies as we change T and V is easy to see from the above, but we want to adapt the tools of one-variable calculus to help us investigate functions of more than one variable. For the most part we shall concentrate on functions of two variables such as $z = x^2 + y^2$ or $z = xsin(y + e^x)$. Graphically $z = f(x, y)$ describes a surface in 3D space — varying the x- and y-coordinates gives the z-coordinate, producing the surface:



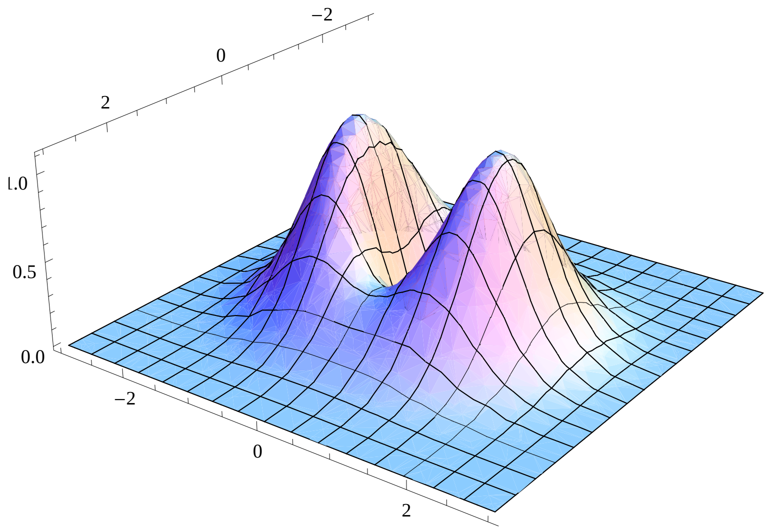


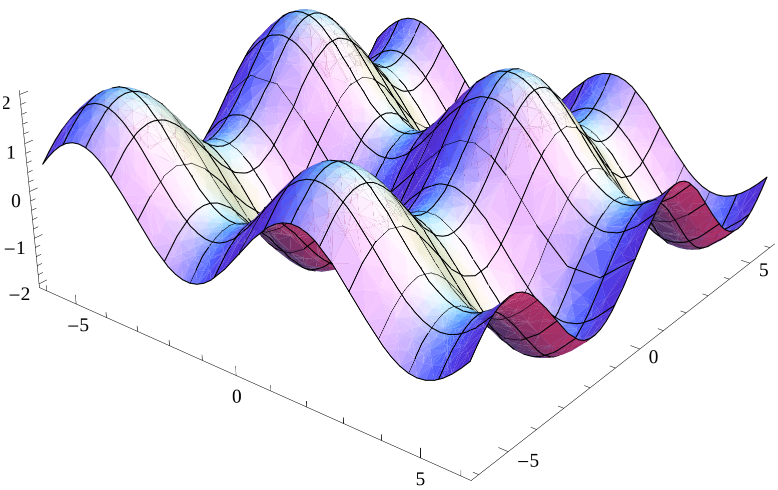
As an example, consider the function $z = x^2 + y^2$. If we choose a positive value for z, for example $z = 4$, then the points (x, y) that can give rise to this value are those satisfying $x^2 + y^2 = 4 = 2^2$, i.e. those on the circle centred on the origin of radius 2. Note that at $(x, y) = (0, 0)$, $z = 0$, but if $x \neq 0$ or $y \neq 0$, then $x^2 > 0$ or $y^2 > 0$, and it follows that $z > 0$. Thus the minimum value taken by this function is $z = 0$, at the origin:



Three examples. Which are which?

$$f(x,y) = (x^2 + 3y^2)e^{-x^2-y^2}$$
$$g(x,y) = \frac{\sin x \sin y}{xy}$$
$$h(x,y) = \sin x + \sin y$$

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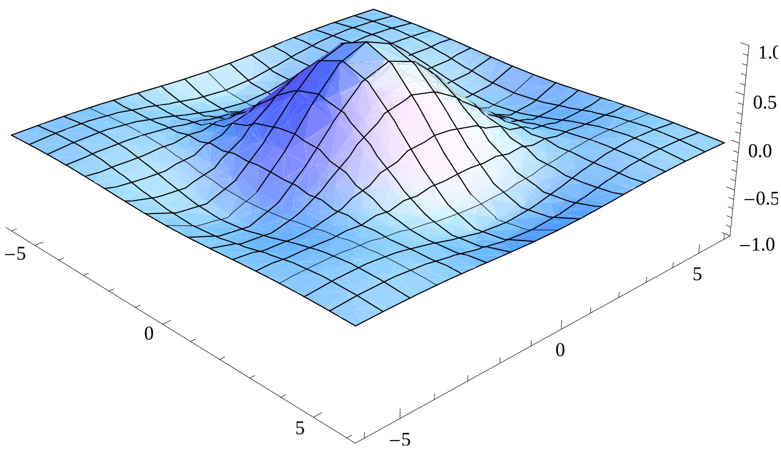
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Partial Derivatives

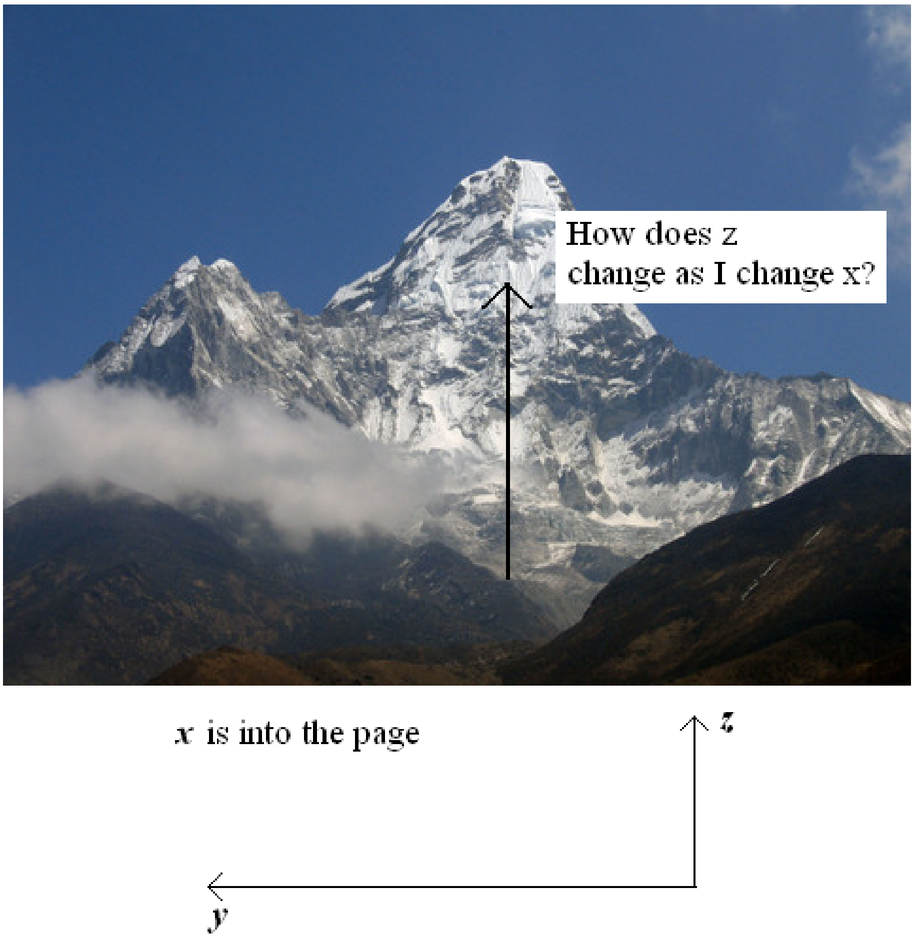

Figure 1.8: What is the rate of change in z as I keep y constant

If we were to look at this from side on:

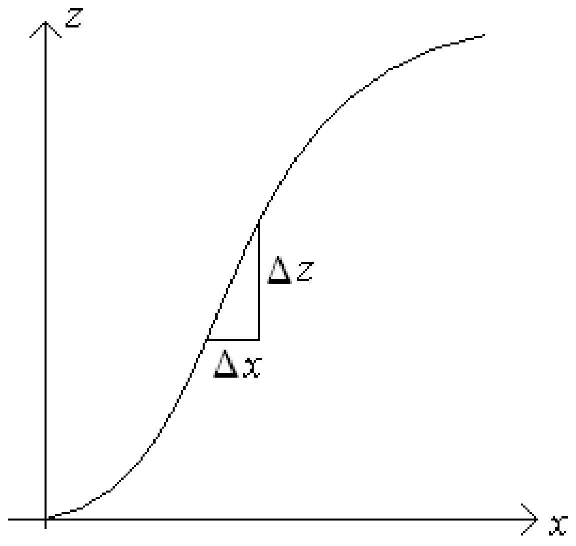

Figure 1.9: When y is a constant z can be considered a function of x only.

In general we have that $z = f(x, y)$; but if $y = b$ is fixed (constant):

We can view $f(x, b)$ as a function of x alone. Now what is the rate of change of a single-variable function with respect to x:

Which is also the slope of the tangent to f at x. Hence the rate of change of $f(x, y)$ with respect to x at $x = a$ when y is fixed at $y = b$ is the slope of the surface in the x-direction.

Example

Let $z = f(x, y) = x^3 + x^2y^3 - 2y^3$. What is the rate of change of z when $y = 2$?

Hence the rate of change of z with respect to x, when y is fixed at $y = b$, is given by:

More generally, we fix $y = y$ and define

as *the partial derivative of f with respect to x.*
We define the partial derivative of f with respect to y in exactly the same way.

Example

What are the partial derivatives of

$$z = x^2 + xy^5 - 6x^3y + y^4$$

with respect to x and y respectively?

There are many alternative notations for partial derivatives. For instance, instead of $\frac{\partial f}{\partial x}$ we can write f_x or f_1. In fact,

$$\frac{\partial f}{\partial x} \equiv \frac{\partial z}{\partial x} \equiv f_x(x,y) \equiv f_1(x,y)$$
$$\frac{\partial f}{\partial y} \equiv \frac{\partial z}{\partial y} \equiv f_y(x,y) \equiv f_2(x,y)$$

To compute partial derivatives, all we have to do is remember that the partial derivative of a function with respect to x is the same as the *ordinary* derivative of the function g of a single variable that we get by keeping y fixed. Thus we have the following:

1. To find $\frac{\partial f}{\partial x}$, regard y as a constant and differentiate $f(x,y)$ with respect to x.

2.

Example

If $f(x,y) = 4 - x^2 - 2y^2$, find $f_x(1,1)$ and $f_y(1,1)$ and interpret these numbers as slopes.

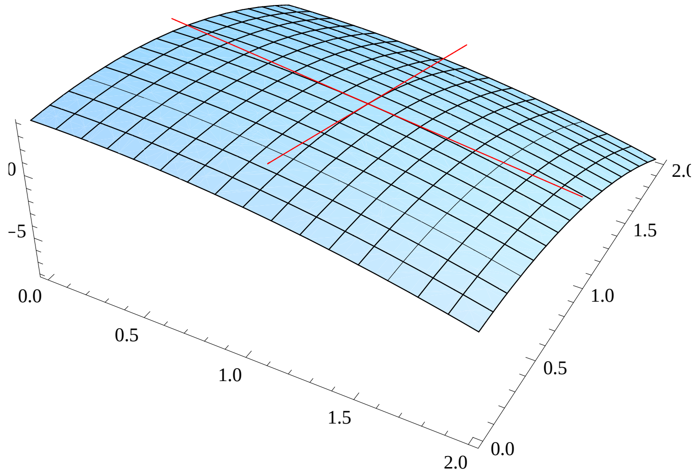

Figure 1.10: $f_x(1,1)$ and $f_y(1,1)$ are the slopes of the tangents to $(1,1)$ in the x and y directions respectively.

Using this technique we can make use of known results from one-variable theory such as the product, quotient and chain rules (Careful — the Chain rule only works if we are differentiating with respect to one of the variables — we may have more to say on this in the next section).

Examples

Find the partial derivative with respect to y of the function

$$f(x,y) = \sin(xy)e^{x+y}$$

Compute f_1 and f_2 when $z = x^2y + 3x\sin(x-2y)$.

Functions of More Variables

We can extend the notion of partial derivatives to functions of any (finite number) of variables in a natural way. For example if $w = \sin(x + y) + z^2 e^x$ then:

Higher Order Derivatives

Suppose $z = x \sin y + x^2 y$. Then

Both of these partial derivatives are again functions of x and y, so we can differentiate both of them, either with respect to x, or with respect to y. This gives us a total of four *second order partial derivatives:*

Remark: The mixed partial derivatives in this case are equal:

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}.$$

This is not something special about our particular example — it will be true for all *reasonably behaved functions.* This is the *symmetry of second derivatives.* Note the notation:

$$\frac{\partial}{\partial x \partial y} = f_{yx} \quad \text{etc.} \tag{1.11}$$

Examples

Compute

$$\frac{\partial z}{\partial x}\,,\ \frac{\partial z}{\partial y}\ ,\ \text{and}\ \frac{\partial^2 z}{\partial x^2}$$

when $z = x^3y + e^{x+y^2} + y\sin x$.

Compute all the second order partial derivatives of the function $f(x, y) = \sin(x + xy)$.

Exercises

1. *Find all the first order derivatives of the following functions:*

$$(i)\ f(x,y) = x^3 - 4xy^2 + y^4 \qquad (ii)\ f(x,y) = x^2e^y - 4y$$
$$(iii)\ f(x,y) = x^2\sin xy - 3y^2 \quad (iv)\ f(x,y,z) = 3x\sin y + 4x^3y^2z$$

2. *Find the indicated partial derivatives: (i)* $f(x,y) = x^3 - 4xy^2 + 3y$*:* f_{xx}*,* f_{yy}*,* f_{xy}
(ii) $f(x,y) = x^4 - 3x^2y^3 + 5y$: f_{xx}, f_{xy}, f_{xyy}
(iii) $f(x,y,z) = e^{2xy} - \frac{z^2}{y} + xz\sin y$: f_{xx}, f_{yy}, f_{yyzz}

1.5 Applications to Error Analysis

Differentials

For a differentiable function $y = f(x)$ of a single variable x, we define the differential 'dx' to be an independent variable; that is, dx can be given the value of any real number. The differential of y is then defined by:

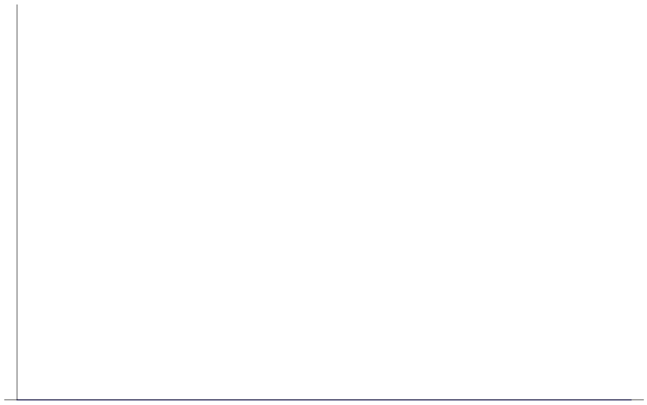

Figure 1.11: The differential estimates the actual change in y, Δy, due to a change in x: $x \to \Delta x$. For small changes in x, the differential is approximately equal to the actual change in y: $dy \approx \Delta y$.

For a differentiable function of two variables $z = f(x, y)$, we define the differentials dx and dy to be independent variables and the differential dz estimates the change in z when x changes to $x + \Delta x$ and y changes to $y + \Delta y$:

Example

If $z = f(x, y) = x^2 + 3xy - y^2$, find the differential dz. If x changes from 2 to 2.05 and y changes from 3 to 2.96, compute the values of dz and Δz (the actual change in z).

Example

The pressure, volume and temperature of a mole of an ideal gas are related by the equation $PV = 8.31T$*, where* P *is measured in kilopascals,* V *in litres and* T *in kelvins. Use differentials to find the approximate change in the pressure if the volume increases from 12 L to 12.3 L and the temperature decreases from 310 K to 305 K.*

Propagation of Errors

Suppose we have a physical property P related to two other properties A and B by:

Now suppose we measure A and B and record values A_0 and B_0 with associated errors ΔA and ΔB. We can now keep track of the errors in P due to errors in A and B by knowing "*how much P will change due to small changes in A (and/ or B) between $A - \Delta A$ and $A + \Delta A$ (and $B - \Delta B$ and $B + \Delta B$)*". The differential of P gives an estimate of this:

Now we don't want errors to cancel each other out so we write:

Example

The base radius and height of a right circular cone are measured as 10 cm and 25 cm, respectively, with a possible error in measurement of as much as 0.1 cm in each. Use differentials to estimate the maximum error in the calculated volume of the cone.

This procedure generalises in the obvious way.

Example

The dimensions of a rectangular box are measured to be 75 cm, 60 cm, and 40 cm, and each measurement is correct within 0.2 cm. Use differentials to estimate the largest possible error when the volume of the box is calculated from these measurements.

Exercises

1. *Use differentials to estimate the amount of tin in a closed tin closed tin can with diameter 8 cm and height 12 cm if the can is 0.04 cm thick.*

2. *Use differentials to estimate the amount of metal in a closed cylindrical can that is 10 cm high and 4 cm is diameter if the metal in the wall is 0.05 cm thick and the metal in the top and bottom is 0.1 cm thick.*

3. *If R is the total resistance of three resistors, connected in parallel, with the resistances R_1, R_2 and R_3, then*
$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$
. *If the resistances are measured as $R_1 = 25\ \Omega$, $R_2 = 40\ \Omega$ and $R_3 = 50\ \Omega$, with possible errors of 5% in each case, estimate the maximum error in the calculated value of R.*

4. *The moment of inertia of a body about an axis is given by $I = kbD^3$ where k is a constant and B and D are the dimensions of the body. If B and D are measured as 2 m and 0.8 m respectively, and the measurement errors are 10 cm in B and 8 mm in D, determine the error in the calculated value of the moment of inertia using the measured values, in terms of k.*

5. *The volume, V, of a liquid of viscosity coefficient η delivered after a time t when passed through a tube of length l and diameter d by a pressure p is given by*
$$V = \frac{pd^4t}{128\eta l}.$$
If the errors in V, p and l are 1%, 2% and 3% respectively, determine the error in η. HINT: If the error in A is x% then the error is $xA_0/100$ when $A = A_0$.

Chapter 2

Numerical Methods

2.0.1 Outline of Chapter

- Solving equations using the Bisection Method and the Newton-Raphson Method
- Approximate definite integrals using the Midpoint, Trapezoidal and Simpson's Rules.
- Euler's Method

2.1 Root Approximation using the Bisection and Newton-Raphson Methods

Suppose that you want to solve an equation such as

$$48x(1+x)^{60} - (1+x)^{60} + 1 = 0$$

How would you solve such an equation?
For the quadratic equation $ax^2 + bx + c = 0$ there is a well-known formula for the roots. For third- and forth- degree equations there are also formulas for the roots, but they are extremely complicated. If f is a polynomial of degree 5 or higher, there is no such formula. Likewise, there is no formula that will enable us to find solutions to so-called *transcendental equations* such as:

This section will outline two approximation methods — first some theory.

Continuous Functions and The Intermediate Value Theorem

Consider a function with continuous graph:

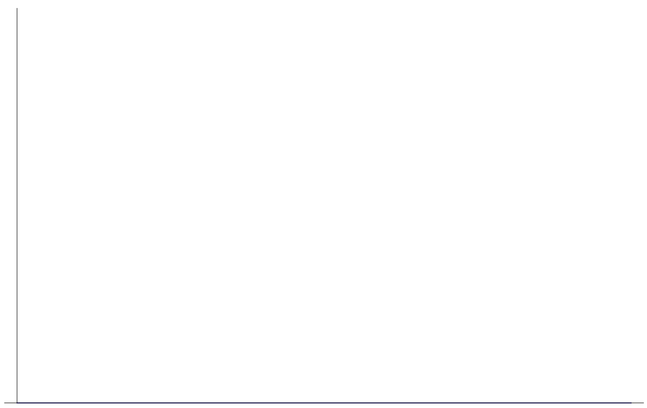

Figure 2.1: A function with a continuous graph can be drawn without lifting the pen off the page.

Mathematicians can abstract this class of function but for MATH6037 we define a continuous function as follows:

Definition

Let $I \subset \mathbb{R}$ be an interval and suppose that $f : I \to \mathbb{R}$, $x \mapsto f(x)$. Then we say that f is *continuous* if the graph of f is continuous.

Examples of Continuous Functions

The following functions are all continuous — where defined!

1.

2.

3.

4.

5.

Theorem

Suppose that f and g are continuous functions and $k \in \mathbb{R}$. Then the following are also continuous functions

1.

2.

3.

4.

5.

6.

Now consider the following situation:

Figure 2.2: Suppose a continuous function f changes sign over an interval (a, b) — then f must cut the x-axis at some point between a and b — that is f must have a *root* between a and b.

Intermediate Value Theorem: MATH6037 Version

Examples

Use the Intermediate Value Theorem to show that the equation $x^3 - 4x^2 + x + 3 = 0$ *has a root between* 1 *and* 2.

Use the Intermediate Value Theorem to show that the equation $(\cos x)x^3 + 5\sin^4 x - 4 = 0$ *has a root between* 0 *and* 2π.

Apply the Intermediate Value Theorem to find an interval in which $x^2 + x = 1$ *has a root.*

Apply the Intermediate Value Theorem to find an interval in which $3\sin x + \cos^2 x = 2$ *has a root.*

We use this theorem to estimate the location of roots. The following two methods then zoom in on the root. The first is a repeated application of the Intermediate Value Theorem — the second uses tangents to the curve.

The Bisection Method

Tthe first step is to take the equation, bring all the terms over to left-hand side and re-write the equation as $f(x) = 0$, where $f(x)$ is the terms on the left-hand side. Solutions to $f(x) = 0$ are known as *roots* of the function.
Once this is done, the second step is to evaluate the function f at various points (usually $x = 0, 1, 2, 3, ..., -1, -2$) until we find that the sign changes — e.g. if f is continuous and $f(2) = 1$ and $f(3) = -4$ then there is a root between 2 and 3, in the interval $(2, 3)$:

Figure 2.3: If f is continuous and changes sign between 2 and 3, then there is a root between 2 and 3. Next we evaluate $f(2.5)$ to see is the root in $(2, 2.5)$ or $(2.5, 3)$

Once we have found an interval (a, b) in which we know there is a root — we evaluate at the midpoint of (a, b) to see whether there is a root in the left or the right of (a, b). We can keep continuing this process until we are as close to the root as we choose.

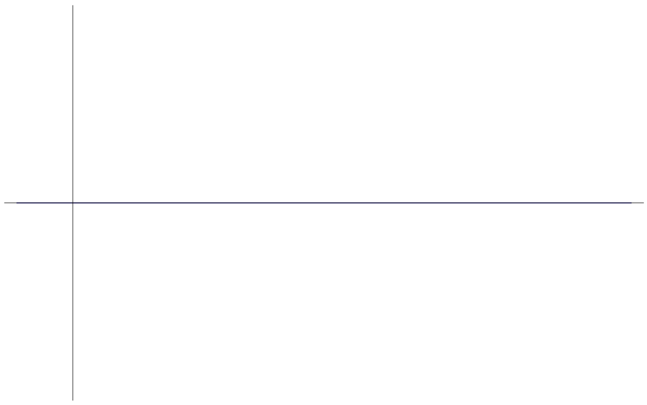

Figure 2.4: We can iterate this process to find smaller and smaller intervals in which there must be a root.

Examples

Show that the polynomial $p(x) = x^4 - 2x^3 - 2x^2 + 1$ *has a root* r *satisfying* $0 < r < 2$ *and use four iterations of the bisection method to find an approximation of* r.

Find an interval of length less than 0.05 *which contains a root of* $\sin x = x$.

The Newton-Raphson Method

Another such method is the *Newton-Raphson method.* As before, the first step is to take the equation, bring all the terms over to left-hand side and re-write the equation as $f(x) = 0$, where $f(x)$ is the terms on the left-hand side. Solutions to $f(x) = 0$ are known as *roots* of the function. For example, finding the solutions to the equation

is equivalent to finding the roots of the function:

Using a quick application of the Intermediate Value Theorem, we find an interval (a, b) on which $f(x)$ has a root. Now as a rough approximation to the root, we can choose any x_0 between a and b (usually $(a + b)/2$ - the midpoint). Now what we do is the following:

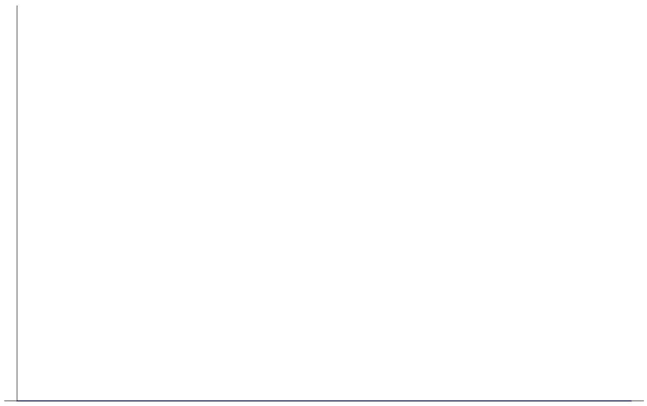

Figure 2.5: We use the tangent to the curve at x_0 to get a better approximation to the root r. Not that at all times we will require that $f'(x_0) \neq 0$.

To find a formula for x_1 in terms of x_0, we use the fact that the slope of the tangent to the curve at x_0 is $f'(x_0)$. A point on the tangent is given by $(x_0, f(x_0))$ and using the formula for the equation of a line:

Now, the equation of the line is like a membership card for the line — if a point satisfies the equation it's on the line, otherwise it's not. Now the point $(x_1, 0)$ is certainly on the line so it satisfies the equation:

We use x_1 as a first approximation to r. Next we repeat this procedure with x_0 replaced by x_1, using the tangent line at $(x_1, f(x_1))$:

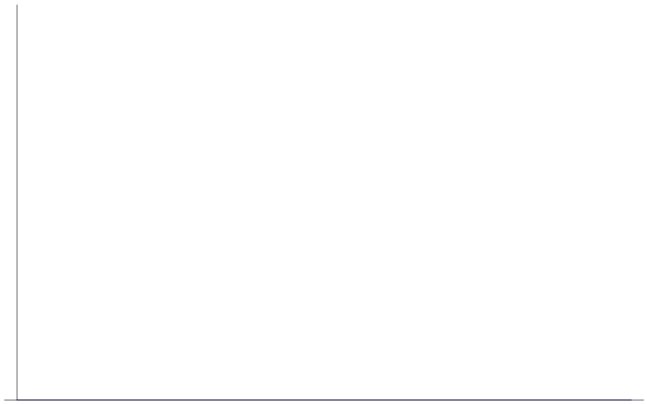

Figure 2.6: We use the tangent to the curve at x_1 to get an even better approximation to the root r, x_2.

This gives a second approximation:

If we keep repeating this procedure, we obtain a sequence of approximations $x_1, x_2, x_3, \ldots$. In general, if the nth approximation is x_n (and $f'(x_n) \neq 0$), then the next approximation is given by:

If the sequence x_n gets closer and closer to r as n gets large, we say that the sequence *converges to* r and we write:

Remarks

Although the sequence of successive approximations converges in a great many cases, in certain circumstances the sequence may not converge. However, except in pathological examples which we will not encounter, if the sequence of approximations converges, it will do so to a root.
Suppose we want to achieve a given accuracy, say to eight decimal places, using the Newton-Raphson Method. How do we know when to stop? A good rule of thumb, backed up by a theorem, is that we can stop if two successive approximations x_n and x_{n+1} agree to eight decimal places.
Notice that the procedure in going from x_n to x_{n+1} is the same. It is called an *iterative process* and is particularly convenient for use with a computer.

Examples

Starting with $x_0 = 2$, find the second approximation to the root of the equation $x^3 - 2x - 5 = 0$.

Use Newton's method to find $\sqrt[6]{2}$ correct to eight decimal places.

Find, correct to six decimal places, the root of the equation $\cos x = x$.

Exercises

1. *If* $f(x) = x^3 - x^2 + x$*, show that there is a number* c *such that* $f(c) = 10$.

2. *Use the Intermediate Value Theorem to prove that there is positive number* c *such that* $c^2 = 2$ *(this proves existence of the number* $\sqrt{2}$*).*

3. *Use the Intermediate Value Theorem to show that there is a root of the given equation in the specified interval (i)* $x^4 + x - 3 = 0$, $(1, 2)$ *(ii)* $\sqrt[3]{x} = 1 - x$, $(0, 2)$ *(iii)* $\cos x = x$, $(0, 1)$ *(iv)* $\tan x = 2x$, $(0, 1.4)$

4. *Use the Intermediate Value Theorem to locate an interval of length* 1 *in which each of the following equations have a roots (note that in general a polynomial of degree* n *has* n *roots — I just want ye to find a location of one of them.).*

   (i) $x^3 + 2x - 4 = 0$.

   (ii) $x^5 + 2 = 0$.

   (iii) $x^3 = 30$.

   (iv) $x^4 + x - 4 = 0$.

   (v) $x^4 = 1 + x$.

   (vi) $\sqrt{x + 3} = x^2$.

   (vii) $x^5 - x^4 - 5x^3 - x^2 + 4x + 3 = 0$.

   (viii) $3\sin(x^2) = 2x$.

   Now use the Bisection Method to find intervals of length less than 0.1 *(this will require four iterations of the Bisection Method — after four iterations the interval on which we know there is a root will have length* $1/2^4 = 1/16 < 0.1$*)*
   Remarks in italics are by me for extra explanation. These comments would not be necessary for full marks in an exam situation. Exercises taken from p.75 of these notes..

5. *Use the Newton-Raphson Method to find a root of* $e^{-x} = x$ *to five decimal places.*

2.2 Approximate Definite Integrals using the Midpoint, Trapezoidal and Simpson's Rules

There are two situations in which it is impossible to find the exact value of of a definite integral.
The first situation arises from the fact that in order to evaluate a definite integral $\int_a^b f(x)\,dx$ using the Fundamental Theorem of Calculus we need to know an anti-derivative of f. Sometimes, however, it is difficult, or even impossible to find an antiderivative. For example, it is impossible to evaluate the following exactly:

$$\int_0^1 e^{x^2}\,dx \quad \text{, and} \quad \int_{-1}^1 \sqrt{1+x^3}\,dx$$

The second situation arises when the function is determined from a scientific experiment through instrument readings or collected data.
In both cases we need to find approximate values of definite integrals. We know that a definite integral represents the area under a curve so we use rectangles the approximate the area under the curve.

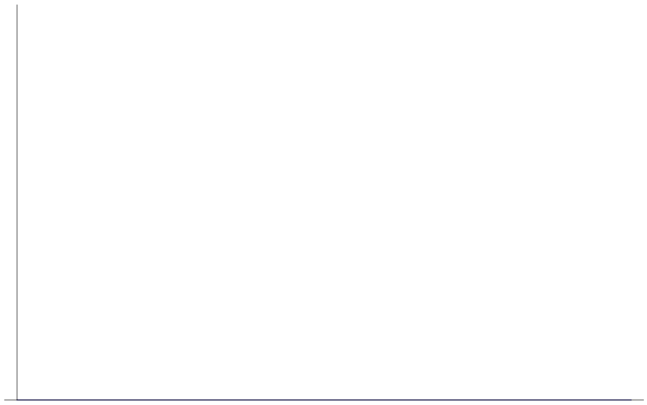

Figure 2.7: Suppose we want to integrate the function $f(x)$ over the interval (a, b). We can approximate the integral by a rectangle of width $(b - a)$ and height $f((b - a)/2)$. This corresponds to the Midpoint Rule.

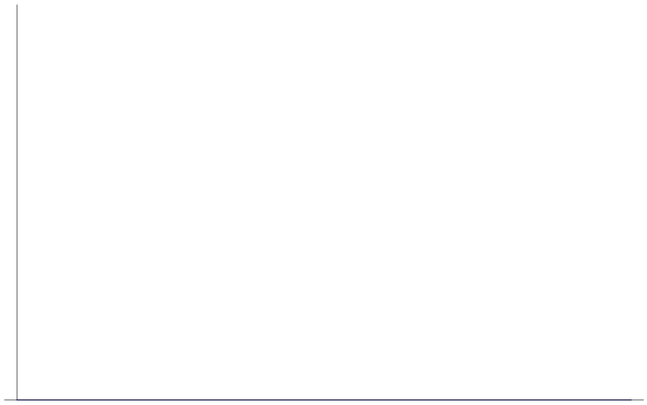

Figure 2.8: We could also approximate the integral by a rectangle of width $(b-a)$ and height $f(a)$. This corresponds to the Left Endpoint Rule.

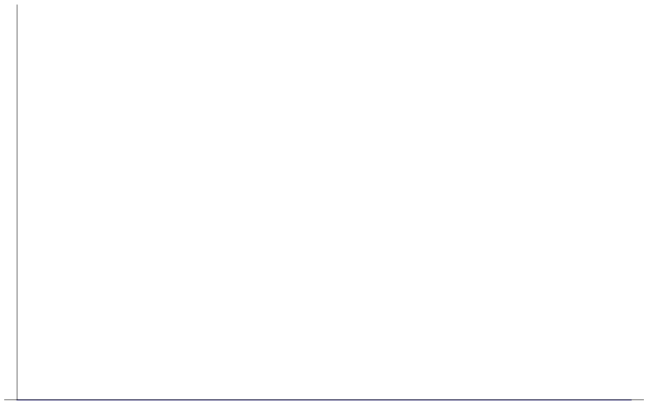

Figure 2.9: We could also approximate the integral by a rectangle of width $(b-a)$ and height $f(b)$. This corresponds to the Right Endpoint Rule Rule.

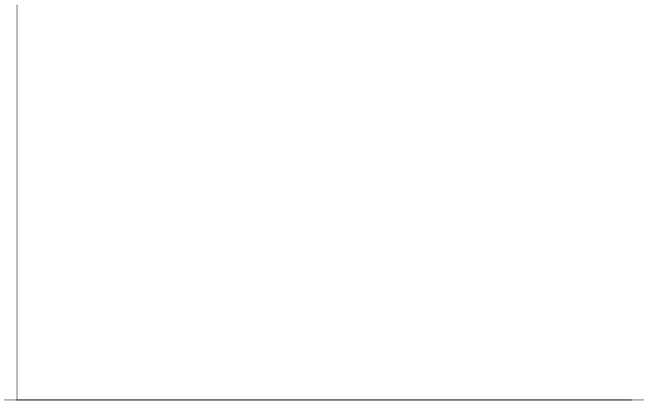

Figure 2.10: We could also approximate the integral by a trapezoid of width $(b - a)$ and heights $f(a)$, $f(b)$. This corresponds to the Trapezoidal Rule. As an exercise, show that the Trapezoidal Rule gives the average of the Left- and Right-Endpoint Rules.

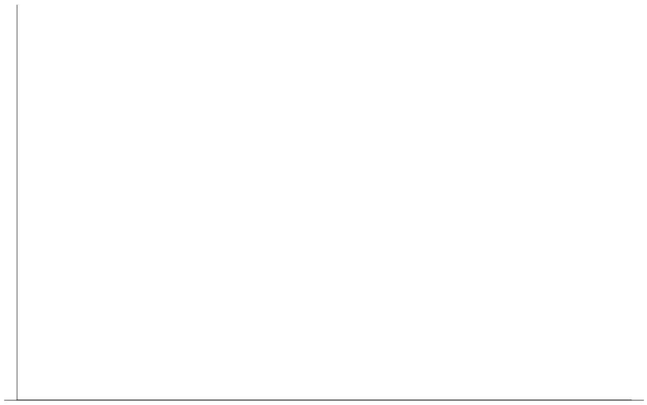

Figure 2.11: Finally, we could approximate the integral by the area under a quadratic function passing through the points $\{(a, f(a)), ((b - a)/2, f((b - a)/2)), (b, f(b))\}$. This corresponds to Simpson's Rule.

What we can do is first divide the integral into n "rectangles" and use one of the methods outlined above to approximate each of the rectangles separately.

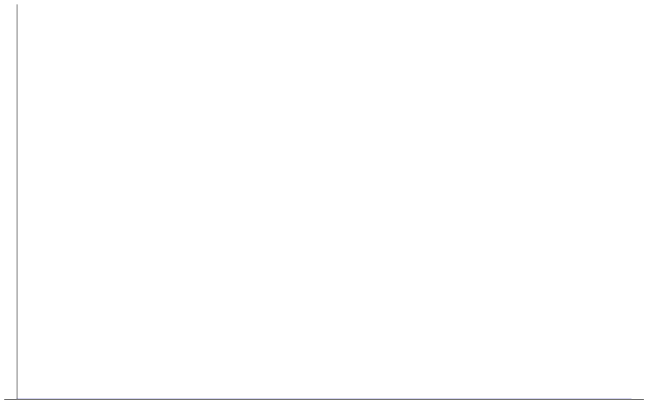

Figure 2.12: The idea of approximate integration is to break up the area into manageable chunks which we can then approximate separately.

The Midpoint Rule

Consider, once again the problem of finding the area underneath the curve of a function, between two points a and b:

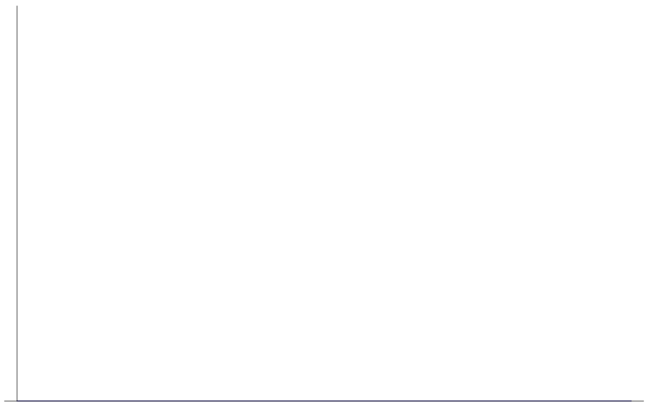

Figure 2.13: We can approximate the area under the curve by rectangles. In particular, if we choose the height of the rectangles to be the value of the function at the midpoint of the width, we have an approximation known as the Midpoint Rule.

Now each of the rectangles S_i has area width by height:

$$S_i = f(\overline{x_i})\frac{b-a}{n}. \tag{2.1}$$

Hence we can approximate the area by adding them up: $A \approx S_1 + S_2 + \cdots + S_n$.

Midpoint Rule

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\bar{x_i})\Delta x = \Delta x[f(\bar{x_1}) + f(\bar{x_2}) + \cdots + f(\bar{x_n})]$$

where

$$\Delta x = \frac{b-a}{n}$$

and

$$\bar{x_i} = \frac{1}{2}(x_{i-1} + x_i) = \text{ midpoint of } [x_{i-1}, x_i].$$

Example

Use the Midpoint Rule with $n = 5$ to approximate

$$\int_1^2 \frac{1}{x}\,dx.$$

Compare this with the actual value of the integral.
The midpoints of the five subintervals are $1.1, 1.3, 1.5, 1.7, 1.9$ so the Midpoint Rule gives:

$$\begin{aligned}\int_1^2 \frac{1}{x}\,dx \approx M_5 &= \Delta x[f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9)] \\ &= \frac{1}{5}\left(\frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \frac{1}{1.7} + \frac{1}{1.9}\right) \\ &\approx 0.691908.\end{aligned}$$

Now the actual value of the integral:

$$\int_1^2 \frac{1}{x}\,dx = [\log|x|]_1^2 = \log 2 - \log 1 = \log 2 \approx 0.693147.$$

The difference between them is given by:

$$E_M = \left|\int_a^b f(x)\,dx - M_5\right| \approx 0.00123918.$$

The Trapezoidal Rule

Consider, once again the problem of finding the area underneath the curve of a function, between two points a and b:

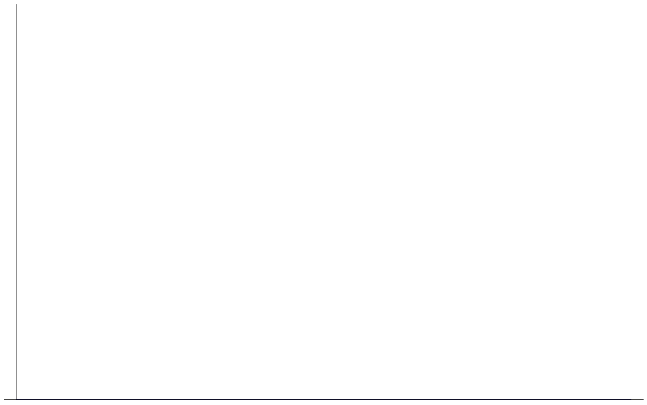

Figure 2.14: We can approximate the area under the curve by trapezoids. Remember all of the subintervals are length $\Delta x = (b-a)/n$.

Now each of the trapezoids S_i has area width by height for the rectangular part, plus half the base by the height for the triangular 'hat', hence[1]:

$$\begin{aligned} T_i &= f(x_{i-1})\Delta x + \frac{1}{2}\Delta x(f(x_i) - f(x_{i-1})) \\ &= \frac{1}{2}\Delta x[f(x_{i-1}) + f(x_i)]. \end{aligned}$$

Hence we can approximate the area by adding them up:

$$\begin{aligned} \int_a^b f(x)\,dx &\approx \frac{1}{2}\Delta x\left[(f(x_0)+f(x_1)) + (f(x_1)+f(x_2)) + \cdots + (f(x_{n-1})+f(x_n))\right] \\ &= \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]. \end{aligned}$$

Trapezoidal Rule

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

where

$$\Delta x = \frac{b-a}{n}$$

and

$$\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i) = \text{ midpoint of } [x_{i-1}, x_i].$$

[1] as an exercise show that this calculation is the same if $f(x_i) > f(x_{i-1})$.

Example

Use the Trapezoidal Rule with $n = 5$ to approximate

$$\int_1^2 \frac{1}{x}\,dx.$$

Compare this with the actual value of the integral.
With $n = 5$, and $b - a = 1$, we have $\Delta x = 1/5$ and so the Trapezoidal Rule gives:

$$\begin{aligned}\int_1^2 \frac{1}{x}\,dx \approx T_5 &= \frac{1/5}{2}[f(1) + 2f(1.2) + 2f(1.4) + 2f(1.6) + 2f(1.8) + f(2)] \\ &= \frac{1}{10}\left(\frac{1}{1} + \frac{2}{1.2} + \frac{2}{1.4} + \frac{2}{1.6} + \frac{2}{1.8} + \frac{1}{2}\right) \\ &\approx 0.695635.\end{aligned}$$

Now the actual value of the integral:

$$\int_1^2 \frac{1}{x}\,dx = [\log|x|]_1^2 = \log 2 - \log 1 = \log 2 \approx 0.693147.$$

The difference between them is given by:

$$E_T = \left|\int_a^b f(x)\,dx - T_5\right| \approx 0.00248782.$$

Simpson's Rule

Another rule for approximating definite integrals is by using quadratic functions instead of straightline segments to approximate a curve:

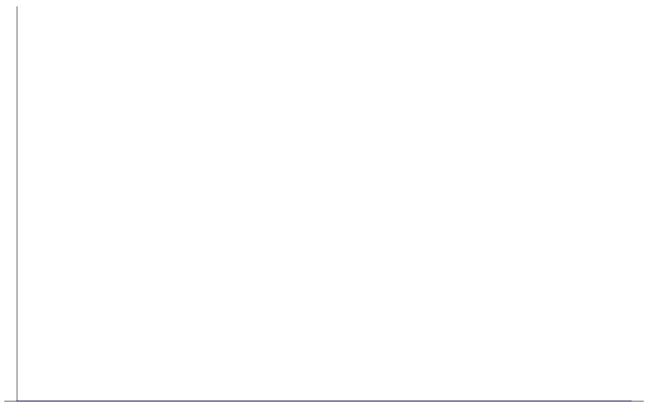

Figure 2.15: We can approximate the area under the curve by the area under a quadratic. Remember all of the subintervals are length $\Delta x = (b - a)/n$ and in this case we actually have an even number of subintervals.

If we follow this analysis carefully we can show:

Simpson's Rule

$$\int_a^b f(x)\,dx \approx S_n = \frac{\Delta x}{3}[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)]$$

where n is even and $\Delta x = (b-a)/n$.

Error Analysis

Error Bounds for the Trapezoidal and Midpoint Rules

Suppose $K = \max_{x\in[a,b]} f''(x)$. If E_M and E_T are the errors in the Midpoint and Trapezoidal Rules:

$$|E_M| \leq \frac{K(b-a)^3}{24n^2} \tag{2.2}$$

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} \tag{2.3}$$

Examples

Give an upper bound for the error involved when we approximate $\int_0^1 e^{x^2}\,dx$ by M_{10}.

How large should we take n *to ensure that the Trapezoidal and Midpoint Rule approximations to* $\int_1^2 \frac{1}{x}\,dx$ *is accurate to within* 0.0001.

Error Bound for Simpson's Rule

Suppose that $K = \max_{x\in[a,b]} |f^{(iv)}(x)|$. *If* E_S *is the error in using Simpson's Rule, then*

$$|E_S| \leq \frac{K(b-a)^5}{180n^4}. \tag{2.4}$$

Examples

Give an upper bound for the error involved when we approximate $\int_0^1 e^{x^2}\,dx$ *by* S_{10}.

How large should we take n *to ensure that the Simpson Rule approximation to* $\int_1^2 \frac{1}{x}\,dx$ *is accurate to within* 0.0001.

Exercises

1. *Estimate* $\int_0^1 \cos(x^2)\,dx$ *using (a) the Trapezoidal Rule and (b) the Midpoint Rule, each with* $n = 4$.

2. *Use (a) the Midpoint Rule and (b) Simpson's Rule to approximate to six decimal places.*

$$\int_0^{\pi} x^2 \sin x\,dx,\ n = 8.$$

$$\int_0^1 e^{-\sqrt{x}}\,dx,\ n = 6.$$

   Integrate the first integral by parts and compare these approximate values with the real value.

3. *Use (a) the Trapezoidal Rule, (b) the Midpoint Rule and (c) Simpson's Rule to approximate the given integral with the specified value of* n *(Round to six decimal places).*

$$\int_0^4 \sqrt{1+\sqrt{x}}\,dx,\ n = 8.$$

$$\int_0^4 \sqrt{x}\sin x\,dx,\ n = 8.$$

$$\int_0^3 \frac{1}{1+y^5}\,dy,\ n = 6.$$

4. *Find the approximations* T_8 *and* M_8 *for* $\int_0^1 \cos(x^2)\,dx$. *Estimate the errors involved in the approximations. How large do we have to choose* n *so that the approximations* T_n *and* M_n *are accurate to within* 0.00001*?*

5. *Find the approximations* T_{10} *and* S_{10} *for* $\int_0^1 e^x\,dx$ *and the corresponding errors* E_T *and* E_S. *Compare the actual errors (in comparison to the true value of the integral) with error estimates* E_T *and* E_S. *How large should* n *be to guarantee that the approximations* T_N *and* M_n *are accurate to* 0.00001*?*

6. *How large should* n *be to guarantee that the Simpson's Rule approximation to* $\int_0^1 e^{x^2}$ *us accurate to within* 0.00001*?*

7. * *Show that if* p *is a polynomial of degree 3 or lower, then Simpson's Rule gives the exact value of* $\int_a^b p(x)\,dx$.

2.3 Euler's Method

Differential Equations

Perhaps the most important of all the applications of calculus is to differential equations. When scientists — both physical and social — use calculus, more often than not it is to analyse a differential equation that has arisen in the process of modeling some phenomenon that they are studying. Although it is often impossible to find an explicit formula for the solution of a differential equation, we will see that graphical and numerical approaches provide approximations.

Mathematical models often take the form of differential equations — that is, an equation that contains an unknown function and some of it's derivatives. The aim is to find the function that satisfies the equation — that solves the differential equation. Examples of processes that have been successfully modeled by differential equations include population growth, the motion of a spring and electrical circuits.

First Order Differential Equations

A *first order differential equation* is an equation that contains an unknown function and it's first derivative. Examples of first order differential equations:

$$\frac{dP}{dt} = kP$$

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right)$$

$$\frac{dy}{dx} = xy$$

In the first two equations k (and K) are constants and the 'solution' will be a function $P = P(t)$ — a function of t. In the third the solution will be a y, a function of x, $y(x)$. In general, the *solution* of a differential equation is not unique — usually there will be an infinite family of solutions, as a whole called the *general solution.*

When applying differential equations, we are usually not as interested in finding a family of solutions (the *general solution*) as we are in finding a solution that satisfies some additional requirement. In many physical problems we need to find the particular solution that satisfies a condition of the form $y(t_0) = y_0$, for known constant t_0 and y_0. This is called an *initial* or *boundary* condition and the differential equation can be referred to as an *initial value problem* or a *boundary value problem.*

Direction Fields

Unfortunately, it's impossible to solve most differential equations in the sense of obtaining an explicit formula for the solution. In this section, we show that, despite the absence of an explicit solution, we can still learn a lot about the solution through a graphical approach (direction fields) or a numerical approach (Euler's Method).

Suppose we are asked to sketch the graph of the solution of the initial value problem:

$$\frac{dy}{dx} = x + y \ , \ \ y(0) = 1.$$

We don't know a formula for the solution, so how can we possibly sketch its graph? Let's think about what the differential equation means. The equation $y' = x + y$ tells us that the slope at any point (x, y) on the graph of $y(x)$ is equal to the sum of the x- and y-coordinates at that point. In particular, because the curve passes through the point $(0, 1)$, its slope there must be $0 + 1 = 1$. So a small portion of the solution curve near the point $(0, 1)$ looks like a short line segment through $(0, 1)$ with slope 1:

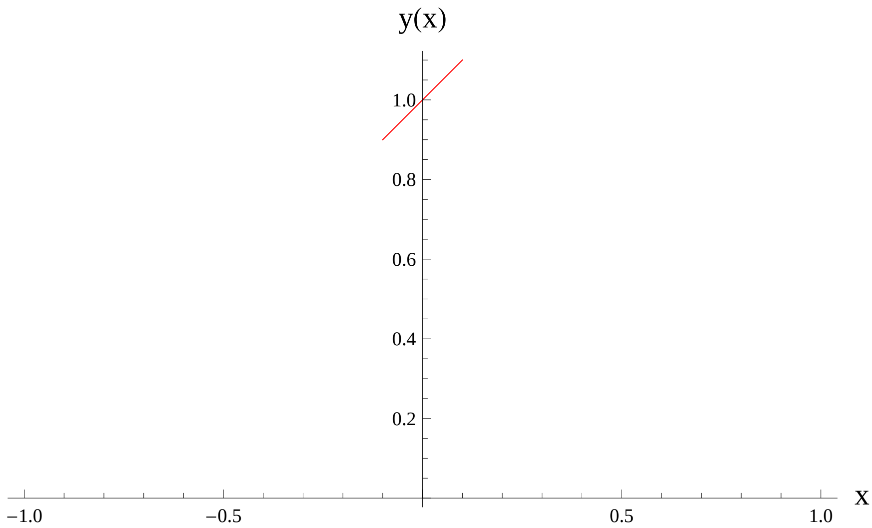

Figure 2.16: Near the point $(0, 1)$, the slope of the solution curve is 1.

As a guide to sketching the rest of the curve, let's draw short line segments at a number of points (x, y) with slope $x + y$. The result is called a *direction field* and is shown below:

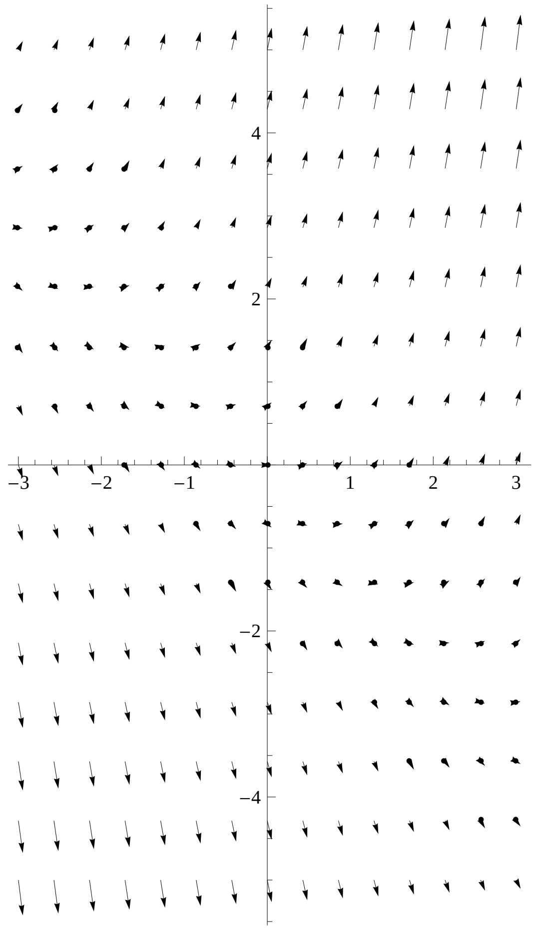

Figure 2.17: For example, the line segment at the point $(1, 2)$ has slope $1 + 2 = 3$. The direction field allows us to visualise the general shape of the solution by indicating the direction in which the curve proceeds at each point.

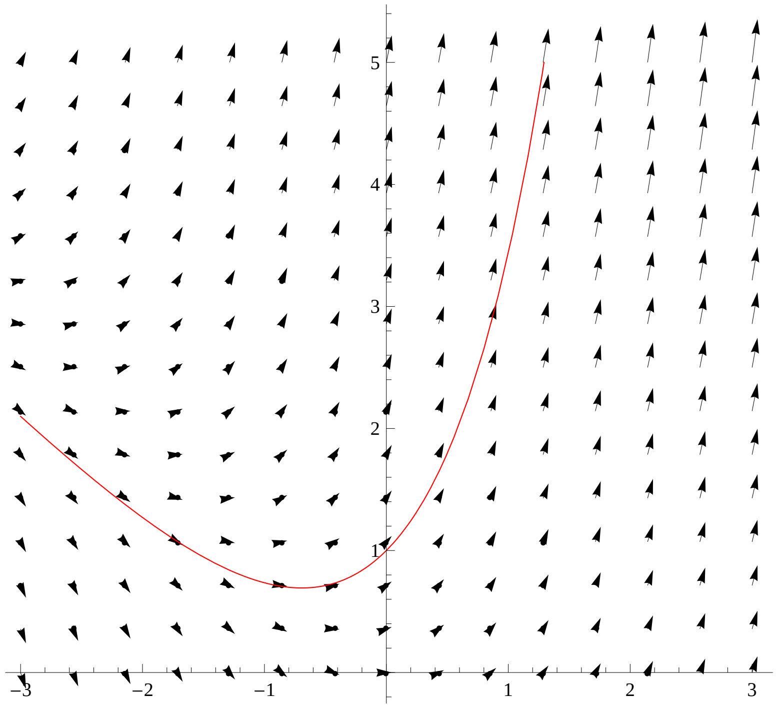

Figure 2.18: We can sketch the solution curve through the point $(0, 1)$ by following the direction field. Notice that we have drawn the curve so that it is parallel to nearby line segments.

Euler's Method

The basic idea behind direction fields can be used to find numerical approximations to solutions of differential equations. We illustrate the methods on the initial-value problem that we used to introduce direction fields:

$$\frac{dy}{dx} = x + y \ , \quad y(0) = 1.$$

The differential equation tells us that $y'(0) = 0 + 1 = 1$, so the solution curve has slope 1 at the point $(0, 1)$. As a first approximation to the solution we could use the linear approximation $L(x) = 1x + 1$. In other words we could use the tangent line at $(0, 1)$ as a rough approximation to the solution curve.

Euler's idea was to improve on this approximation by proceeding only a short distance along this tangent line and then making a correction by changing direction according to the direction field:

Euler's method says to start at the point given by the initial value and proceed in the direction indicated by the direction field. Stop after a short time, look at the slope at the new location, and proceed in that direction. Keep stopping and changing direction according to the direction field. Euler's method does not produce an exact solution to the initial-value problem — it gives approximations. But by decreasing the step size (and therefore increasing the amount of corrections), we obtain successively better approximations to the correct solution.

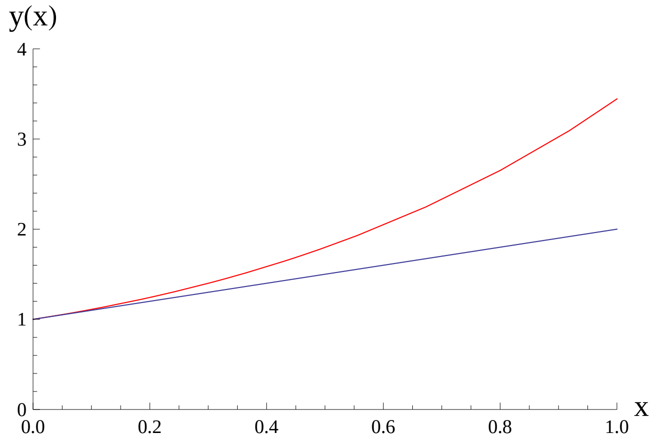

Figure 2.19: The tangent at $(0, 1)$ approximates the solution curve for values near $x = 0$.

For the general first-order initial-value problem $y' = F(x, y)$, $y(x_0) = y_0$, our aim is to find approximate values for the solution at equally spaced numbers x_0, $x_1 = x_0 + h$,$x_2 = x_0 + 2h = x_1 + h$..., where h is the step size. The differential equation tells us that the slope at (x_0, y_0) is $y' = F(x_0, y_0)$:

This shows us that the approximate value of the solution when $x = x_1$ is

$$y_1 = y_0 + hF(x_0, y_0)$$

Similarly,

$$y_1 = y_0 + hF(x_0, y_0)$$

Euler's Method

If

$$\frac{dy}{dx} = F(x, y) \ , \ \ y(x_0) = y_0$$

is an initial value problem. If we are using Euler's method with step size h then

$$y(x_{n+1}) \approx y_{n+1} = y_n + hF(x_n, y_n) \tag{2.5}$$

for $n \geq 0$.

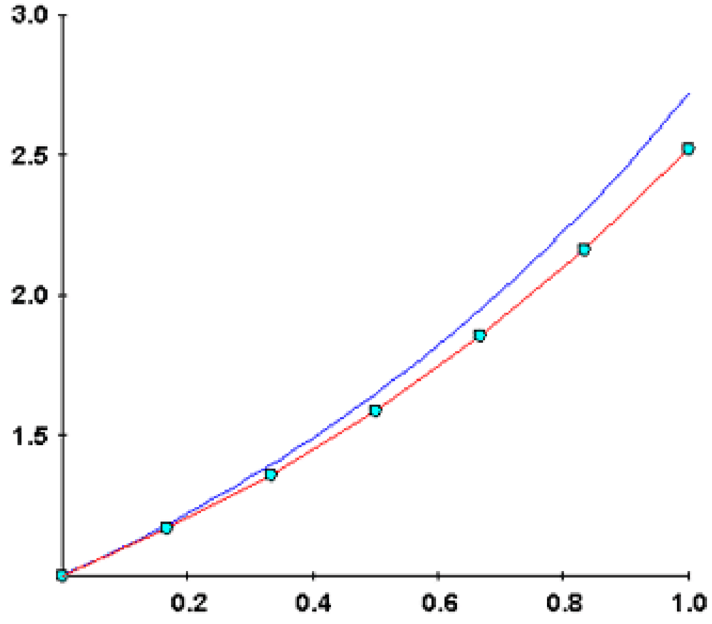

Figure 2.20: Euler's Method starts at some initial point (here $(x_0, y_0) = (0, 1)$), and proceeds for a distance h (in this plot $h = 1/6$.) at a slope that is equal to the slope at that point $y' = x_0 + y_0$. At the point $(x_1, y_1) = (x_0 + h, y_1)$, the slope is changed to what it is at (x_1, y_1), namely $x_1 + y + 1$, and proceeds for another distance h until it changes direction again.

Example

Use Euler's method with step size $h = 0.1$ to approximate $y(1)$, where $y(1)$ is the solution of the initial value problem:

$$\frac{dy}{dx} = x + y \ , \quad y(0) = 1$$

Solution: We are given that $h = 0.1$, $x_0 = 0$ and $y_0 = 1$, and $F(x, y) = x + y$. So we have

$$\begin{aligned} y_1 &= y_0 + F(x_0, y_0) = 1 + 0.1(0 + 1) = 1.1 \\ y_2 &= y_1 + F(x_1, y_1) = 1.1 + 0.1(0.1 + 1.1) = 1.22 \\ y_3 &= y_2 + F(x_2, y_2) = 1.22 + 0.1(0.2 + 1.22) = 1.362 \end{aligned}$$

Continue this process [Exercise] to get $y_{10} = 3.187485$, which approximates $y(x_{10}) = y(x_0 + 10(0.1)) = y(1)$, as required.

Exercises

1. *Use Euler's method with step size 0.5 to compute the approximate* y*-values* y_1, y_2, y_3 *and* y_4 *of the initial value problem* $y' = y - 2x$, $y(1) = 0$.

2. *Use Euler's method with step size to estimate* $y(1)$, *where* $y(x)$ *is the solution of the initial value problem* $y' = 1 - xy$, $y(0) = 0$.

3. *Use Euler's method with step size* 0.1 *to estimate* $y(0.5)$, *where* $y(x)$ *is the solution of the initial value problem* $y' - y = xy$, $y(0) = 1$.

4. *Use Euler's method with step size* 0.2 *to estimate* $y(1.4)$, *where* $y(x)$ *is the solution of the initial-value problem* $y' - x + xy = 0$, $y(1) = 0$.

Chapter 3

Introduction to Laplace Transforms

3.0.1 Outline of Chapter

- Definition to transform
- Determining the Laplace transform of basic functions
- Development of rules
- First shift theorem
- Transform of a derivative
- Inverse transforms
- Applications to solving Differential Equations
- Applications to include the Damped Harmonic Oscillator

3.1 Definition of Transform

Improper Integrals

Consider the infinite region S that lies under the curve $y = 1/x^2$, above the x axis and to the right of $x = 1$:

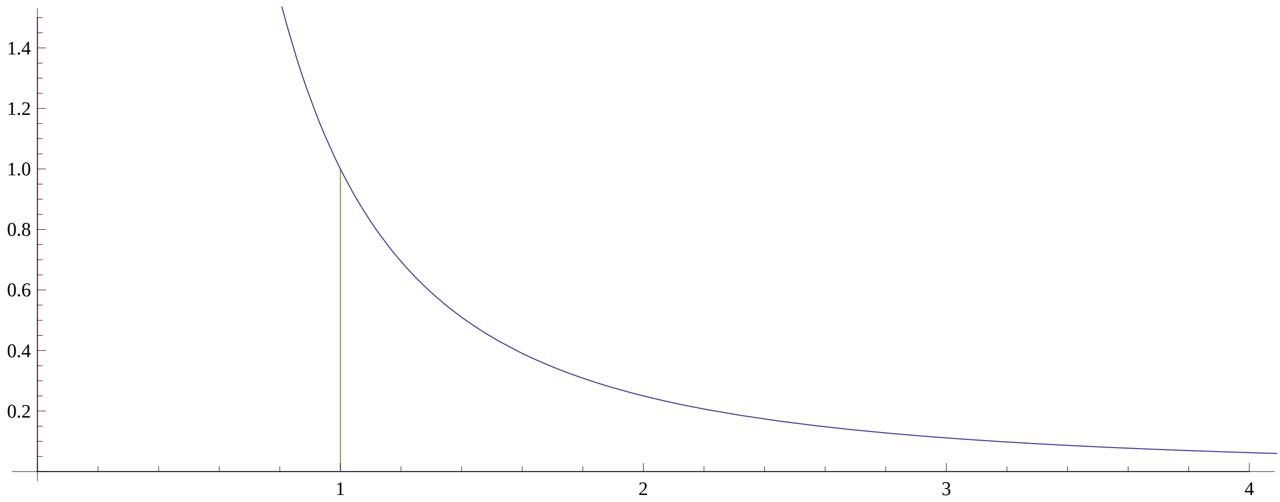

Figure 3.1: You might think that because S is infinite in extent, that it's area must be infinite.

Lets take a closer look. The area of the part of S that lies to the left of $x = R$ is:

We also observe that

The area $A(R)$ approaches 1 as $t \to \infty$, so we say that the area of the infinite region S equals 1 and we write:

Using this example as a guide, we define the integral of f over an infinite interval as the limit of integrals over finite intervals.

Definition

If $\int_a^R f(x)\,dx$ exists for every $R \geq a$, then

The integral $\int_a^\infty f(x)\,dx$ is called *convergent* if it exists; otherwise it is *divergent.*

Examples

Determine whether the integral $\int_1^\infty 1/x\,dx$ *is convergent or divergent.*

Evaluate

$$\int_0^\infty xe^{-x}\,dx.$$

Evaluate

$$\int_0^\infty \frac{1}{1+x^2}\,dx$$

The Laplace Transform — Formal Definition

At the very roughest level, the Laplace Transform is a function:

The difference between functions of the type $f(x) = x^2$ is that the inputs are functions of a real variable — and the outputs are functions of a complex variable. By and large we will be suppressing the fact that $\mathcal{L}\{f(t)\} = F(s)$ is a function defined on the complex numbers, and we will usually just treat s as a real number. An interpretation is that t is a time variable and s is a frequency variable.

Definition

Consider a function $f(t)$ defined for $t \geq 0$. Then, if the integral:

exists at s, it is called the *Laplace transform* of $f(t)$ and we write:

Exercises

1. *Determine whether the integrals are convergent or divergent. Evaluate those that are convergent.*

$$(i) \quad \int_1^\infty \frac{1}{(3x+1)^2}\, dx.$$

$$(ii) \quad \int_0^\infty \frac{x}{(x^2+2)^2}\, dx.$$

$$(iii) \quad \int_4^\infty e^{-y/2}\, dy.$$

$$(iv) \quad \int_{2\pi}^\infty \sin\theta \, d\theta.$$

$$(v) \quad \int_0^\infty \frac{dz}{z^2+3z+2}.$$

2. *Find the Laplace transform of the zero function $f(t) = 0$.*

3. *Find the Laplace transform of the following function, $g(t)$:*

$$g(t) := \begin{cases} 1 & \text{if } t \in [0,1] \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

3.2 The Laplace transform of basic functions

Analysis Fact

Let $a, b > 0$*;*

$$\lim_{x\to\infty} \frac{x^a}{e^{bx}} = 0.$$

A Constant Function

Let $f(t) = k$, a constant. What about $\mathcal{L}\{f(t)\}$?

In the region where $s > 0$:

In particular, if $f(t) = 1$;

$$\mathcal{L}\{f(t)\} = \frac{1}{s}\ , \text{ for } s > 0. \tag{3.2}$$

Exponential Function

Let $f(t) = e^{at}$ where $a \in \mathbb{R}$.

Now if $s > a$, then $a - s < 0$ hence $e^{(a-s)R} \to 0$:

So if $f(t) = e^{at}$ with $s > a$;

$$\mathcal{L}\{f(t)\} = \frac{1}{s-a}\ , \text{ for } s > a. \tag{3.3}$$

Identity Function

Let $f(t) = t$.

Let $I = \int te^{-st}\, dt$;

Now putting in the limits:

Now assume that $s > 0$. Now as $R \to \infty$:

So if $f(t) = t$ with $s > 0$;

$$\mathcal{L}\{f(t)\} = \frac{1}{s^2}\ , \text{ for } s > 0. \tag{3.4}$$

Powers

Let $f_n(t) = t^n$ and $s > 0$. We do a proof by induction.

Let $P(n)$ be the proposition that $\mathcal{L}\{f_n(t)\} = n!/s^{n+1}$.

Consider $P(1)$. We have just shown that:

Hence $P(1)$ is true.
Assume that $P(k)$ is true; that is:

Consider now $P(n+1)$; i.e. evaluate $\mathcal{L}\{t^{k+1}\}$.

Now putting in the limits:

$$\mathcal{L}\{f_{k+1}(t)\} = \underbrace{\left[-\frac{t^{k+1}e^{-st}}{s}\right]_0^\infty}_{=:J} + \frac{k+1}{s}\underbrace{\int_0^\infty t^k e^{-st}\,dt}_{=\mathcal{L}\{t^k\}=k!/s^{k+1}}$$

Now looking at J:

Therefore, $\mathcal{L}\{t^{k+1}\}$:

Hence by the inductive hypothesis $P(n)$ is true for all $n \in \mathbb{N}$. So if $f(t) = t^n$ with $s > 0$;

$$\mathcal{L}\{f(t)\} = \frac{n!}{s^{n+1}}\ , \text{ for } s > 0. \tag{3.5}$$

Polynomials and Trigonometric Functions

Let $p(t) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$ be a polynomial. What about $\mathcal{L}\{p(t)\}$? How can we do the Laplace transform of this sum?

Complex Analysis Fact

We have the following:

$$\cos \omega t =$$

$$\sin \omega t =$$

So to look at the Laplace Transform of say, $\cos 2t$, we can consider the Laplace transform of:

The next section will show us how to do this.

Exercises

1. *Find, from first principles,* $\mathcal{L}\{e^{2t}\}$ *and state it's region of convergence.*

2. *Find, from first principle,* $\mathcal{L}\{1+t\}$.

3. *Find, from fist principles,* $\mathcal{L}\{t^2\}$. *Use this result to find* $\mathcal{L}\{10t^2\}$.

3.3 Properties of the Laplace Transform

Linearity

Suppose that $a, b \in \mathbb{R}$ and $g, h : \mathbb{R} \to \mathbb{R}$ are real functions. Define

$$f(t) = ag(t) + bh(t).$$

What can we say about $\mathcal{L}\{f(t)\}$ — i.e. about sums $f + g$ and linear combinations $f + \lambda g$?

Proposition

The Laplace transform is linear:

$$\mathcal{L}\{ag(t) + bh(t)\} = a\mathcal{L}\{g(t)\} + b\mathcal{L}\{h(t)\} = aG(s) + bH(s) \tag{3.6}$$

Proof: This all hinges on the fact that integration is linear:

•

Examples

Find the Laplace transform of

$$q(t) = at^2 + bt + c.$$

Solution:

Find the Laplace transform of $f(t) = 4e^{-t} - 1t^3$.

Solution:

Sine and Cos

1. We have that

   Hence, $\mathcal{L}\{\cos \omega t\}$:

2. Now looking at $\sin t$:

Examples

1. *Find* $\mathcal{L}\{\cos 5t\}$.

2. *Find* $\mathcal{L}\{\cos 3t \sin t\}$.
   We don't have a product rule for Laplace transforms so we must endeavour to write this product as a sum. We can do this using a formula from the tables:

$$2 \sin A \cos B = \sin(A + B) + \sin(A - B)$$
$$\Rightarrow \sin A \cos B = \frac{1}{2} \sin(A + B) + \frac{1}{2} \sin(A - B)$$

First Shift Theorem

Proposition

Suppose that $a \in \mathbb{R}$ *and* $\mathcal{L}\{f(t)\} = F(s)$. *Then*

$$\mathcal{L}\{e^{-at} f(t)\} = F(s+a).$$

Proof:

•

Examples

1. *Find* $\mathcal{L}\{e^t t^2\}$.

2. *Find* $\mathcal{L}\{e^{-2t} \cos 3t\}$.

Differentiation

Can we say anything about the Laplace transform of a derivative? First we have to say a little something about functions with a Laplace transform.

Integration Fact

Suppose that $\int_0^\infty g(x)\,dx$ *exists; then* $\lim_{x\to\infty} g(x) = 0$.

Justification

Proposition

Suppose that $f(t)$ *has Laplace transform* $F(s)$. *Then*

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0).$$

Proof

Example

Suppose that $y(x)$ is the solution of the differential equation:

$$\frac{dy}{dx} - 5y = 5e^{-x},$$

and $y(0) = 1$. Find $\mathcal{L}\{y\}$.

Proposition

Suppose that $f(t)$ has Laplace transform $F(s)$. Then

$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0).$$

Proof

Example

Find the Laplace transform of the door closer; i.e. the function $x(t)$ satisfying the differential equation:

$$m\frac{d^2x}{dt^2} + \lambda\frac{dx}{dt} + kx = 0.$$

You can assume that the initial displacement is A and initial speed is 0.

Exercises

1. *Find the Laplace Transform of the following functions:*

$$
\begin{aligned}
(i)&\ \ 6\cos 4t + t^3\\
(ii)&\ \ 2\sin 3t\cos t\\
(iii)&\ \ t^3e^{-t}\\
(iv)&\ \ 6t^2 + 3t - 8\\
(v)&\ \ t^3e^{-5t}\\
(vi)&\ \ 3\cos 4t\sin 2t\\
(vii)&\ \ 3\sin 5t + 4\cos 5t\\
(viii)&\ \ 15t^2e^{-4t}\\
(ix)&\ \ 50t - 250(1 - e^{-t/5})\\
(x)&\ \ t^2(1-t)\\
(xii)&\ \ 4\sin 2t + 5e^{-5t}\cos 2t\\
(xiii)&\ \ (t+1)^2\\
(xiv)&\ \ \frac{(t^2+3t)^3}{t}\\
(xv)&\ \ 4e^{-3t}\sin(5t + \pi/3)
\end{aligned}
$$

For parts (xiv) and (xv) please simplify by multiplying out and canceling and by using the $\sin(A+B)$ *formula.*

2. *Find the Laplace transforms of the functions which satisfy the following differential equations:*

$$
\begin{aligned}
(i)&\ \ 4\frac{dI}{dt} + 12I = 60\,,\ I(0) = 0.\\
(ii)&\ \ y'' + 2y' + 4y = 0\,,\ y(0) = 1,\ y'(0) = 0.\\
(iii)&\ \ y'' - 2y' - 5y = 0\,,\ y(0) = 0,\ y'(0) = 3.\\
(iv)&\ \ \frac{dx^2}{dt^2} + 4\frac{dx}{dt} + 5x(t) = 0\,,\ x(0) = 1,\ x'(0) = 2\\
(v)&\ \ \frac{d^2\theta}{dt^2} - 4\frac{d\theta}{dt} + 13\theta(t) = 3e^{2x} - 5e^{3x}\,,\ \theta(0) = \theta'(0) = 1.\\
(vi)&\ f''(t) - 3f'(t) = 2e^{2x}\sin x\,,\ f(0) = 1\,,\ f'(0) = 2.
\end{aligned}
$$

3.4 Inverse transforms

A Review of Partial Fractions

1.

2.

3.

4.

Example

Find the partial fraction expansion of

$$\frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x}$$

Now comparing the numerators:

The Cover Up Method

There is a cheeky little method for doing partial fractions *when $q(x)$ factors has two distinct linear terms.* Suppose we are trying to find the partial fraction decomposition of

$$\frac{5x+10}{(x+1)(x+6)}.$$

1. Cover-up the $x+6$ with your hand and put $x=-6$ into what is left:

2. Cover-up the $x+1$ with your hand and put $x=-1$ into what is left:

3. These *residues* are the constants:

Theorem

When we have $q(x)$ factorising into distinct linear terms the cover-up method always works.

Proof

Multiplying as shown to get common denominators:

$$\frac{a\beta+b}{\beta-\alpha}\cdot\frac{1}{x-\beta}\cdot\frac{x-\alpha}{x-\alpha}-\frac{a\alpha+b}{\beta-\alpha}\cdot\frac{1}{x-\alpha}\cdot\frac{x-\beta}{x-\beta}$$

Multiplying out:

$$\frac{(a\beta - a\alpha)x + (b\beta - b\alpha) + (bx - bx) + (a\alpha\beta - a\alpha\beta)}{(\beta - \alpha)(x - \alpha)(x - \beta)}$$

Examples

Use the cover-up method to find the partial fraction expansion of:

$$\frac{1}{(x-8)(x-6)}$$

Use the cover-up method to find the partial fraction expansion of:

$$\frac{5x-2}{x^2-4}.$$

Definition of the Inverse Transform

We have seen that the Laplace Transform takes functions from the t-domain to the s-domain. Is there an *inverse* transform, $\mathcal{L}^{-1}$ that can take us back (faithfully)? The answer is yes — we will not be examining this transform in detail but you can believe me that it does exist. Of course it has the special property that

Straight away looking at the table of Laplace transforms we can see the following (in all cases $F(x) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$):

Proposition

- Linearity: For any constants $a, b \in \mathbb{R}$, $\mathcal{L}^{-1}\{aF(s) + bG(s)\} = af(t) + bg(t)$.
- First Shift Theorem: $\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t)$.
- Powers: $\mathcal{L}^{-1}\{1/s^n\} = t^{n-1}/(n-1)!$.
- Linear Partial Fractions:
$$\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}.$$
- Quadratic Partial Fractions:
$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos kt$$
and
$$\mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin kt$$

Loads of Examples

Find the inverse Laplace transforms of the following functions:

1. $$\frac{12s}{(s+3)(s-2)}$$

2. $$\frac{4s}{36s^2+1}$$

3.

$$\frac{2}{(s+3)^5}$$

4.

$$\frac{3s^2 + 2}{(s-2)(s+3)(s^2+25)}$$

5.

$$\frac{(s+1)^2}{s^4}$$

6.

$$\frac{2s+1}{s^2+6s+13}$$

7.

$$\frac{6s}{s^2 + 8s + 25}$$

Exercises

Find the Inverse Laplace transforms of the following functions:

$$(i)\ \frac{(s+1)^3}{s^4}$$

$$(ii)\ \frac{1}{4s+1}$$

$$(iii)\ \frac{4s}{4s^2+1}$$

$$(iv)\ \frac{1}{s^2+3s}$$

$$(v)\ \frac{2s+4}{(s-2)(s^2+4s+3)}$$

$$(vi)\ \frac{s}{(s-1)(s^2+1)}$$

$$(vii)\ \frac{s}{(s^2-4)(s+2)}$$

$$(viii)\ \frac{4}{s(s^2+4)}$$

$$(ix)\ \frac{s}{s^2+6s+13}$$

$$(x)\ \frac{12}{s(s+2)^2}$$

3.5 Differential Equations

We introduced the idea of a differential equation in the section on Euler's Method. A differential equation is an equation involving an unknown function $y(x)$ and it's derivatives:

The aim is to solve for the function $y(x)$. In this section we deal with *ordinary differential equations* — that is differential equations involving a function of a single variable; e.g. $y(x)$. In subsequent modules we will encounter *partial differential equations*. For example, the *Laplace Equation* asks for a function of two variables $z = f(x, y)$ which satisfies:

The following tidbit is examinable in MATH6037:

Definition

A *harmonic function* is a function $z = f(x, y)$ which satisfies the Laplace equation.

Solving Linear Differential Equations using Laplace Transforms

Consider a linear differential equation:

1. Take the Laplace transform of both sides.
2. Solve for $\mathcal{L}\{y\} =: Y(s)$, the transformed solution.
3. Find the solution $y(t)$ by applying the inverse transform to $Y(s)$.

All of our differential equations in MATH6037 will look like:

where $a, b, c \in \mathbb{R}$. In many examples $\phi(t)$ will be zero. We will now present a number of examples of differential equations which can be solved using this method.

Radioactive Decay

The decay of an unstable nucleus is entirely random and it is impossible to predict when a particular atom will decay. However, it is equally likely to decay at any time. Therefore, given a sample of a particular radioisotope, the rate of decay is proportional to the number of atoms present, where λ is a positive constant known as the *decay constant*:

Example: A First Order, General Solution

Derive an expression for the number of atoms present in a radioactive particle with decay constant λ if the initial number of atoms is given by N_0.

That is,

$$\mathcal{N}(s) = \frac{N_0}{s+\lambda}$$

The Charging of a Capacitor

It can be shown that the charge $q(t)$ on a capacitor of capacitance C, in series with a resistor of resistance R and connected to a battery of voltage E satisfies the first order differential equation:

$$R\frac{dq}{dt} + \frac{1}{C}q = E. \tag{3.7}$$

Please find an expression for the charge on the capacitor plate if it is initially uncharged. In the limit as $t \to \infty$ what does the charge look like?

Of course the capacitor is initially uncharged — i.e. $q(0) = 0$;

We can do a cover up method with this:

Now applying the inverse transform:

Now as times tends to infinity, this tends to EC.

Pendulum

The angle a pendulum of length L makes with the vertical, θ, can be shown to satisfy the nonlinear differential equation:

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0.$$

For small angles θ, we can make the approximation $\sin\theta \approx \theta$. Hence we can approximate solutions to the original differential equation by solutions to

Using the initial conditions, $d\theta/dt(0) = 0$ and $\theta(0) = 0.1$ rad, and taking $g/L = 1/10$, use Laplace transforms to solve the differential equation for $\theta(t)$.

Describe the long term behaviour of the pendulum. How realistic do you think this model is?

Applying the boundary conditions:

Now applying the inverse transform:

Abstract Example

Solve the initial value problem:

$$y''(x) + y'(x) - 6y(x) = 0$$

subject to the initial values $y(0) = 1$, $y'(0) = 0$.

Apply the boundary conditions:

Expand using partial fractions — we can use the cover-up method here:

Now applying the inverse transform:

Free-Falling

Newton's Second Law states:

To look at a particular example of this consider a particle of mass m falling from the sky. The only forces on the particle are gravity and air resistance. The gravitational force is given by $+mg$ and the air resistance, or drag, can be modeled by $-\lambda v$. Hence the *equation of motion* is given by:

Example

Consider a particle of mass 1 kg dropped from a height of 10,000 m. Let $\lambda = 10$ *and use the approximation* $g \approx 10\ m\ s^{-2}$*. Derive an expression for it's height above earth as a function of time,* $h(t)$*. Also consider it's speed as time tends to infinity.*

Now dividing across by $s^2 + 10s$:

We will need to transform back — so use partial fractions:

Now looking at $10/(s^2(s+10))$:

Now putting everything back together we get:

$$\begin{aligned} H(s) &= -\frac{1000}{s+10} + \frac{1000}{s} + \frac{1000}{s+10} - \frac{1}{10}.\frac{1}{s} + \frac{1}{s^2} + \frac{1}{10} \cdot \frac{1}{s+10} \\ &= \frac{90001}{10} \cdot \frac{1}{s+10} + \frac{9999}{10} \cdot \frac{1}{s} + \frac{1}{s^2} \end{aligned}$$

Now applying the inverse transform:

$$\mathcal{L}^{-1}\{H(s)\} = h(t) = \frac{1}{10}e^{-10t} + \frac{9999}{10} + t$$

Now speed is the rate of change of position — i.e. the derivative. Now:

That is as time grows indefinitely the speed approaches 1. In fact had we analysed the original, general, equation we would have found that this *terminal* velocity is given by mg/λ.

Exercises

Solve the differential equations at the end of section 3.3.

3.6 The Damped Harmonic Oscillator

Figure 3.2: A good door closer should close automatically, close in a gentle manner and close as fast as possible.

One possible design would be to put a mass on the door and attach a spring to it (just for ease of explanation we'll only worry about one dimension).

Assuming that the door is swinging freely the only force closing the door is the force of the spring. Now *Hooke's Law* states that the force of a spring is directly proportion to it's distance from the equilibrium position. If the door is designed so that the equilibrium position of the spring corresponds to when the door is closed flush, then if $x(t)$ is the position of the door t seconds after release, then the force of the spring at time t is given by:

where $k \in \mathbb{R}$ is known as the spring constant. We can show by solving the differential equation that this system *does* close the door automatically but the balance between closing the door gently and and closing the door quickly is lost.

Clearly we need to slow down the door as it approaches the door-frame. A simple model uses a *hydraulic damper*:

Figure 3.3: A hydraulic damper increases its resistance to motion in direct proportion to speed.

With the force due to the hydraulic damper proportional to speed, the force of the hydraulic damper at time t will be:

for some $\lambda \in \mathbb{R}$. Now by Newton's Second Law:

and the fact that speed is the first derivative of distance, and in turn acceleration is the first derivative of speed, means that the *equation of motion* is given by:

We will now solve this equation for general parameters m, λ and k. We will use the initial conditions that $x(0) = a$ — the initial distance from equilibrium, and $x'(0)$. The door closer is released from rest.

The Full Analysis — Damped Harmonic Oscillator*

We begin by defining two parameters γ and ω_0. These are related to the damping force and the spring force by:

$$\gamma = \frac{\lambda}{2m}.$$
$$\omega_0 = \sqrt{\frac{k}{m}}.$$

Now dividing across by m:

We can rewrite this using the new parameters:

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = 0$$

. Taking the Laplace transform of both sides, using Linearity:

Now using the differentiation theorems:

Now apply the boundary conditions:

Solve for $X(s)$:

We need to apply the inverse transform to this however as it is it's too difficult. We break the analysis up into *large*, *small* and *critical* damping.

Large Damping: $\gamma > \omega_0$

Consider the denominator of $X(s)$:

If this could be factorised as $(s-\alpha)(s+\alpha)$ we could use partial fractions to expand $X(s)$. Remember, if α and β are the roots of $q(x) = ax^2 + bx + c$, then we can write $q(x) = a(x-\alpha)(x-\beta)$ — so if we can find the roots we can factorise (normally you would factorise to find the roots). We have a formula for the roots of a quadratics:

Hence applied to the denominator of $X(s)$:

Now in this case the roots are real and distinct — as $\gamma > \omega_0$ implies that $\gamma^2 > \omega_0^2$. Hence we can factorise the bottom as (in any case):

$$\begin{aligned} s^2 + 2\gamma s + \omega_0^2 &= (s - (-\gamma - \sqrt{\gamma^2 - \omega_0^2}))(s - (-\gamma + \sqrt{\gamma^2 - \omega_0^2})) \\ &= (s - \gamma_-)(s - \gamma_+) \end{aligned}$$

where $\gamma_\pm = -\gamma + \sqrt{\gamma^2 - \omega_0^2}$. Now we have that:

$$X(s) = \frac{as + 2a\gamma}{(s-\gamma_-)(s-\gamma_+)} = a \underbrace{\frac{s + 2\gamma}{(s-\gamma_-)(s-\gamma_+)}}_{h(s)}$$

We want to find a partial fraction expansion for :

$$h(s) = \frac{A}{s - \gamma_-} + \frac{B}{s - \gamma_+}.$$

We can actually use the cover-up method on this one. The A 'residue' occurs where $s = \gamma_-$:

Which is:

$$\begin{aligned} A &= \frac{-\gamma - \sqrt{\gamma^2 - \omega_0^2} + 2\gamma}{-\gamma - \sqrt{\gamma^2 - \omega_0^2} - (-\gamma + \sqrt{\gamma^2 - \omega_0^2})} \\ &= \frac{\gamma - \sqrt{\gamma^2 - \omega_0^2}}{-2\sqrt{\gamma^2 - \omega_0^2}} = \frac{\sqrt{\gamma^2 - \omega_0^2} - \gamma}{2\sqrt{\gamma^2 - \omega_0^2}} \end{aligned}$$

Similarly we can show that:

$$B = \frac{\sqrt{\gamma^2 - \omega_0^2} + \gamma}{2\sqrt{\gamma^2 - \omega_0^2}}$$

Finally we have that:

$$X(s) = aA\frac{1}{(s - \gamma_-)} + aB\frac{a}{(s - \gamma_+)}.$$

Now take the inverse Laplace transform, using linearity:

$$x(t) = aA\mathcal{L}^{-1}\left\{\frac{1}{s - \gamma_-}\right\} + aB\mathcal{L}^{-1}\left\{\frac{1}{s - \gamma_+}\right\}$$

Using the tables we have that:

$$x(t) = aAe^{\gamma_- t} + aBe^{\gamma_+ t}. \tag{3.8}$$

The displacement tends exponentially to zero. For large times, the dominant term is that containing the larger coefficient, γ_+:

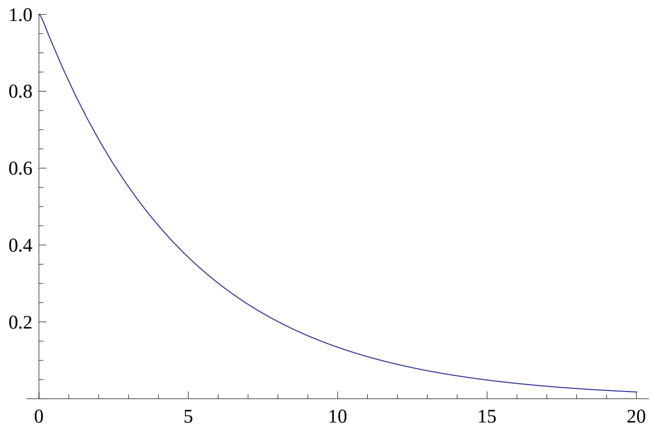

Figure 3.4: In large damping, the door closer fails to close quickly.

Small Damping: $\gamma < \omega_0$

In the case of small damping, the denominator of $X(s)$ has no real roots. The roots are given by:

$$\gamma_\pm = -\gamma \pm \sqrt{\gamma^2 - \omega^2},$$

and hence we cannot use a partial fraction expansion to evaluate (well not in the way we did it anyway). However, if we could complete the square we might be somewhere on our way (something looking like the final term the final destination):

$$X(s) = \frac{(s+2\gamma)}{(s+p)^2+q^2} \approx \frac{s+p}{(s+p)^2+q^2} + \frac{q}{(s+p)^2+q^2},$$

Multiplying out:

Let $\omega = \sqrt{\omega_0^2 - \gamma^2}$. This leaves us with:

$$X(s) = \frac{s+2\gamma}{(s+\gamma)^2+\omega^2}$$

If we split this as follows:

then they look like the Laplace transforms of sine and cos after a shift. Hence, using linearity, apply the inverse Laplace transform:

$$x(t) = a\mathcal{L}^{-1}\left\{\frac{s+\gamma}{(s+\gamma)^2+\omega^2}\right\} + a\frac{\gamma}{\omega}\mathcal{L}^{-1}\left\{\frac{\omega}{(s+\gamma)^2+\omega^2}\right\}$$

Now using the shift theorem we are actually looking at terms of the form $F(s-(-\gamma))$ so we must multiply each term by $e^{-\gamma t}$:

Finally, it is left as an exercise (Tech Maths 1) to show that this can be rewritten as:

$$x(t) = ae^{-\gamma t}\cos(\omega t - \alpha). \tag{3.9}$$

We see that the solution represents an oscillation with decreasing amplitude $ae^{-\gamma t}$, and angular frequency ω. Note that ω is always less than the frequency of the undamped oscillator, ω_0.

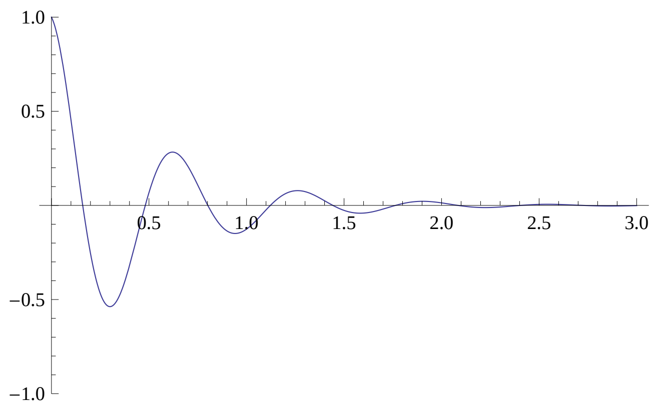

Figure 3.5: In small damping, the door closes too quickly and we have oscillations.

Critical Damping: $\gamma = \omega_0$

In this case the roots are repeated:

so that

$$X(s) = \frac{2s + \gamma}{(s + \gamma)^2}$$

Now we can certainly handle the inverse transforms of $1/(s+\gamma)$ — and $\gamma/(s+\gamma)^2$ looks like $1/s^2$ except with a shift. We'll do them separately. First up:

Secondly, using linearity,

$$\gamma \mathcal{L}^{-1}\left\{\frac{1}{(s+\gamma)^2}\right\}.$$

Now this is the same as $1/s^2$ except there has been a shift $s \to s - (-\gamma)$, so we'll have to multiply the inverse transform of $1/s^2$ by $e^{-\gamma t}$. Now using our formula:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{(n-1)!} \tag{3.10}$$

we have:

Hence adding together, and not forgetting the a at the beginning:

$$= ae^{-\gamma t} + a\gamma e^{-\gamma t} t = (a + a\gamma t)e^{-\gamma t}.$$

Critical damping is the ideal as there are no oscillations and the door closes quickly:

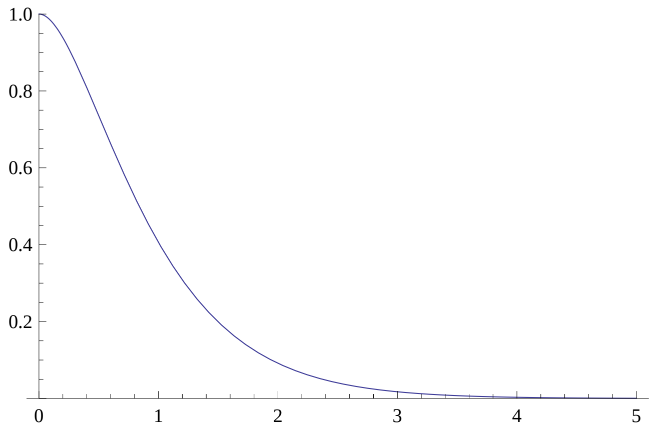

Figure 3.6: In critical damping, the door closes quickly but without oscillations.

Summary

The equation of a damped harmonic oscillator is given by:

Define parameters

$$\gamma = \frac{\lambda}{2m}.$$
$$\omega_0 = \sqrt{\frac{k}{m}}.$$

There are three types of behaviour:

$\gamma < \omega_0$ Small Damping categorised by the door swinging[1].

$\gamma > \omega_0$ Large Damping categorised by the door closing too slowly[2].

$\gamma = \omega_0$ Critical Damping categorised by the door closing quickly but without oscillations[3].

[1] too hot
[2] too cold
[3] just right

An Exam Style Question

The differential equation governing the displacement $x(t)$ of a door closer is given by:

$$x''(t) + 8x'(t) + 41x(t) = 0,$$

with the initial conditions $x(0) = 2$ and $x'(0) = 0$.

(i) Solve the differential equation using *Laplace transforms*.

(ii) Is the door closer well designed? Justify your answer by making a suitable analysis of the original differential equation, or otherwise.

Exercises

Solve the following differential equations:

$$(i)\ \frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 29y = 0\,,\ y(0) = 1, y'(0) = 0.$$

$$(ii)\ y'' - 4y' + y = 0\,,\ y(0) = 1,\ y'(0) = 0.$$

$$(iii)\ y'' + 2y' + 4y = 0\,,\ y(0) = 3,\ y'(0) = 6.$$

$$(iv)\ y'' + 4y = 0\,,\ y(0) = -1,\ y'(0) = -8.$$

$$(v)\ y'' - 16y = 0\,,\ y(0) = 5,\ y'(0) = e.$$