MS2001 Summer 2011 Solutions

Question 1

(a) Let $x \in \mathbb{R}$ be a real number such that $x > 1$ and let $n \in \mathbb{N}$ be a natural number such that $n \geq 2$. Using the properties of the inequality relation, or induction, prove carefully that

$$x^n > x.$$

(b) Use the Calculus of Limits to evaluate:

$$\lim_{x \to 1} \frac{\sqrt{x+8} - 3}{1 - x}$$

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by:

$$f(x) = (\sqrt{x^2+1} + \sin x)^{50}$$

Consider the statement:

The function $f(x)$ is differentiable on $\mathbb{R}$.

Is this statement true or false? Give reasons for you answer. Please find $f'(x)$ where f is differentiable.

(d) A cylinder is to be made such that the sum of its radius r, and its height, h, is 6 cm. Find, in terms of π, the maximum possible volume of such a cylindar.

Solution

(a) **Direct Method:** By assumption, $x > 1$. We can multiply both sides by x as $x > 0$ (*If $a, b, c \in \mathbb{R}$ and $c > 0$ then $a > b$ implies $ca > cb$*). That is we have

$$x^2 > x.$$

Now was can multiply the LHS by x and the RHS by 1 (*If $a, b, c, d \in \mathbb{R}$, $a > b > 0$ and $c > d > 0$, then $ac > bd$*). Hence

$$\begin{aligned} x^3 &> x \\ \Rightarrow \quad \underbrace{x \cdots x}_{\text{repeat until } n \text{ `}x\text{'s.}} &> x \\ \Rightarrow x^n &> x \end{aligned}$$

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Inductive Method: Let $P(n)$ be the proposition that if $x > 1$ and $n \in \mathbb{N}$, that $x^n > x$.
Consider $P(2)$. Is $x^2 > x$? Well $x^2 - x = x(x-1)$. Clearly $x > 0$ and $x > 1$ implies $x - 1 > 0$. Hence $x(x-1)$ is the product of positive real numbers so is positive:

$$x^2 - x > 0,$$
$$\Rightarrow x^2 > x;$$

that is $P(2)$ is true[1].
Assume that $P(k)$ is true:

$$x^k > x.$$

Consider $P(k+1)$. Is $x^{k+1} > x$? Consider $x^{k+1} - x = x(x^k - 1)$. Once again $x > 0$ and by the inductive hypothesis, $x^k > x > 1$ and hence $x^k > 1$. That means $x^k - 1 > 0$ and thus $x^{k+1} - x$ is a product of positive terms and hence positive. That is

$$x^{k+1} - x > 0,$$
$$\Rightarrow x^{k+1} > x.$$

Hence by the Axiom of Induction the statement $P(n)$ is true for all $n \in \mathbb{N}$ •

(b) Firstly plugging in $x = 1$ results in $0/0$. Mutliplying by the conjugate of the numerator:

$$\begin{aligned} \frac{\sqrt{x+8}-3}{1-x} &= \frac{\sqrt{x+8}-3}{1-x} \times \left(\frac{\sqrt{x+8}+3}{\sqrt{x+8}+3}\right), \\ &= \frac{(x+8)-9}{(1-x)(\sqrt{x+8}+3)} = -\frac{(x-1)}{(x-1)(\sqrt{x+8}+3)}, \\ &= -\frac{1}{\sqrt{x+8}+3}, \end{aligned}$$

if $x \neq 1$. Hence

$$\lim_{x\to 1} \frac{\sqrt{x+8}-3}{1-x} = \lim_{x\to 1} \left(-\frac{1}{\sqrt{x+8}+3}\right) = -\frac{1}{6}.$$

(c) This statement is true. $\sin x$ is differentiable[2]. $\sqrt{x}$ is differentiable for $x > 0$. $x^2+1 > 0$ for all x as $x^2 \geq 0 \Rightarrow x^2 > -1 \Rightarrow x^2 + 1 > 0$. Moreover $x^2 + 1$ is differentiable as it is a polynomial. By the Chain Rule $\sqrt{x^2+1}$ is differentiable. By the Sum Rule $\sqrt{x^2+1} + \sin x$ is differentiable. x^{50} is differentiable as x^{50} is a polynomial. By the Chain Rule $f(x)$ is differentiable.

$$\begin{aligned} f'(x) &= 50(\sqrt{x^2+1} + \sin x)^{49} \left[\frac{d}{dx}[(x^2+1)^{1/2} + \sin x]\right] \\ &= 50(\sqrt{x^2+1} + \sin x)^{49} \left[\frac{1}{2}(x^2+1)^{-1/2}.(2x) + \cos x\right] \\ &= 50(\sqrt{x^2+1} + \sin x)^{49} \left[\frac{x}{\sqrt{x^2+1}} + \cos x\right] \end{aligned}$$

[1]Less than three people proved the base case $P(2)$. You can't just say "$x^2 > x$ true" — you must prove it. This is one of many ways of proving it.

[2]if you say differentiable it means *differentiable everywhere*

(d) The volume of a cylinder is given by

$$V(r,h) = \pi r^2 h. \tag{1}$$

From the question we a know that $r+h=6$ that is $h=6-r$ so that we can write the volume as a function of a r alone:

$$V(r) = \pi r^2(6-r) = 6\pi r^2 - \pi r^3. \tag{2}$$

As this function is defined on the closed interval $[0,6]$, we can analyse this function using the Closed Interval Method[3]. This theroem states that the absolute extrema of a continuous function are found at the critical points. The critical points are the endpoints, the stationary points and where the function is not differentiable. Clearly $r=0$ and $r=6$ are critical points. Next we find points where the derivative equals zero:

$$\frac{dV}{dr} = 12\pi r - 3\pi r^2 \stackrel{?}{=} 0;$$
$$\Rightarrow 3\pi r(4-r) = 0,$$

That is $r=0$ or $r=4$. As $V(r)$ is a polynomial it is differentiable everywhere so the critical points are $r=0,4,6$.

$$V(0) = \pi(0)^2(6) = 0\pi\,\text{cm}^3$$
$$V(4) = \pi(4)^2(2) = 32\pi\,\text{cm}^3$$
$$V(6) = \pi(6)^2(0) = 0\pi\,\text{cm}^3$$

Hence the maximum possible volume is $32\pi\,\text{cm}^3$.

Question 2

(a) Using the Closed Interval Method or otherwise, find a positive upper bound $M \in \mathbb{R}$ such that,

$$|x^2 - 7x + 4| < M.$$

for $x \in [2,4]$.

(b) Hence use the $\varepsilon - \delta$ definition of a limit to prove that:

$$\lim_{x\to 3}(x^3 - 10x^2 + 25x - 6) = 6.$$

Solution

(a) **Closed Interval Method:** Let $f(x) := x^2 - 7x + 4$. As a polynomial, f is continuous and hence satisfies the hypothesis of the Closed Interval Method on the closed interval $[2,4]$. That is the absolute extrema of f occur at the critical points of f. The critical points are the endpoints, the points where $f' = 0$ and the points where f' is undefined.

[3]in fact we can use the First and Second Derivative Tests also if we're careful about the domain of $V(r)$ — namely $(0,6)$ in reality.

As a polynomial, f is differentiable so the only critical points are $x = 2, 4$ and where $f' = 0$.

$$\begin{aligned} f'(x) &= 2x - 7 \stackrel{?}{=} 0, \\ \Rightarrow 2x &= 7, \\ \Rightarrow x &= \frac{7}{2}. \end{aligned}$$

Now

$$\begin{aligned} f(2) &= 4 - 14 + 4 = -6, \\ f(4) &= 16 - 28 + 4 = -8 \\ f(7/2) &= \frac{49}{4} - \frac{49}{2} + 4 = -\frac{33}{4}. \end{aligned}$$

Hence we can say that $|x^2 - 7x + 4| \leq 33/4 < 9 =: M$, for all $x \in [2, 4]$.

Using Inequalities: Using the triangle inequality and the fact that $|xy| = |x||y|$:

$$\begin{aligned} |x^2 - 7x + 4| &\leq |x^2| + |-7x| + |4|, \\ &\leq |x|^2 + 7|x| + 4, \\ &\leq 16 + 28 + 4 = 48, \end{aligned}$$

Hence we can say that $|x^2 - 7x + 4| \leq 48 < 49 =: M$.

(b) Let $g(x) = x^3 - 10x^2 + 25x - 6$ and consider

$$\begin{aligned} |f(x) - 6| &= |(x^3 - 10x^2 + 25x - 6) - 6|, \\ &= |x^3 - 10x^2 + 25x - 12|. \end{aligned}$$

By inspection $g(3) = 0$ hence by the Factor Theorem $(x - 3)$ is a root of $g(x)$:

$$\begin{array}{r|rrrr} & x^2 & -7x & +4 & \\ \hline x-3 & x^3 & -10x^2 & +25x & -12 \\ & x^3 & -3x^2 & & \\ \hline & & -7x^2 & +25x & \\ & & -7x^2 & +21x & \\ \hline & & & 4x & -12 \\ & & & 4x & -12 \\ \hline & & & & 0 \end{array}.$$

Hence (using either $M = 9, 49$ or similar)

$$\begin{aligned} |g(x) - 6| &= |(x^2 - 7x + 4)(x - 3)|, \\ &\leq |x^2 - 7x + 4||x - 3|, \\ &< M|x - 3|. \end{aligned}$$

Suppose that $\varepsilon > 0$. Then if we choose $\delta := \varepsilon/M$ and $0 < |x - 3| < \varepsilon/M$:

$$|g(x) - 6| < M|x - 3| < M \cdot \frac{\varepsilon}{M} < \varepsilon.$$

i.e.

$$\lim_{x \to 3}(x^3 - 10x^2 + 25x - 6) = 6.$$

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Question 3

(a) Let $a \in \mathbb{R}$ and consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by:

$$f(x) = \begin{cases} |x - a| & \text{if } x < 0 \\ x - a & \text{if } x \geq 0 \end{cases}.$$

For what value(s) of a is f continuous?
Suppose $a = 1$. Is f differentiable at $x = 0$? Justify your answer.

(b) The *Folium of Descartes* is a plane curve with the equation

$$x^3 + y^3 - 3xy = 0$$

It passes through the origin, has a single loop, and has two branches that are asymptotic to the straight line $y = -x - a$. The Folium of Descartes has a horizontal tangent at the origin. Find the x-coordinate of the other point where it has a horizontal tangent.

Solution

(a) Away from 0, f is continuous. For $x < 0$, $f(x)$ is the composition of the continuous functions $|\cdot|$ and $x - a$; and for $x > 0$, $f(x)$ is a polynomial. Hence we examine the limit as $x \to 0$.

$$\begin{aligned} \lim_{x \to 0^-} f(x) &= \lim_{x \to 0^-} |x - a| = |-a| = |a|, \\ \lim_{x \to 0^+} f(x) &= \lim_{x \to 0^+} (x - a) = -a. \end{aligned}$$

So for f to be continuous we require that

$$|a| = -a. \tag{3}$$

The only real numbers that satisfy these conditions are zero and the negative numbers. Hence f is continuous for $a \in (-\infty, 0]$.

No it is not. If $a = 1$ then f is not continuous at 0. Not continuous implies not differentiable.

(b) For a horizontal tangent we must have

$$\frac{dy}{dx} = 0. \tag{4}$$

Differentiating across with respect to x:

$$\begin{aligned} \frac{d}{dx}(x^3 + [y(x)]^3 - 3x[y(x)]) &= \frac{d}{dx}0, \\ \Rightarrow 3x^2 + 3y^2\frac{dy}{dx} - 3x\frac{dy}{dx} - 3y &= 0, \\ \Rightarrow \frac{dy}{dx}(3y^2 - 3x) &= 3y - 3x^2, \\ \Rightarrow \frac{dy}{dx} &= \frac{3y - 3x^2}{3y^2 - 3x}. \end{aligned}$$

We know that $a/b = 0 \Rightarrow a = 0$. Hence we require

$$3y - 3x^2 = 0 \Rightarrow y = x^2.$$

To see which points on the curve satisfy this condition, substitute into the equation of the curve:

$$\begin{aligned} x^3 + (x^2)^3 - 3x(x^2) &= 0, \\ \Rightarrow x^3 + x^6 - 3x^3 &= 0, \\ \Rightarrow x^6 - 2x^3 &= 0, \\ \Rightarrow x^3(x^3 - 2) &= 0. \end{aligned}$$

Hence we either have $x^3 = 0$ or $x^3 - 2 = 0$. The first of these refers to the origin hence we require:

$$\begin{aligned} x^3 - 2 &= 0, \\ \Rightarrow x^3 &= 2, \\ \Rightarrow x &= \sqrt[3]{2}. \end{aligned}$$

Question 4

(a) State Rolle's Theorem.

(b) Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous and differentiable. Prove that if there exist distinct points $x_1, x_2 \in \mathbb{R}$ with

$$f(x_1) = g(x_1) \text{ , and } f(x_2) = g(x_2),$$

then there exists a point $c \in (x_1, x_2)$ such that the tangent line to $f(x)$ at c is parallel to the tangent line to $g(x)$ at c.
[HINT: Consider the function $h(x) := f(x) - g(x)$.]

(c) For $a, b, c \in \mathbb{R}$ and $a \neq 0$, the function

$$p(x) = ax^2 + bx + c$$

is continuous and differentiable and so satisfies the hypothesis of the the Mean Value Theorem on any (bounded) closed interval. Verify the Mean Value Theorem for $p(x)$ on the closed interval $[0, 1]$.

Solution

(a) If[4] $f : [a, b] \to \mathbb{R}$ **is continuous on** $[a, b]$**, differentiable on** (a, b) **and** $f(a) = f(b)$, then *there exists a* $c \in (a, b)$ *such that* $f'(c) = 0$.

[4] a lot of us mixed up the **hypothesis** and the *conclusion*. In general, a theorem will read "If some object **satisfies these conditions**... then the object *has these properties*.""

(b) Following the hint, let $h(x) := f(x) - g(x)$. Now as a sum of continuous and differentiable functions, h is continuous and differentiable. Now

$$\begin{aligned} h(x_1) &= f(x_1) - g(x_1) = 0, \\ h(x_2) &= f(x_2) - g(x_2) = 0, \\ \Rightarrow h(x_1) &= h(x_2). \end{aligned}$$

Hence h satisfies the hypothesis of Rolle's Theorem on the interval $[x_1, x_2]$. That is there exists a $c \in (x_1, x_2)$ such that:

$$\begin{aligned} h'(c) &= 0, \\ \Rightarrow f'(c) - g'(c) &= 0, \\ \Rightarrow f'(c) &= g'(c). \end{aligned}$$

i.e. the tangent line to $f(x)$ at c at c is parallel to the tangent line to $g(x)$ at c •

(c) The Mean Value Theorem implies that there exists a point $c \in (0, 1)$ such that

$$p'(c) = \frac{p(1) - p(a)}{1 - 0} = p(1) - p(0), \tag{5}$$

i.e. a point where the slope is equal to the average slope across $[0, 1]$. Now

$$\begin{aligned} p(1) - p(0) &= a + b + c - (a(0)^2 + b(0) + c), \\ &= a + b. \end{aligned}$$

Also

$$p'(x) = 2ax + b. \tag{6}$$

Hence we are looking for a solution to the equation

$$\begin{aligned} p'(x) &= p(1) - p(0), \\ \Rightarrow 2ax + b &= a + b \\ \Rightarrow x &= \frac{a}{2a} = \frac{1}{2}. \end{aligned}$$

i.e. we have verified the Mean Value Theorem for the function $p(x)$ •

Question 5

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by:

$$f(x) = \frac{x^2 + x + 1}{x + 1}$$

For what values of x is this function defined? Describe the 'horizontal' and vertical asymptotes of $f(x)$. Using the second derivative test, find and classify all local maxima and minima. By using the method of split points, find the intervals where $f(x)$ is concave up and concave down. Find the roots of $f(x)$ if any. Find where $f(x)$ cuts the y-axis.
Use **all** of this information to sketch the graph of $y = f(x)$.

Solution

- **Domain:** The function is defined for all $x \in \mathbb{R}$ such that $x + 1 \neq 0 \Leftrightarrow x \neq -1$.

- **Horizontal Asymptotes:** The 'horizontal' asymptote is got by examining the behaviour as $x \to \infty$:
$$\lim_{x\to\infty} \frac{x^2 + x + 1}{x + 1} \approx \frac{x^2}{x} = x. \tag{7}$$

- **Vertical Asymptotes:** The vertical asymptotes of $f(x)$ occur when $f(x) \to \infty$. It is necessary that the denominator tends to 0: $x + 1 \to 0 \Rightarrow x \to -1$. However, this is not a sufficient condition[5]. Hence evaluate the limit as $x \to -1$:
$$\begin{aligned} \lim_{x\to -1} f(x) &= \left(\lim_{x\to -1} x^2 + x + 1\right)\left(\lim_{x\to -1} \frac{1}{x+1}\right), \\ &= 1 \cdot \infty = \infty. \end{aligned}$$
i.e. there is a vertical asymptote at $x = -1$.

- **Maxima/ Minima:** To use the second derivative test to find maxima and minima first we find the stationary points where $f'(x) = 0$ — and then test whether they are maxima or minima by testing the second derivative ($y'' < 0$ for maxima; $y'' > 0$ for minima.). Using the quotient rule:
$$\begin{aligned} f'(x) &= \frac{(x+1)(2x+1) - (x^2 + x + 1)(1)}{(x+1)^2}, \\ &= \frac{2x^2 + x + 2x + 1 - x^2 - x - 1}{(x+1)^2}, \\ &= \frac{x^2 + 2x}{(x+1)^2} = \frac{x(x+2)}{(x+1)^2}. \end{aligned}$$
Now $f'(x)$ is a fraction so only zero when the top is zero, morryah $x(x+2) = 0 \Rightarrow x = 0$ or $x = -2$. Now using a quotient rule again:
$$f''(x) = \frac{(x+1)^{\cancel{2}}(2x+2) - 2\cancel{(x+1)}(x^2 + 2x)}{(x+1)^{\overset{3}{\cancel{4}}}}.$$
As the function is not defined at $x = -1 \Rightarrow x + 1 = 0$, we can divide above and below by $(x + 1)$:
$$f''(x) = \frac{\cancel{2x^2} + \cancel{2x + 2x} + 2 - \cancel{2x^2} - \cancel{4x}}{(x+1)^3} = \frac{2}{(x+1)^3}.$$
Now $f''(0) = 2 > 0$ so there is a local minimum at $x = 0$ (with y-coordinate $f(0) = 1$); and $f''(-2) = -2 < 0$ so there is a local maximum at $x = -2$ (with y-coordinate $f(-2) = -3$.)

- **Concavity:** A function is concave up for $f''(x) > 0$ and concave down for $f''(x) < 0$. The concavity can only change, therefore, at split points when $f'' = 0$ or undefined. $f''(x) \neq 0$ as $2 \neq 0$ but undefined when $x = -1$. Hence set up the split point diagram:

[5] nearly all students got $x = -1$ is a vertical asymptote but never checked the limit as $x \to -1$. This is vital. For example, $g(x) = (x^2 - 9)/(x - 3)$ seems to have a vertical asymptote at $x \to +3$ but if we in fact evaluate the limit we will find that $g(x)$ doesn't grow infinitely big but instead tends to 6; that is $x = 3$ is *not* a vertical asymptote.

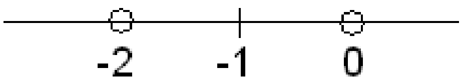

Figure 1: A function's concavity can only change at split points. In this example, to determine the concavity on $(-\infty, -1)$ and $(-1, \infty)$ we choose *test points* in these intervals. Any will do — here we choose $x = -2, 0$.

$f''(-2) < 0$ implies that f is concave down on $(-\infty, -1)$ and $f''(0) > 0$ implies that f is concave up on $(-1, \infty)$.

- **Roots:**
$$f(x) = \frac{x^2 + x + 1}{x + 1} = 0 \Leftrightarrow x^2 + x + 1 = 0.$$
Now
$$x_{\pm} = \frac{-1 \pm \sqrt{1 - 4(1)(1)}}{2} = \frac{-1 \pm \sqrt{-3}}{2}.$$
Hence there are *no* real roots.

- **y-Intercept:** The graph cuts the y-axis when $x = 0$; that is at $f(0) = 1$.

Hence we produce the plot:

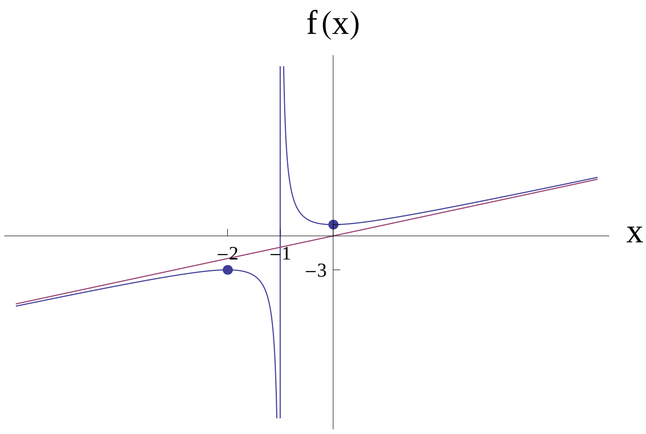

Figure 2: Notice that we include the vertical asymptote $x = -1$ and the 'horizontal' asymptote $y = x$ — and more importantly that the graph of $f(x)$ behaves like them when it gets far from the origin. We show the maxima at $(-2, -3)$ and the minima at $(0, 1)$. We have the graph concave down for $x < -1$ and concave up for $x > -1$; as required. Finally we exhibit that $f(x)$ has no roots.