MS 2001: Additional but Harder Exercises for Definitions I

October 21, 2011

You can test your understanding of the definitions by answering these questions — i.e. you'll only be able to answer these questions if you really, really know your definitions.

Questions:

1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ and define a function $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = 1/f(x)$. Prove that g has no *roots*.
2. Let $n \in \mathbb{N}$ and $f : \mathbb{R} \to \mathbb{R}$. Prove that k is a *root* of $[f(x)]^n$ if and only if k is a *root* of f.
3. Let $f : \mathbb{R} \to \mathbb{R}$. Prove that if k is a *root* of f, then 0 is a *root* of $f(x + k)$.
4. Prove that the product of two *even* functions is an *even* function.
5. Prove that the composition of two *even* functions is an *even* function.
6. Use the unit circle to prove that the cosine function is an *even* function.
7. Prove that the product of two *odd* functions is an *even* function.
8. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is an *odd* function defined on the entire real line. Prove that f has a root.
9. Using the fact that sine is an *odd* function, prove that the tangent function is *odd*.
10. Prove that $g : \mathbb{R} \to \mathbb{R}$, defined by $g(x) = -x$ is *decreasing*.
11. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is both *increasing* and *decreasing* for all $x \in \mathbb{R}$.
12. Give an example of a function that is *strictly increasing* for all $x \in \mathbb{R}$ but has no roots.
13. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is *strictly increasing* on a non-empty closed interval $[a, b] \subset \mathbb{R}$. Show that if $a \leq x_1 < x_2 \leq b$, then the (secant) line joining $(x_1, f(x_1))$ to $(x_2, f(x_2))$ has positive slope.
14. Suppose that $g : \mathbb{R} \to \mathbb{R}$ is *strictly decreasing* and has a root at $a \in \mathbb{R}$. Prove that g has no other roots.

15. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a positive *increasing* function and that $g : \mathbb{R} \to \mathbb{R}$ is a positive *decreasing* function then $q = f/g$, $q(x) = f(x)/g(x)$ is an *increasing* function.

16. Use the formula for the roots of a *quadratic* function $p(x) = ax^2 + bx + c$ to find an expression for the sum of the roots of p; and the product of the roots of p.

17. Suppose that $q(x) = ax^2 + bx + c$ is a *quadratic* function with real roots α and β. Use the fact that quadratic functions are symmetric about the line $x = -b/2a$ — and that their maxima/ minima are found there to find an expression for $\alpha + \beta$.

18. Let $r(x) = ax^2 + bx + c$ be a *quadratic* function. Use the factor theorem to find an expression for the sum of the roots of r; and the product of the roots of r.

19. Prove that all *polynomials* of odd degree have at least one root.

20. Prove the factor theorem for the *polynomial* $c(x) = ax^3 + bx^2 + cx + d$.

21. Give an example of degree 4 *polynomials* p and q such that $p + q$ is a *polynomial* of degree 3.

22. Suppose that p and q are polynomials and let r be the *rational* function defined by $r(x) = p(x)/q(x)$. Prove that if k is a root of r then k is a root of p. By finding a counterexample, show that the converse does not hold.

23. Suppose that p and q are polynomials and let r be the *rational* function $r(x) = p(x)/q(x)$. If $q(a) = 0$, then $r(a)$ is not defined at a and hence discontinuous at a. Find examples of polynomials p and q such that:

    (a) r is continuous.

    (b) r is not continuous — but is bounded (there exists a positive number $M > 0$ such that $|r(x)| < M$ for all $x \in \mathbb{R}$).

24. Prove that $|x^2 + 1| = x^2 + 1$ for all $x \in \mathbb{R}$.

25. Prove that the *absolute value* function is even.

26. Suppose that $f : \mathbb{R} \to \mathbb{R}$ has the property that $f(x) < 0$ for all $x \in \mathbb{R}$. Describe the relationship between the graph of $f(x)$ and the graph of $|f(x)|$.

27. Let $k \in \mathbb{R}$ be a constant and $a \in \mathbb{R}$. Use the $\varepsilon - \delta$ definition of a *limit* to prove that

$$\lim_{x \to a} k = k, \text{ and } \lim_{x \to a} x = a.$$

28. Suppose that $f : \mathbb{R} \to \mathbb{R}$ and

$$\lim_{x \to 1} f(x) = 0.$$

    Does this imply that 1 is a root of f?

29. Show that there are two values of $a \in \mathbb{R}$ such that the *left- and right-handed limits* of $f : \mathbb{R} \to \mathbb{R}$ at $x = 1$ agree where:

$$f(x) = \begin{cases} (ax)^2 & \text{if } x < 1 \\ ax + 6 & \text{if } x \geq 1 \end{cases}$$

    Produce a rough sketch of f in each case.

30. Assuming we know what 2^x *is*, we can define a function $f : \mathbb{R} \to \mathbb{R}$ by:

$$f(x) = \frac{1}{1 + 2^{-1/x}}$$

Sketch an argument that suggests that the *left- and right-hand limits of* $f(x)$ at 0 are, respectively, 0 and 1.

31. Construct a function $f : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{x \to 0^-} f(x) = +\infty \text{ , and } \lim_{x \to 0^+} f(x) = 1.$$

32. Suppose that $f : \mathbb{R} \to \mathbb{R}$ and that

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that if } 0 < |x| < \delta \Rightarrow |f(x)| < \varepsilon.$$

Does this imply that

$$\lim_{x \to 0} f(x) = 0.$$

33. For all real numbers x, $y \in \mathbb{R}$ with $x \neq y$, there exists a fraction between x and y — i.e. a $q \in (x, y)$. Consider the function

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Prove that f is not *continuous* at *any* point.

34. Consider the function

$$g(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Prove that g is *continuous* at 0.

35. Suppose that functions $f_1 : \mathbb{R} \to \mathbb{R}$ and $f_2 : \mathbb{R} \to \mathbb{R}$ have the property that, for $i = 1, 2$

$$f_i(x) = f_i(y) \Rightarrow x = y.$$

Prove that $f = f_1 \circ f_2$ has this property also.

36. Find a set $A \subseteq \mathbb{R}$, and functions $f : A \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that $(g \circ f)(x) = x$ for all $x \in A$ but that there exists a $y \in \mathbb{R}$ such that $(f \circ g)(y) \neq y$.

37. Suppose that $f : [0, 1] \to [0, 1]$ is continuous and strictly increasing on $[0, 1]$ such that $f(0) = 0$ and $f(1) = 1$. Suppose further that $g : [0, 1] \to [0, 1]$ is a function such that

$$(g \circ f)(x) = x$$

for all $x \in [0, 1]$. How is the graph of g related to the graph of f.