MS 2001: Test 1 A and 1 B Q. 3

Answer all questions. Marks may be lost if necessary work is not clearly shown.

The more justification the better but not necessarily required for full marks — J.P.

Question 1

(a) Find the solution set of the following inequality:

$$\left|\frac{x+1}{3x-2}\right| < 5\,,\ \ x \neq 2/3.$$

(b) Evaluate the following using the Calculus of Limits.

$$\lim_{t\to -2}\frac{t^3+8}{t+2}.$$

Solution

(a) First we split the absolute value of the fraction:

$$\begin{aligned}\frac{|x+1|}{|3x-2|} &< 5,\\ \Rightarrow |x+1| &< 5|3x-2|,\\ \Rightarrow |x+1|^2 &< [5|3x-2|]^2,\\ \Rightarrow (15x-10)^2-(x+1)^2 &> 0,\\ \Rightarrow [(15x-10)+(x+1)][(15x-10)-(x+1)] &> 0,\\ \Rightarrow (16x-9)(14x-11) &> 0,\end{aligned}$$

where we used the fact that we could multiply across by $|3x-2|>0$, the fact that $|x|^2=x^2$ and the difference of two squares. Now $q(x)=(16x-9)(14x-11)$ is a $+ax^2$ or $\bigcup$ quadratic functions so positive *outside* the roots. The roots of q are $x=9/16$ and $11/14$ $(11/14>9/16)$ hence the solution set is given by

$$(-\infty,9/16)\cup(11/14,\infty).$$

(b) Firstly we check the value of the function at $t=-2$:

$$\frac{(-2)^3+8}{-2+2}=\frac{0}{0},$$

undefined. Now -2 is a root of t^3+8 so by the Factor Theorem $t+2$ is a factor:

$$\begin{array}{r|llll} & t^2 & -2t & +4 & \\ \hline t+2 & t^3 & 0t^2 & 0t & +8\\ & t^3 & +2t^2 & & \\ \hline & & -2t^2 & +0t & \\ & & -2t^2 & +4t & \\ \hline & & & 4t & +8\\ & & & 4t & +8\\ \hline & & & & 0\end{array}\ .$$

Hence we may write:

$$\frac{t^3+8}{t+2} = \frac{(t+2)(t^2-2t+4)}{t+2},$$
$$= t^2 - 2t + 4,$$

as we can cancel the $t+2$s when $t \neq -2$ — which is the case here as we are looking at the limit as $t \to -2$ — which is not concerned with $t = -2$. Therefore we have

$$\lim_{t\to -2} \frac{t^3+8}{t+2} = \lim_{t\to -2}(t^2 - 2t + 4)$$
$$= (-2)^2 - 2(-2) + 4 = 12.$$

Question 2

Define what it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be continuous at a point $a \in \mathbb{R}$.

For a constant $k \in \mathbb{R}$, consider the function $g : \mathbb{R} \to \mathbb{R}$ defined by:

$$g(x) := \begin{cases} x^2 & \text{for } x > 0 \\ x+1 & \text{for } -1 < x \leq 0 \\ 3x+k & \text{for } x \leq -1 \end{cases}$$

Is g continuous at 0? Justify your answer. For what value(s) of $k \in \mathbb{R}$ is g continuous at $x = -1$? Justify your answer.

Solution

A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at a point $a \in \mathbb{R}$ if

$$\lim_{x\to a} f(x) = f(a).$$

A rough sketch of the function suggests that the function is *not* continuous at 0. We can show this by showing that the left- and right-hand limits at 0 disagree.

$$\lim_{x\to 0^-} g(x) = \lim_{x\to 0}(x+1) = 1,$$
$$\lim_{x\to 0^+} g(x) = \lim_{x\to 0}(x^2) = 0.$$

As the left- and right-hand limits disagree, the limit at 0 does not exist so the function is not continuous at 0.

To find a k such that g is continuous we examine the left- and right-hand limits at -1:

$$\lim_{x\to -1^-} g(x) = \lim_{x\to -1}(3x+k) = -3+k,$$
$$\lim_{x\to -1^+} = \lim_{x\to -1}(x+1) = 0.$$

For g to be continuous at -1, it is necessary that these agree:

$$-3 + k = 0,$$
$$\Rightarrow k = 3.$$

Now the one-sided limits agree so the limit at -1 is zero — and equal to the value of the function there as required. Ans: $k = 3$.

Question 3

1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is an *odd* function. Which of the following statements are true? (Circle the correct statement)

   (a) The graph of $f(x)$ is symmetric about the line $x = 0$.
   A symmetry about the line $x = 0$ is equivalent to a symmetry about the y-axis yielding $f(-x) = f(x)$ — this is an even function. A counterexample is given by $f(x) = x$. An odd function has symmetry about the line $y = -x$.

   (b) f has no real roots.
   $f(x) = x$ is odd but has a real root.

   (c) f is a polynomial of the form

   $$f(x) = a_n x^n + \cdots + a_5 x^5 + a_3 x^3 + a_1 x,$$

   where all the powers of x are odd.
   $f(x) = \sin x$ is odd and doesn't have this form. It does have a power series *of odd degree powers but power series are not polynomials as we've defined them.*

   (d) For all $x \in \mathbb{R}$, $f(-x) = -f(x)$. ✓

2. Suppose that $r : \mathbb{R} \to \mathbb{R}$ is a *rational* function. Which of the following statements are true? (Circle the correct statement)

   (a) $r(x) = p(x)/q(x)$ for some polynomials $p(x)$, $q(x)$. ✓

   (b) $r : \mathbb{Q} \to \mathbb{Q}$.
   Define a rational function by

   $$r(x) = \frac{x + (\sqrt{2} - 1)}{x}.$$

   Now r is a rational function such that $r(1) = \sqrt{2}$ — hence it doesn't send fractions to fractions necessarily.

   (c) r has no real roots.
   Define a rational function by

   $$r(x) = \frac{x - 1}{x}.$$

   Now r is a rational function with a root at $x = 1$.

   (d) For some $n \in \mathbb{N}$, and $a_n, a_{n-1}, \ldots, a_1, a_0 \in \mathbb{R}$, with $a_n \neq 0$.

   $$r(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

   This is the definition of a polynomial. In particular, the rational function

   $$r(x) = \frac{x^2 + 1}{x}$$

   doesn't have this form.

3. Consider the *left-hand limit* of a function $f : \mathbb{R} \to \mathbb{R}$? Which of the following statements are true? (Circle the correct statement)

(a) If the left-hand limit at $a \in \mathbb{R}$,

$$\lim_{x \to a^-} f(x) = L,$$

then

$$\lim_{x \to a^+} f(x) = L.$$

Define $f : \mathbb{R} \to \mathbb{R}$

$$f(x) := \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

Then the left-hand limit at 0 *is* 0 *but the right-hand limit at* 0 *is not* 0 *also — it's* 1.

(b) If the left-hand limit at $a \in \mathbb{R}$ exists, and furthermore we have that

$$\lim_{x \to a^-} f(x) = f(a),$$

then the limit

$$\lim_{x \to a} f(x)$$

exists also.
The function defined in part (a) has this property — the left-hand limit at 0 *is equal to the value of the function at* 0 *— but the limit at* 0 *does not exist.*

(c) If

$$\lim_{x \to a^-} f(x) = 10,$$

then there exists a positive real number d such that whenever $0 < a - x < d$, then $|f(x) - 10| < 1/100$.
✓ *Take the definition of a left-handed limit* $\lim_{x \to a^-} f(x) = L$. *Given* any $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that whenever* $0 < a - x < \delta$, *we have* $|f(x) - L| < \varepsilon$. *Now in this case we are told* $\lim_{x \to a^-} = 10$. *It is true that* $\varepsilon = 1/100 > 0$ *and our* $L = 10$. *So given this* $\varepsilon = 1/100$, *we know that there exists a* $\delta = d > 0$ *such that whenever* $0 < a - x < d$, *we have* $|f(x) - 10| < 1/100$.

(d) If the left-hand limit at $a \in \mathbb{R}$ exists then f is continuous at a.
Again the function defined in part (a) is a counterexample.

Question 3: Test 1 B

1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is an *even* function. Which of the following statements are true? (Circle the correct statement)

   (a) f is a polynomial of the form

   $$f(x) = a_n x^n + \cdots + a_4 x^4 + a_2 x^2 + a_0,$$

   where all the powers of x are even.
   $f(x) = \cos x$ *is even but not of this form (although it has a power series of even powers of* x*).*

(b) For all $x \in \mathbb{R}$, $f(-x) = f(x)$. ✓

(c) The graph of $f(x)$ has a symmetry through the line $y = 0$.
No function has a symmetry through the line $y = 0$ as this would imply that $f(x)$ takes two values whenever f is non-zero: functions map uniquely. A counter-example is $f(x) = x^2$ — which doesn't have a symmetry through $y = 0$. Even functions have a symmetry in the line $x = 0$: the y-axis.

(d) f has real roots.
$f(x) = x^2 + 1$ is even but has no real roots.

2. Suppose that $h : \mathbb{R} \to \mathbb{R}$ is the *composition* $g \circ f$ of functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. Which of the following statements are true? (Circle the correct statement)

(a) If f and g are continuous, it does not necessarily imply that h is continuous.
Yes it does — this is Proposition 2.3.6

(b) If $f(x) < 0$ then $h(x) < 0$.
Let $f(x) = -1 < 0$ and $g(x) = x^2$. Now $h(x) = g(f(x)) = g(-1) = +1$ which is not negative.

(c) If k is a root of f, then k is a root of h also.
Let $f(x) = 0$ and $g(x) = 1$. Now any x is a root of f, but $h(x) = g(f(x)) = g(0) = 1$ — hence not any x is a root of h.

(d) For all $x \in \mathbb{R}$, $h(x) = g(f(x))$. ✓

3. Suppose that a function $f : \mathbb{R} \to \mathbb{R}$ is *continuous at a point* $a \in \mathbb{R}$? Which are the following statements are true? (Circle the correct statement)

(a) Then the function defined by $g(x) = 1/f(x)$ is continuous at a.
Let $f(x) = x$. Now f is continuous as a polynomial — in particular at $x = 0$ — but $g(x) = 1/x$ is not continuous at 0 as it is not defined there.

(b) There exists a positive real number d such that whenever $|x - a| < d$, then $|f(x) - f(a)| < 1/100$.
✓ *To be continuous at a you need $\lim_{x \to a} f(x) = f(a)$. This is equivalent to*

$$\forall \varepsilon > 0, \exists \delta > 0 : |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

Here $\varepsilon = 1/100$ is an $\varepsilon > 0$. Therefore there exists a $\delta = d > 0$ such that whenever $|x - a| < d$, we have that $|f(x) - f(a)| < 1/100$.

(c) a is a root of f.
$f(x) = 1$ is continuous as a line so in particular is continuous at 0 — but 0 is not a root of f.

(d) f is continuous at all points $x \in \mathbb{R}$.
The function defined in Test 1 A Q. 3(a) is continuous at $x = 1$ but is not continuous at $x = 0$.