Examples

1. Complete the square:

$$4x^2 + 8x + 28$$

Solution: Do what you want to do:

$$4x^2 + 8x + 28 = (px + q)^2 + r^2$$
$$= p^2x + 2pqx + q^2 + r^2$$

Now equate coefficients:

$$p^2 = 4 \Rightarrow p = 2$$
$$2pq = 4q = 8 \Rightarrow q = 2$$
$$q^2 + r^2 = 4 + r^2 = 28 \Rightarrow r^2 = 24 \Rightarrow r = \sqrt{24} = \sqrt{4(6)} = 2\sqrt{6}$$
$$4x^2 + 8x + 28 = (2x + 2)^2 + (\sqrt{24})^2$$

2. Evaluate

$$\int \sqrt{5 - 4x - x^2}\, dx,$$

given that

$$\int \sqrt{a^2 - u^2}\, du = \frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2} \arcsin \frac{u}{a} + C, \qquad \text{for } a > 0.$$

Solution: Here we want to make the integrand look like $\sqrt{a^2 - u^2}$ so we begin by completing the square thus:

$$5 - 4x - x^2 = r^2 - (x + p)^2 = r^2 - (x^2 + 2px + p^2)$$
$$= -x^2 - 2px + (r^2 - p^2)$$

Now equate coefficients and rewrite the integrand:

$$-2p = -4 \Rightarrow p = 2$$
$$r^2 - p^2 = r^2 - 4 = 5$$
$$\Rightarrow r^2 = 9$$
$$\Rightarrow r = 3$$
$$5 - 4x - x^2 = 3^2 - (x + 2)^2$$

Now make the substitution $u = x + 2$:

$du = dx$

$$I = \int \sqrt{3^2 - u^2}\, du$$

$$= \frac{u}{2}\sqrt{3^2-u^2} + \frac{9}{2}\arcsin\left(\frac{u}{3}\right) + C$$

$$= \frac{(x+2)}{2}\sqrt{3^2-(x+2)^2} + \frac{9}{2}\arcsin\left(\frac{(x+2)}{3}\right) + C$$

Summer 2011 Question 1(b)(iii)

Evaluate

$$\int \frac{dx}{\sqrt{15+2x-x^2}}.$$

Solution: There is no direct integration but there is a possible manipulation. There is a quadratic under a square-root in the denominator. Complete the square:

$$15 + 2x - x^2 = r^2 - (x+p)^2$$

$$= r^2 - (x^2 + 2px + p^2)$$

$$= -x^2 - 2px + r^2 - p^2$$

Now equate coefficients: $-2p = 2 \Rightarrow p = -1$

$r^2 - p^2 = r^2 - 1 = 15 \Rightarrow r^2 = 16 \Rightarrow r = 4.$

Now rewrite the integral and note that it looks like the integral of $\arcsin(x)$:

$$I = \int \frac{dx}{\sqrt{4^2-(x-1)^2}} \qquad \text{Let } u = x-1 \Rightarrow du = dx$$

$$= \int \frac{du}{\sqrt{4^2-u^2}} = \arcsin\left(\frac{u}{4}\right) + C = \arcsin\left(\frac{x-1}{4}\right) + C.$$

Exercises

Q. 1-3: Complete the Square. Q. 4-6 Evaluate the integral.

1. $x^2 + x + 1$.
2. $-x^2 + 5x - 2$.
3. $3x^2 - 2x + 4$.
4. $\displaystyle\int \frac{dx}{\sqrt{15 + 2x - x^2}}$ Ans: $\arcsin\left(\frac{x-1}{4}\right) + C$
5. $\displaystyle\int \frac{dx}{x^2 - x + 2}$ Ans: $\frac{2}{\sqrt{7}}\arctan\left(\frac{2x-1}{\sqrt{7}}\right) + C$
6. $\displaystyle\int \frac{dx}{3x^2 - 2x + 5}$ Ans: $\frac{1}{\sqrt{14}}\arctan\left(\frac{3x-1}{\sqrt{14}}\right) + C$

2.4 Trigonometric Integrals

For $\int \sin^m x \cos^n x\, dx$, if m or n is an odd positive integer, make the substitution

$$u = \textit{other trigonometric function}.$$

and possibly use the well-known trigonometric identity $\sin^2 x + \cos^2 x = 1$. This should work because say we have $\sin^{2n}(x)\cos^{2m+1}(x)$ as the integrand, this can also be written as

$$\underbrace{(\sin x)^{2n}}_{\text{function}} \underbrace{\cos x}_{\text{derivative}} \underbrace{\cos^{2m}(x)}_{\text{back-substitution with } \sin^2+\cos^2=1}.$$

Examples

Find the following integrals:

1. $\displaystyle\int \sin x \cos^2 x\, dx$.

   Solution: Here the power of $\sin x$ is odd so we let $u = \cos x$:

$$\frac{du}{dx} = -\sin x \Rightarrow dx = -\frac{du}{\sin x}$$

$$I = \int \sin x\, u^2 \left(-\frac{du}{\sin x}\right) = -\int u^2\, du$$

$$= -\frac{u^3}{3} + C$$

$$= -\frac{\cos^3 x}{3} + C$$

2. $\displaystyle\int \frac{\sin^3 x}{\cos^4 x}\,dx.$

Solution: First we have

$$\int \sin^3 x \cos^{-4} x\,dx,$$

so as sin is the odd power we let $u = \cos x$:

But $\sin^2 = 1 - \cos^2$
$= 1 - u^2$

$$\frac{du}{dx} = -\sin x \Rightarrow dx = -\frac{du}{\sin x}$$

$$I = \int \sin^{\cancel{3}2} x\, u^{-4}\left(-\frac{du}{\cancel{\sin x}}\right) = -\int \sin^2 x\, u^{-4}\,du$$

$$= -\int (1-u^2)u^{-4}\,du = -\int (u^{-4} - u^{-2})\,du = -\frac{u^{-3}}{-3} + \frac{u^{-1}}{-1} + C$$

$$= \frac{1}{3}\cos^3 x - \frac{1}{\cos x} + C.$$

Autumn 2011 Question 1(b)(ii)

Evaluate

$$\int \sin^3(x)\,dx.$$

Solution: There is no direct integration but a number of possible manipulations. The method here suggests we use $u = \cos x$ as the power of sin is odd:

$$\frac{du}{dx} = -\sin x \Rightarrow dx = -\frac{du}{\sin x}$$

$$I = \int \sin^{\cancel{3}2} x\left(-\frac{du}{\cancel{\sin x}}\right) = -\int \sin^2 x\,du$$

Note that we have a back-substitution via the relationship between $\sin^2 x$ and $\cos^2 x$:

$$= -\int (1 - \cos^2 x)\,du = -\int (1-u^2)\,du$$

$$= -u + \frac{u^3}{3} + C$$

$$= \frac{\cos^3 x}{3} - \cos x + C.$$

Exercises

1. Evaluate the following integrals:

   (a) $\int \sin^2 x \cos x \, dx$ Ans: $\frac{1}{3}\sin^3 x + C$

   (b) $\int \sin^3 x \cos x \, dx$ Ans: $\frac{1}{4}\sin^4 x + C$

   (c) $\int \sin^3 x \cos^4 x \, dx$ Ans: $\frac{1}{7}\cos^7 x - \frac{1}{5}\cos^5 x + C$

   (d) $\int \sin^3 x \, dx$ Ans: $\frac{1}{3}\cos^3 x - \cos x + C$

   (e) $\int \sin^5 x \, dx$ Ans: $-\frac{1}{5}\cos^5 x + \frac{2}{3}\cos^3 x - \cos x + C$

   (f) $\int \frac{\cos^3 x}{\sqrt{\sin x}} \, dx$ Ans: $2\sin^{1/2} x - \frac{2}{5}\sin^{5/2} x + C$

2. (a) Consider $\int \sin x \cos x \, dx$. Here both $\sin x$ and $\cos x$ have odd positive integer powers, so one has a choice of substitutions that work: $u = \cos x$ or $v = \sin x$. Evaluate this integral using both methods. The answers that you get will look different from each other at first sight, but show that in fact they are the same. Now use an identity from the tables to integrate.

   (b) Similarly to the previous exercise, evaluate the integral $\int \sin x \cos^3 x \, dx$ in two different ways then show that the answers you get are the same.

2.5 Integration by Parts

The chain rule for differentiation leads to the substitution method for integration. The product rule for differentiation leads to a new integration technique: *integration by parts*:

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\Rightarrow u\frac{dv}{dx} = \frac{d(uv)}{dx} - v\frac{du}{dx}$$

Integrating, we get the integration by parts formula:

$$\int u \, dv = uv - \int v \, du.$$

To use this formula to evaluate an integral, you must make a double substitution: choose u and dv so that $u\,dv$ equals the integrand, then apply the formula. (Clearly this is more complicated than the ordinary substitution method, which you would normally try first.)

It can help if one follows the LIATE guideline in choosing u. The reason for this is once you choose u, dv is determined and the LIATE rule is in order of increasing ease of integration

Examples

Evaluate each of the following:

1. $I = \int x \ln x \, dx$. You may assume the derivative of $\ln x$ is $1/x$:
   Solution: Choose $u = \ln x$ by LIATE and therefore $dv = x\,dx$:

$$\frac{du}{dx} = \frac{1}{x} \Rightarrow du = \frac{dx}{x} \, ; \int dv = v = \int x\,dx = \frac{x^2}{2}$$

$$I = \ln x\left(\frac{x^2}{2}\right) - \int \frac{x^{\cancel{2}}}{2} \frac{dx}{\cancel{x}} = \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x\,dx$$

$$= \frac{1}{2}\left(x^2 \ln x - \frac{x^2}{2}\right) + C$$

   Check this answer by differentiating it—observe that this needs the product rule, as one would expect. This example is typical: one does not need to insert "$+C$" at the early stage of going from dv to v, but it is needed when the final integral is evaluated.

2. $I = \int_0^{2\pi} x \sin 2x \, dx$.
   Solution: Ignore the limits of integration for the moment; we'll substitute them in the usual way after we have evaluated the indefinite integral $\int x \sin 2x \, dx$.

Let $u = x \Rightarrow du = dx$ | $\int dv = \int \sin 2x\,dx \Rightarrow v = -\frac{\cos(2x)}{2} + C$ [Ex].

$$I = x\left(-\frac{\cos 2x}{2}\right) - \int\left(-\frac{1}{2}\cos 2x\right)dx$$

$$= -\frac{x}{2}\cos 2x + \frac{1}{4}\sin 2x$$

Check this answer by differentiating it—this needs the product rule. Consequently,

$$\int_0^{2\pi} x \sin 2x \, dx = \left(-\frac{2\pi}{2}\cos 4\pi + \frac{1}{4}\sin 4\pi\right)$$

$$-\left(-\frac{0}{2}\cos 0 + \frac{1}{4}\sin 0\right) = -\pi.$$

Sometimes one needs to apply integration by parts more than once, as the next example illustrates.

3. $I = \displaystyle\int x^2 e^x \, dx$.

Solution: Let $u = x^2$, $dv = e^x \, dx$:

$$\frac{du}{dx} = 2x \Rightarrow du = 2x\,dx$$

$$v = e^x$$

$$I = x^2 . e^x - \int e^x 2x\,dx = x^2 . e^x - 2\int x e^x \, dx.$$

We do not know immediately how to evaluate $\int xe^x \, dx$. Has the integration by parts failed? No, because we have replaced $\int x^2 e^x \, dx$ by the simpler integral $\int xe^x \, dx$. Thus continue down the same road by applying integration by parts to this new integral, hoping to simplify it still further: set

$J = \int xe^x \, dx$. Let $u = x$, $dv = e^x \, dx$, $du = dx$ $v = e^x$

$$J = xe^x - \int e^x dx$$

$$= xe^x - e^x + C$$

Substituting this formula into I gives

$$I = x^2 e^x - 2(xe^x - e^x + C)$$

$$= x^2 e^x - 2xe^x + 2e^x - 2C$$

$$= e^x(x^2 - 2x + 2) + C$$

since $-2C$ is again "any constant" and can be written more simply as C.

Integration by parts, which comes from the product rule, is usually applied to integrands that are products of different types of functions. Our three examples above are such products: a polynomial times a log function, a polynomial times a trigonometric function, and a polynomial times an exponential function. But sometimes the product nature of the integrand is not immediately apparent, as in the next example.

4. $I = \int \arccos x \, dx$. Use the fact that $\frac{d}{dx}\arccos(x) = -\frac{1}{\sqrt{1-x^2}}$.
Solution: The hint should tell us to try $u = \arccos(x)$:

$$\frac{du}{dx} = -\frac{1}{\sqrt{1-x^2}} \Rightarrow du = -\frac{dx}{\sqrt{1-x^2}}$$

$$dv = dx$$
$$v = x$$

Now apply the formula

$$I = \arccos(x).x - \int x\left(-\frac{dx}{\sqrt{1-x^2}}\right)$$

$$= x.\arccos(x) + \underbrace{\int \frac{x}{\sqrt{1-x^2}}\,dx}_{J}$$

Now for J, let $w = 1 - x^2$ (function-derivative)

$$\frac{dw}{dx} = -2x \Rightarrow dx = -\frac{1}{2}\frac{dw}{x}$$

$$J = \int \frac{\cancel{x}}{\sqrt{w}}\left(-\frac{1}{2}\frac{dw}{\cancel{x}}\right) = -\frac{1}{2}\int w^{-1/2}\,dw = -\frac{1}{\cancel{2}}\frac{w^{1/2}}{1/\cancel{2}} + C$$
$$= -\sqrt{1-x^2}$$

$$\Rightarrow I = x.\arccos(x) - \sqrt{1-x^2} + C$$

One has to choose u and dv correctly for the method to work. In the last example, if we had taken $u = 1$ and $dv = \arccos(x)\,dx$, we would have been stuck because to continue we need to know v, and finding this is the same problem as evaluating the original integral, so we cannot proceed further.

While this equation is true, it is of no help to us since we have replaced the original integral by a more difficult one. If instead we had started by choosing $u = x$ and $dv = e^x$, then integration by parts works.

Summer 2011: Question 3 (d)

Use integration by parts[1] to evaluate $\int x \sec^2 x \, dx$.

Solution: By LIATE choose[2] $u = x$ and $dv = \sec^2 x \, dx$: $du = dx \Rightarrow v = \tan x$

$I = x\tan(x) - \int \tan x \, dx$

$= x\tan(x) - \ln|\sec x| + C$

Exercises Evaluate the following integrals.

1. $\int x \sec^2 x \, dx$ Ans: $x \tan x + \ln|\cos x| + C$
2. $\int_0^2 x e^{2x} \, dx$ Ans: $\frac{1}{4}(3e^4 + 1)$
3. $\int x \arctan x \, dx$ Ans: $\frac{1}{2}(x^2 + 1)\arctan x - \frac{x}{2} + C$
4. $\int \frac{\ln x}{x^2} \, dx$ Ans: $-\frac{1 + \ln x}{x} + C$
5. $\int \arcsin x \, dx$ Ans: $x \arcsin x + \sqrt{1 - x^2} + C$
6. $\int \ln x \, dx$ Ans: $x(\ln x - 1) + C$
7. $\int \sec^3 x \, dx$ Ans: $\frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$
8. $\int x^2 \sin x \, dx$ Ans: $(2 - x^2)\cos x + 2x \sin x + C$

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[1] it is normal in an exam that you will be prompted to use integration by parts

[2] the aware among you should note that $\sec^2$ is the derivative of tan so should definitely be in the dv as we know how to integrate it