Project 1: Riemann Sums

In this project you prove some summation identities and then apply them to calculate various integrals from first principles. You may assume the following identities about summation (here c, the a_i and b_i are assumed real numbers):

$$\sum_{i=1}^{n} c = cn. \tag{1}$$

$$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i. \tag{2}$$

$$\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i. \tag{3}$$

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1. A famous story of mathematical lore concerns the great German mathematician Carl Friedrich Gauss. The story has it that after the young Gauss misbehaved, his teacher, J.G. Büttner, gave him a task: add from 1 to 100. The young Gauss reputedly produced the correct answer within seconds, to the astonishment of his teacher and his assistant Martin Bartels. What Gauss did was add up 1 to 100 like this:

$$\begin{array}{ccccc} 1 & 2 & 3 & \dots & 100 \\ 100 & 99 & 98 & \dots & 1 \end{array}$$

   Gauss then added the pairs vertically and got 101 each time. As there were 100 such pairs — and this was the sum added up twice the answer was

$$\frac{1}{2} \times 101 \times 100 = 5050.$$

   Retrace the steps of the legend Gauss to show that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}. \tag{4}$$

   Nice tricks and indeed induction can be used to show that:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}. \tag{5}$$

   and

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2. \tag{6}$$

2. To compute the integral $\int_1^2 x^2\,dx$ follow the following steps:

   (a) If we divide the closed interval $[1,2]$ into n equal-length subintervals what is the width, Δx, of each subinterval?

   (b) Draw a sketch of the function $y = x^2$ on the closed interval $[1,2]$. Construct rectangles with width Δx and height $f(x_i^*)$ where x_i^* is the right-endpoint of the ith subinterval.

   (c) Now write down the finite Riemann sum by writing an expression for the area of the n rectangles. Use the identities above to show that the area of n rectangles is given by:
$$1 + \frac{n+1}{n} + \frac{2n^2+3n+1}{6n^2}.$$

   (d) Take the limit as $n \to \infty$ to evaluate the integral.

3. Use the same method & steps to evaluate the following:
$$\int_0^1 x^3\,dx.$$
$$\int_a^b 2\,dx.$$
$$\int_a^b x\,dx.$$
   In the last two integrals $a,\ b \in \mathbb{R}$ with $a < b$.

Project 2: Further Techniques of Trigonometric Integration

In this project you explore trigonometric substitutions for integrands containing square root expressions such as $\sqrt{a^2 - x^2}$, $\sqrt{x^2 - a^2}$ and $\sqrt{a^2 + x^2}$ where $a \in \mathbb{R}$ is always a positive constant. Recall Pythagoras Theorem:

> **In a right-angled triangle,** *the square of the* hypotenuse *is equal to the sum of the squares of the other two sides.*

Also, for all angles, $\sin^2 A + \cos^2 A = 1$. Also recall the integration of trigonometric functions and of the inverse trigonometric functions from this chapter. Finally note that $\sec A = 1/\cos A$.

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1. Use Pythagoras Theorem to draw right-angled triangles with side lengths:

   (a) x, a, $\sqrt{x^2 + a^2}$.

(b) x, a, $\sqrt{x^2 - a^2}$.

(c) x, a, $\sqrt{a^2 - x^2}$.

2. The general method here is depending on which of the square root forms occurs in the integrand, we define an angle $\theta \neq \pi/2$ by letting $\frac{x}{a} = \sin / \tan / \sec \theta$. Find the appropriate angle/substitution for cases (a), (b) and (c).

3. Consider the integral $\int x^3 \sqrt{4 - x^2}\, dx$.

   (a) Draw the triangle for the substitution
$$\frac{x}{2} = \sin \theta.$$

   (b) Does this substitution agree with your answer to question 2?

   (c) Show that we can re-write the integrand using this substitution as
$$\int (8 \sin^3 \theta)(2 \cos \theta)(2 \cos \theta\, d\theta) = 32 \int \sin^3 \theta \cos^2 \theta\, d\theta.$$

   (d) Use the techniques of this chapter (and not differentiation) to show that, in terms of θ, this evaluates to
$$-32 \left(\frac{\cos^3 \theta}{3} - \frac{\cos^5 \theta}{5} \right) + C.$$

   (e) Now use the triangle from part (a) to show that
$$\int x^3 \sqrt{4 - x^2}\, dx = -32 \left(\frac{(4 - x^2)^{3/2}}{24} - \frac{(4 - x^2)^{5/2}}{160} \right) + C.$$
   where $(4 - x^2)^{3/2} = ((4 - x^2)^{1/2})^3 = (\sqrt{4 - x^2})^3$ etc.

   Note that if the integral in terms of θ contained a θ, then all we could say about θ is that it is the angle whose sin is $x/2$:
$$\theta = \arcsin(x/2). \tag{7}$$

   This kind of idea will be required below, namely in the second integral where θ will have to be called $\text{arcsec}(x/3)$.

4. Use the same method & steps to show that
$$\int \frac{dx}{(\sqrt{9 - x^2})^3} = \frac{1}{9} \frac{x}{\sqrt{9 - x^2}} + C, \text{ and} \tag{8}$$
$$\int \frac{dx}{x^3 \sqrt{x^2 - 9}} = \frac{1}{54} \text{arcsec} \left(\frac{x}{3} \right) + \frac{1}{18} \frac{\sqrt{x^2 - 9}}{x^2} + C\,, \text{ where } x > 3. \tag{9}$$

   [HINT]: You will need the identities $\cos^2 A = (1 + \cos 2A)/2$ and $\sin 2A = 2 \sin A \cos A$.

Project 3: The Natural Exponential Function as a Power Series and The Most Beautiful Formula in Mathematic

In this project we pre-empt elements of MS 3003 by defining the natural exponential function as a power series. Then we go further and see a beautiful formula involving all of 1, 0, π, e and $i = \sqrt{-1}$. Define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \tag{10}$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{11}$$

This is an *infinite power series*. Of course many infinite series do not make any sense. For example,

$$1 + 1 + 1 + 1 + \cdots$$

is nonsense. An infinite series $\sum_{n=0}^{\infty} a_n$ is said to converge if

$$\lim_{R\to\infty} \sum_{n=1}^{R} a_n \tag{12}$$

is a converges.

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1. Let $a \in \mathbb{R}$ be non-zero and $r \in \mathbb{R}$ be constants. Consider a finite *geometric series* of R terms.

$$S = a + ar + ar^2 + ar^3 + \cdots + ar^{R-1}. \tag{13}$$

It can be shown that if $|r| < 1$ then an *infinite* geometric series is convergent (to $a/(1-r)$). A theorem called the *Ratio Test* says that if an infinite series is *eventually* geometric then it's convergence is the same as that of an infinite geometric series:

0.0.1 Ratio Test

Suppose $S = \sum_{n=0}^{\infty} a_n$ **is an infinite series and**

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = r.$$

Then if $r < 1$ *the series converges.*

In the case of (11), we have $a_n = x^n/n!$. Evaluate $\frac{a_{n+1}}{a_n}$ for $x \neq 0$ and write an expression for its absolute value.

2. Now show that, for any $x \in \mathbb{R}$,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 0.$$

3. Explain, using the Ratio Test, why this implies that $f(x)$ is defined for all $x \in \mathbb{R}$.

4. Assuming that we can differentiate term-by-term show that

$$\frac{d}{dx} f(x) = f(x).$$

   Note that this implies that

$$\int f(x)\, dx = f(x) + C.$$

   [HINT: It might be better to use (10) — this is actually in your MS 2001 notes]

5. Show that $f(0) = 1$. Use this and part 4. to show that f is strictly increasing at 0 and explain with the aid of a sketch why this suggests that $f(x)$ is strictly increasing for all $x > 0$. In fact, it can be shown that $f(x) > 0$ and strictly increasing for all $x \in \mathbb{R}$. Hence it has an inverse function $f^{-1}(x) : (0, \infty) \to \mathbb{R}$.

6. In fact $f(x) = e^x$ and $f^{-1}(x) = \log_e(x)$. How does the approach of this project contrast with that of this chapter?

7. Now we go complex and consider $e^{i\theta}$ (which we are still defining via (11)). Note that $i = \sqrt{-1}$. Hence

$$\begin{aligned} i^0 &= 1 \\ i &= i \\ i^2 &= -1 \\ i^3 &= -i \\ i^4 &= 1 \\ i^5 &= i \quad \text{etc.} \end{aligned}$$

   Use this and (10) to show that the first *nine* terms of the series expansion of $e^{i\theta}$ can be written as:

$$\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!}\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!}\right). \qquad (14)$$

8. You will see next year that the series in the brackets, if continued, are actually those of $\cos\theta$ and $\sin\theta$ so we have:

$$e^{i\theta} = \cos\theta + i\sin\theta. \qquad (15)$$

   Let $\theta = \pi$ and derive the (arguably) most beautiful formula in all of mathematics:

$$e^{i\pi} + 1 = 0. \qquad (16)$$

Project 4: The Area & Volume of familiar Shapes & Solids

Here we derive most of the area and volume formulas in the tables. Please use the tables to ensure you are getting the correct answers! Note that the real numbers b, h and r are *constants* and should be treated as such when integrating; i.e. they can be pulled out in front of the integral sign.

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1. Sketch the function $t(x) = \frac{h}{b}x$ for $0 \leq x \leq b$. Integrate $t(x)$ from 0 to b. What does this represent?

2. A circle of radius r (and centre at the origin) is a curve with equation
$$x^2 + y^2 = r^2. \tag{17}$$
Alternatively, the circle is got by gluing together the graphs of two functions $u(x) = +\sqrt{r^2 - x^2}$ (upper semicircle) and $l(x) = -\sqrt{r^2 - x^2}$ (lower semicircle — both defined for $-r \leq x \leq r$). Hence sketch $u(x)$ for $-r \leq x \leq r$. Integrate $u(x)$ from 0 to r. [HINT: Use the substitution $x = r\sin\theta$ and recall that $1 - \sin^2 A = \cos^2 A$, $\cos^2 A = (1 + \cos 2A)/2$ and $\sin 2A = \sin A \cos A$. Be careful with your limits — you can only use the limits 0 and r when you have transformed back to x. To transform from θ to x note that θ is the angle whose sin is x/r — $\theta = \arcsin(x/r)$. Finally it can be shown that $\cos\theta = \sqrt{r^2 - x^2}/r$.]

   What does this represent? Hence find the area of a circle of radius r.

3. Sketch the constant function $c(x) = r$ for $0 \leq x \leq h$. What solid is generated when this curve is rotated about the x-axis? Find the volume of this solid by using a formula from the notes. What is 'dodgy' about using this particular formula to calculate this volume?

4. Sketch the function $g(x) = \frac{r}{h}x$ for $0 \leq x \leq h$. What solid is generated when this curve is rotated about the x-axis? Find the volume of this solid by using a formula from the notes.

5. Again sketch $u(x)$ for $-r \leq x \leq r$. What solid is generated when this curve is rotated about the x-axis? Find the volume of this solid by using a formula from the notes.

Project 5: Euler's Method of Numerical Solution of Differential Equations

Here we develop a method of numerically solving first order differential equations of the form
$$\frac{dy}{dx} = F(x, y) \text{ given that } y(x_0) = y_0. \tag{18}$$

We must realise that although not all functions have anti-derivatives, most *nice* differential equations have solutions. Here we find numerical approximations to such solutions.

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1. Consider the differential equation

$$\frac{dy}{dx} = x + y \text{ given that } y(0) = 1. \tag{19}$$

   with solution $y(x)$. What does $\frac{dy}{dx}$ here represent?

2. Euler realised that if you knew the slope of the solution and the initial condition you could approximately sketch the solution. The following is a graph of the *slope field* of the above differential equation.

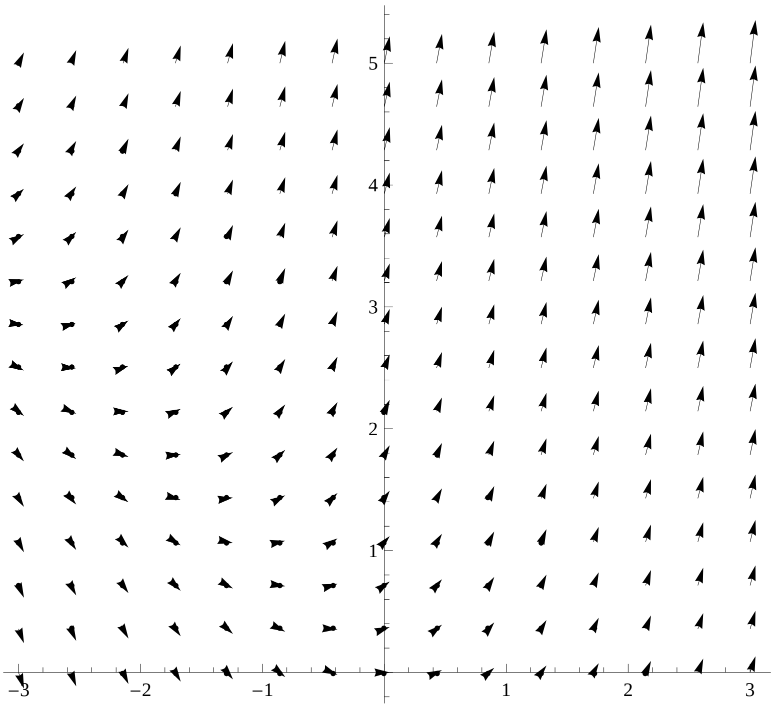

Figure 1: Copy or photocopy this to your handout. Starting at $y(0) = 1$, follow the slope field and sketch the solution.

3. The basic idea behind slope fields can be used to find numerical approximations to solutions of differential equations. We develop the methods on the differential equation (19). The differential equation tells us that $y'(0) = 0 + 1 = 1$, so the solution curve has slope 1 at the point $(0, 1)$. As a first approximation to the solution we could use the linear approximation $y = x + 1$. In other words we could use the tangent line at

$(0,1)$ as a rough approximation to the solution curve. Euler's idea was to improve on this approximation by proceeding only a short distance along this tangent line and then making a correction by changing direction according to the slope field. Euler's method says to start at the point given by the initial value and proceed in the direction indicated by the slope field. Stop after a short space — the *step size* — look at the slope at the new location, and proceed in that direction. Keep stopping and changing direction according to the slope field. Euler's method does not produce an exact solution to the differential equation — it gives approximations. But by decreasing the step size (and therefore increasing the amount of corrections), we obtain successively better approximations to the correct solution. Set up the problem for (19) by sketching the interval $[0,3]$ onto a plane with an x- and y-axis such that $[0,3]$ is divided into six equal subintervals. Make sure that the y-axis continues up to at least $y = 3$. This yields a step size of $h = 0.5$.

4. Draw a line segment of slope 1 from $x = 0$ to $x = 0 + h = 0.5$. Show using the coordinate geometry of the line or otherwise that this line segment has y-value 1.5 at $x = 0.5$.

5. At $x = 0.5$ the slope of the solution to (19) is given by $y'(0.5) = 0.5 + 1.5 = 2$. Now draw a line segment of slope 2 from $x = 0.5$ to $x = 0.5 + h = 1$. Show using the coordinate geometry of the line or otherwise that this line segment has y-value 2.5 at $x = 1$. Compare this with your graphical solution.

6. Note that we have gotten three coordinates on the graph of the approximate solution to (19), namely:

$$(x_0, y_0) = (0, 1),\ (x_1, y_1) = (0.5, 1.5),\ (x_2, y_2) = (1, 2.5).$$

Show that the method we used here is the same as Euler's Method:

0.0.2 Euler's Method

Suppose that

$$\frac{dy}{dx} = F(x, y)\,,\ y(x_0) = y_0 \tag{20}$$

is a differential equation. If we are using Euler's Method with step size h *then*

$$y(x_{n+1}) \approx y_{n+1} = y_n + hF(x_n, y_n)\ \text{, for } n \geq 0. \tag{21}$$

i.e. show that the sequence of coordinates (x_0, y_0), (x_1, y_1), (x_2, y_2) is generated by this method. Note we have $h = 0.5$ and $F(x, y) = x + y$.

7. Hence approximate the solution to (19) at $x = 3$. How might this approximation be improved (assuming we can't solve the differential equation analytically — which we actually can in this case)?

Project 6: The Length of the Monza Circuit

In this project we estimate the length of Monza Circuit by numerical integration techniques. Note that of all the 'projects' this is the one most open to your own ideas.

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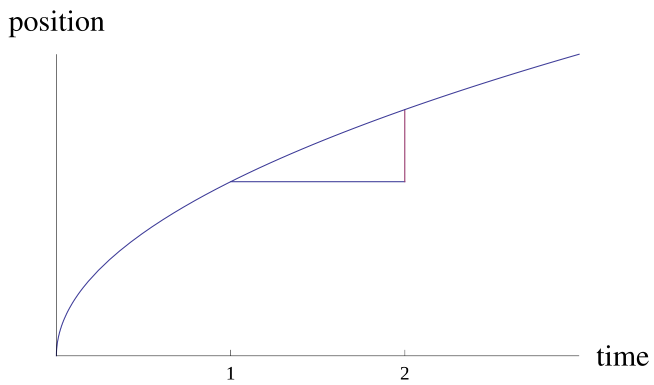
1. Consider the triangle in the graph of position, $s(t)$; vs time, t on the next page. Show that the length of the perpendicular height is $\Delta s = s(2) - s(1)$. Clearly the length of the base $\Delta t = 1$. What does the quantity $\dfrac{\Delta s}{\Delta t}$ represent?

2. What does the quantity
$$\lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt} \tag{22}$$
represent?



3. Let $v(t)$ be the *velocity*/speed of a particle. Explain why
$$\int v(t)\, dt = s(t) + C.$$

4. Therefore if we know the speed of a particle is described by a function $v : \mathbb{R} \to \mathbb{R}$ then we can calculate the distance travelled between times t_1 and t_2 by
$$\int_{t_1}^{t_2} v(t)\, dt. \tag{23}$$
Consider the following data:

   From this data, estimate the length of Monza's track. Note that the seconds will have to be converted to hours first. Find out the length of Monza's track and compare your answer with this. Comment on the accuracy.

Time/ s	Speed/ kmph	Time/ s	Speed/ kmph
0	325	45	275
5	327	50	322
10	69	55	190
15	236	60	280
20	299	65	321
25	327	70	199
30	155	75	266
35	200	80	315
40	200		

Figure 2: This data has been taken from a video on the internet. It is footage of a flying lap by Sebastian Vettel of the Monza racetrack in Italy. I paused the video at intervals of 5 seconds and recorded the driver's speed at these times.