MATH7019 — Technological Maths 311

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Contents

0.1 Introduction

Lecturer

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This page will comprise the webpage for this module and as such shall be the venue for course announcements including definitive dates for the tests. This page shall also house such resources as links (such as to exam papers), as well supplementary material. Please note that not all items here are relevant to MATH7019; only those in the category 'MATH7019'. Feel free to use the comment function therein as a point of contact.

Module Objective

This module covers: Taylor series in one and two variables; first and second order differential equations with constant coefficients; probability distributions, statistical inference and control charts.

Module Content

Further Calculus

Maclaurin and Taylor series of a function of a single variable. Review of partial differentiation. Taylor series expansions of functions of two variables. Differentials.

Differential Equations

Review of the solution of first order differential equations using the method of Separable Variables. Euler's method and the Three Term Taylor method for obtaining numerical solutions to first order differential equations. Solution of second order differential equations using the method of Separable Variables and the method of Undetermined Coefficients. Step functions. Solution of differential equations to include those that occur in the theory of beams and beam struts.

Probability

Laws of probability. Probability distributions such as the Binomial, Poisson Distribution and Normal distributions. Applications of these distributions to engineering problems.

Sampling Theory

Sampling from a Normal population. Confidence intervals for the population mean. Hypothesis tests for population means using the z-test and the t-test.

Quality Control

Control charts for sample means and sample ranges. Process capability.

Assessment

Total Marks 100: End of Year Written Examination 70 marks; Continuous Assessment 30 marks.

Continuous Assessment

The Continuous Assessment will be divided between two in-class tests, each worth 15%, in weeks 6 & 9.
Absence from a test will not be considered accept in truly extraordinary cases. Plenty of notice will be given of the test date. For example, routine medical and dental appointments will not be considered an adequate excuse for missing the test.

Lectures

It will be vital to attend all lectures as many of the examples, proofs, etc. will be completed by us in class.

Tutorials

The aim of the tutorials will be to help you achieve your best performance in the tests and exam.

Exercises

There are many ways to learn maths. Two methods which arent going to work are

1. reading your notes and hoping it will all sink in

2. learning off a few key examples, solutions, etc.

By far and away the best way to learn maths is by doing exercises, and there are two main reasons for this. The best way to learn a mathematical fact/ theorem/ etc. is by using it in an exercise. Also the doing of maths is a skill as much as anything and requires practise.
There are exercises in the notes for your consumption. The webpage may contain a link to a set of additional exercises. Past exam papers are fair game. Also during lectures there will be some things that will be *left as an exercise*. How much time you can or should devote to doing exercises is a matter of personal taste but be certain that effort is rewarded in maths.

Reading

Your primary study material shall be the material presented in the lectures; i.e. the lecture notes. Exercises done in tutorials may comprise further worked examples. While the lectures will present everything you need to know about MATH6038, they will not detail all there is to know. Further references are to be found in the library. Good references include:

- Douglas C. Montgomery, George C., Runger 2007, *Applied statistics and probability for engineers*, Fourth Ed., John Wiley & Sons Hoboken, NJ.
- J. Bird, 2006, *Higher Engineering Mathematics*, Fifth Ed., Newnes.

The webpage may contain supplementary material, and contains links and pieces about topics that are at or beyond the scope of the course. Finally the internet provides yet another resource. Even Wikipedia isn't too bad for this area of mathematics! You are encouraged to exploit these resources; they will also be useful for further maths modules.

Exam

The exam format will roughly follow last year's. Acceding to the maxim that learning off a few key examples, solutions, etc. is bad and doing exercises is good, solutions to past papers shall not be made available (by me at least). Only by trying to do the exam papers yourself can you guarantee proficiency. If you are still stuck at this stage feel free to ask the question come tutorial time.

0.2 Motivation: When is an Approximation Good Enough?

> *Although this may seem a paradox, all exact science is dominated by the idea of approximation.*
>
> Bertrand Russell

Figure 1: A good door closer should close automatically, close in a gentle manner and close as fast as possible.

One possible design would be to put a mass on the door and attach a spring to it (just for ease of explanation we'll only worry about one dimension).

Assuming that the door is swinging freely the only force closing the door is the force of the spring. Now *Hooke's Law* states that the force of a spring is directly proportion to it's distance from the equilibrium position. If the door is designed so that the equilibrium position of the spring corresponds to when the door is closed flush, then if $x(t)$ is the position of the door t seconds after release, then the force of the spring at time t is given by:

where $k \in \mathbb{R}$ is known as the spring constant.

It can be shown that this system *does* close the door automatically but the balance between closing the door gently and closing the door quickly is lost. Indeed if the door is released from rest at $t = 0$, then the speed of the door will have the following behaviour:

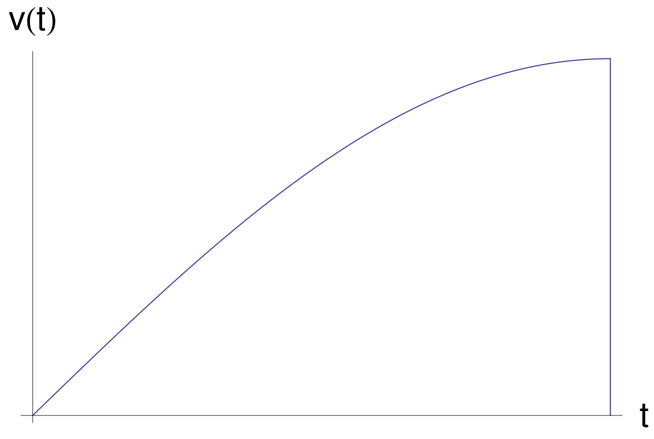


Figure 2: With a spring system alone, the door will quickly pick up speed and slam into the door-frame at maximum speed.

Clearly we need to slow down the door as it approaches the door-frame. A simple model uses a *hydraulic damper*:

Figure 3: A hydraulic damper increases its resistance to motion in direct proportion to speed.

With the force due to the hydraulic damper proportional to speed, the force of the hydraulic damper at time t will be:

for some $\lambda \in \mathbb{R}$. Now by Newton's Second Law:

and the fact that speed is the first derivative of distance, and in turn acceleration is the first derivative of speed, means that the *equation of motion* is given by:

It can be shown that a suitably chosen k and λ will provide us with a system that closes automatically, closes in a gentle manner and closes as fast as possible.
Equations of this form turn up in many branches of physics and engineering. For example, the oscillations of an electric circuit containing an inductance L, resistance R and capacitance C in series are described by

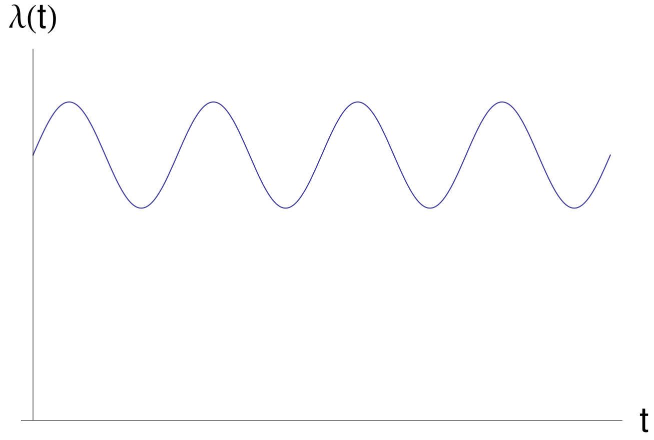
$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0, \tag{1}$$

in which the variable $q(t)$ represents the charge on one plate of the capacitor. These class of equations, *linear differential equations*,

may be solved in various different ways. In this module we will explore one such method — that of *the method of undetermined coefficients*.

Figure 4: Top Gear dropped a VW Beetle from a height of 1 mile and it spun in the air as it fell.

If we are trying to formulate a model for the fall of this car we would have to try and account for the way the roll of the car means that the coefficient of the drag term ($\lambda v(t)$) varies between its maximum and minimum in a wave-like way:



A function with this behaviour is:

$$\lambda(t) = \frac{1}{2}(M + m) + \frac{1}{2}(M - m)\sin \omega t \tag{2}$$

where M and m are the maximum and minimum of $\lambda(t)$ and ω is a constant related to the angular frequency. Then the equation of motion is of the form:

Neither the method of undetermined coefficients nor any other straightforward method I know of solves this differential equation.

Unfortunately this is typical, and for many systems for which a differential equation may be drawn, it may be impossible to solve the equations. There are a number of numerical techniques which can give approximate answers. However if we are participating in some industrial project with millions spent on it we don't want to be chancing our arms on any old estimate or guess. *Approximation Theory* aims to control these errors as follows. Suppose we have a Differential Equation with solution $y(x)$. An approximate solution $A_y(x)$ to the equation can be found using some numerical method. If the approximation method is sufficiently 'nice' we may be able to come up with a measure of the error:

Here $|\cdot|$ is some measure of the *distance* between $y(x)$ and $A_y(x)$. The most common measure here would be maximum error:

We would call the parameter ε here the *control* or the *acceptable error*. Some classes of problem are even nicer in that with increasing computational power we can develop a sequence of approximate solutions $\{A_y^1(x), A_y^2(x), A_y^3(x), \dots\}$ with decreasing errors $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \dots\}$:

Even nicer still from a mathematical point of view if we can find a sequence of approximations with errors decreasing to zero:

In this case we say that the sequence of approximations *converges*.

In this module we will take a first foray into the approximation theory of numerical methods by estimating the solutions of differential equations.

We will not however be measuring *how* accurate our approximate solutions are. However *statistics*, particularly sampling theory, can tell us when our approximations are good enough. For example, suppose we have a business which constructs a machine component. Suppose the company ordering the component wishes to know what 'stress-level' the component can take. Due to natural variations some samples will have a larger tolerance than others — so how can we approach the business and say that our components can take a stress level of S? In practise we can't, but we can make statements along the line of:

> *On average, our components can withstand a stress-level of S.*

However we can't go around testing every single one of the components produced. So what we do is we take a sample of 100 or 1,000 of these components away and have them tested. In this module we will see that we can be 'quite' confident that the average ability to withstand stress of all the components we produce is very well estimated by the sample average. *Sampling Theory* makes precise this idea.

Chapter 1

Differential Equations

What is the origin of the urge, the fascination that drives physicists, mathematicians, and presumably other scientists as well? Psychoanalysis suggests that it is sexual curiosity. You start by asking where little babies come from, one thing leads to another, and you find yourself preparing nitroglycerine or solving differential equations. This explanation is somewhat irritating, and therefore probably basically correct.

David Ruelle

1.1 Review of Separable First Order Differential Equations

A differential equation is an equation containing one or more derivatives, e.g.,

$$y' = x^2,$$
$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} = \sin x.$$

Most laws in physics and engineering are differential equations.

The *order* of a differential equation is the order of the highest derivative that appears;

$$y' = x^2 \quad \text{is first order}$$
$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 1 \quad \text{is second order}$$
$$(\cos x)\left(\frac{d^2y}{dx^2}\right)^3 + \frac{dy}{dx} = y \quad \text{is second order.}$$

A function $y = f(x)$ is a *solution* of a differential equation if when you substitute y and its derivatives into the differential equation, the differential equation is satisfied. For example, $y = \tan x$ is a solution of the differential equation $y' = 1 + y^2$ since if $y = \tan x$, then

A differential equation can have many solutions: $y' = 2$ has a solution $y = 2x + C$ for every constant C. The *general solution* of a differential equation is the set of all possible solutions. The differential equation $y' = 2$ has general solution $y = 2x + C$, where the constant C is arbitrary. It can be very difficult to find the general solution of a differential equation. We shall consider only certain first-order differential equations that can be solved fairly readily.

1.2 First-order Separable Differential Equations

A *separable* first-order differential equation is one that can be written in the form

In this situation we can *separate the variables*:

Each side can now be integrated:

The point of separating the variables is that we cannot usually integrate expressions like $\int y\,dx$ where both variables appear.

Example

Solve the separable first order differential equation:

$$y' = xy.$$

Solution: First separate the variables and integrate:

Usually we want to solve for y:

Here there are two infinite families of solutions. The solution of a first-order differential equation will always contain an unknown constant — and might have different families of solutions also (e.g. the solution $y^2 = x + C$ has the families $y = +\sqrt{x+C}$ and $-\sqrt{x+C}$). However an extra piece of numerical data such as "$y = 2$ when $x = 1$" sometimes reduces this to a unique solution. Note that this will usually be written as $y(1) = 2$ — for the input $x = 1$, the output is $y = 2$. This extra data is called an *initial condition* or *boundary condition* and the entire problem (differential equation and boundary condition) is often called an *initial-value problem* or *boundary-value problem*.

Example

Solve the initial-value problem

$$\frac{dy}{dx} = \frac{1+x}{xy} \quad \text{for } x > 0, \quad \text{where } y(1) = -4.$$

Solution: First separate the variables and integrate:

Now apply the boundary condition:

Now substitute in the constant and hopefully solve for $y(x)$:

Now the fact that $y = -4$ at $x = 1$ and that $\sqrt{x} > 0$ where defined implies that the solution is $y(x) = -\sqrt{2(\log_e x + x + 7)}$. [Ex:] Show that this solves the differential equation and satisfies the boundary condition.

Further Remarks: Picard's Existence Theorem

There is a theorem in the analysis of differential equations which states that if a differential equation is suitably *nice* in an interval about the boundary condition then not only does a solution exist but it is unique. This allows us to define functions as solutions to differential equations. For example, an alternate definition of the exponential function, e^x, is the unique solution to the differential equation:

$$\frac{dy}{dx} = y\,,\ y(0) = 1.$$

Exercises

1. Solve the following differential equations:

   (a) $y' = 3x^2 + 2x - 7$ Ans: $y = x^3 + x^2 - 7x + C$

   (b) $y' = 3xy^2$ Ans: $3x^2y + Cy + 2 = 0$

   (c) $\dfrac{dy}{dx} = \dfrac{3x\sqrt{1+y^2}}{y}$ Ans: $2\sqrt{1+y^2} = 3x^2 + C$

   (d) $\dfrac{dy}{dx} = \dfrac{x}{4y},\ y(4) = -2$ Ans: $x^2 = 4y^2$

2. The point $(3, 2)$ is on a curve, and at any point (x, y) on the curve the tangent line has slope $2x - 3$. Find the equation of the curve. Ans: $y = x^2 - 3x + 2$

3. The slope of the tangent line to a curve at any point (x, y) on the curve is equal to $3x^2y^2$. Find the equation of the curve, given that the point $(2, 1)$ lies on the curve. Ans: $-\dfrac{1}{y} = x^3 - 9$

1.3 Numerical Methods

1.3.1 Direction Fields

Unfortunately, it's impossible to solve most differential equations in the sense of obtaining an explicit formula for the solution. In this section, we show that, despite the absence of an explicit solution, we can still learn a lot about the solution through a graphical approach (direction fields) or a numerical approach (Euler's Method and the Three-Term-Taylor Method).

Suppose we are asked to sketch the graph of the solution of the initial value problem:

$$\frac{dy}{dx} = x + y\ ,\ \ y(0) = 1.$$

We don't know a formula for the solution, so how can we possibly sketch its graph? Let's think about what the differential equation means. The equation $y' = x + y$ tells us that the slope at any point (x, y) on the graph of $y(x)$ is equal to the sum of the x- and y-coordinates at that point. In particular, because the curve passes through the point $(0, 1)$, its slope there must be $0 + 1 = 1$. So a small portion of the solution curve near the point $(0, 1)$ looks like a short line segment through $(0, 1)$ with slope 1:

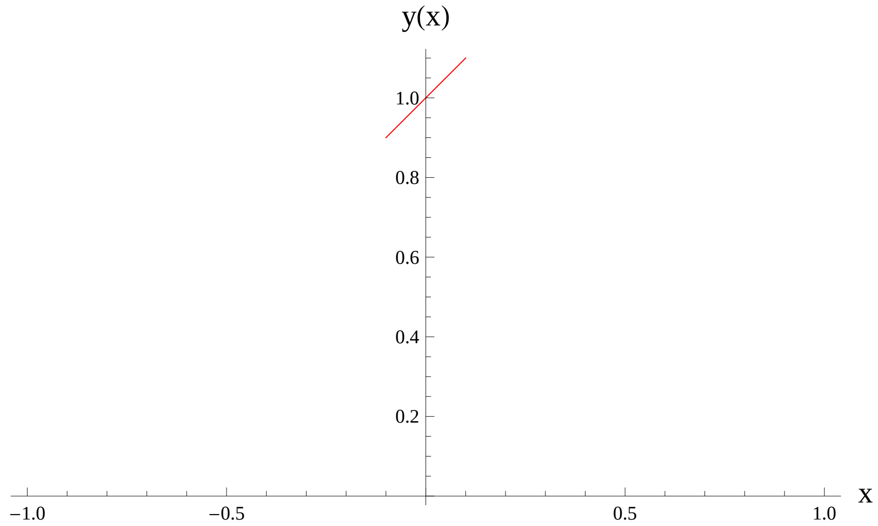


Figure 1.1: Near the point $(0, 1)$, the slope of the solution curve is 1.

As a guide to sketching the rest of the curve, let's draw short line segments at a number of points (x, y) with slope $x + y$. The result is called a *direction field* and is shown below:

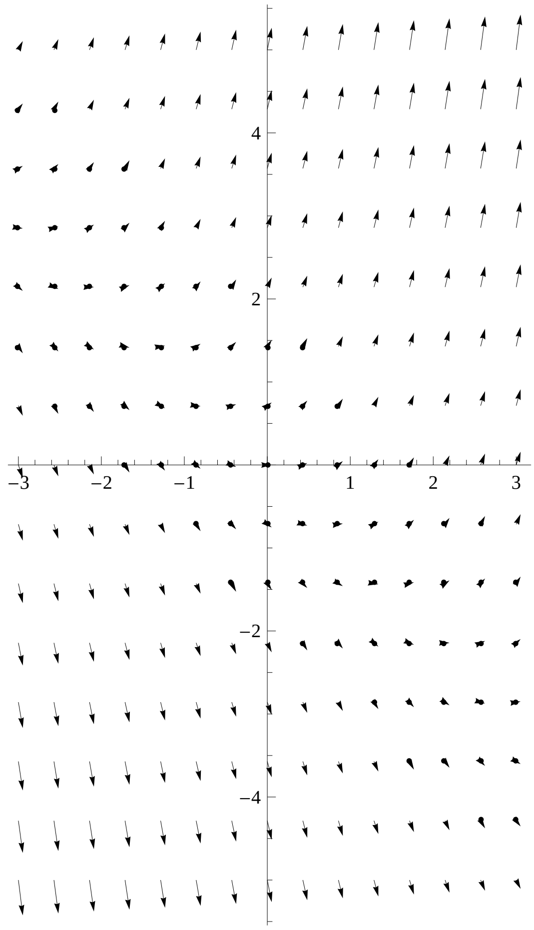


Figure 1.2: For example, the line segment at the point $(1, 2)$ has slope $1+2 = 3$. The direction field allows us to visualise the general shape of the solution by indicating the direction in which the curve proceeds at each point.

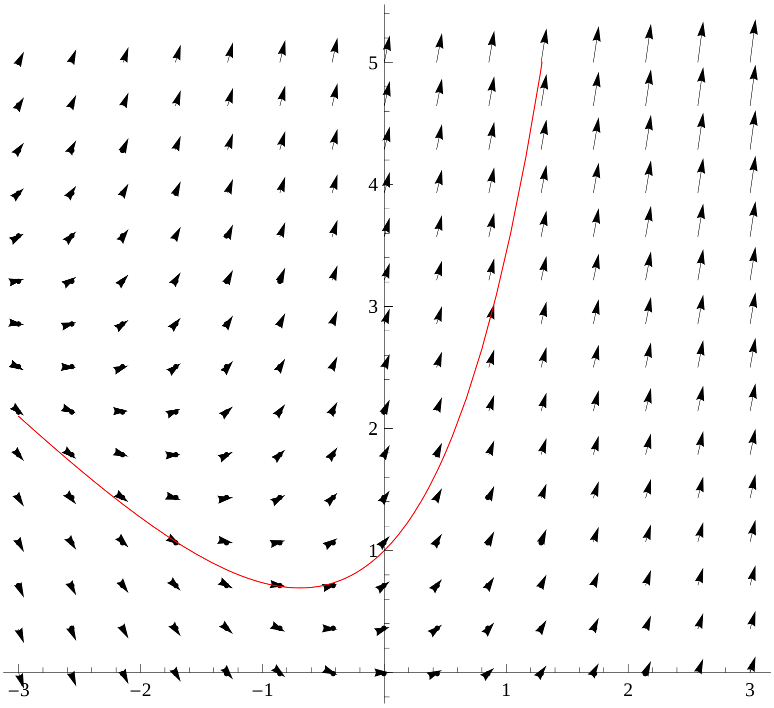


Figure 1.3: We can sketch the solution curve through the point $(0, 1)$ by following the direction field. Notice that we have drawn the curve so that it is parallel to nearby line segments.

1.3.2 Euler's Method

The basic idea behind direction fields can be used to find numerical approximations to solutions of differential equations. We illustrate the methods on the initial-value problem that we used to introduce direction fields:

$$\frac{dy}{dx} = x + y \ , \ \ y(0) = 1.$$

The differential equation tells us that $y'(0) = 0 + 1 = 1$, so the solution curve has slope 1 at the point $(0, 1)$. As a first approximation to the solution we could use the linear approximation $L(x) = 1x + 1$. In other words we could use the tangent line at $(0, 1)$ as a rough approximation to the solution curve.

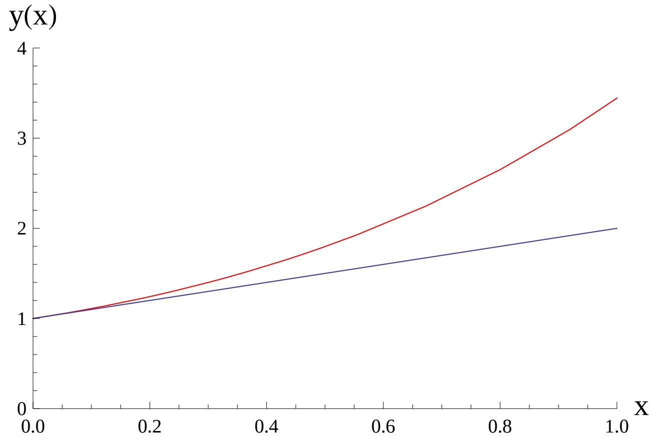


Figure 1.4: The tangent at $(0, 1)$ approximates the solution curve for values near $x = 0$.

Euler's idea was to improve on this approximation by proceeding only a short distance along this tangent line and then making a correction by changing direction according to the direction field:

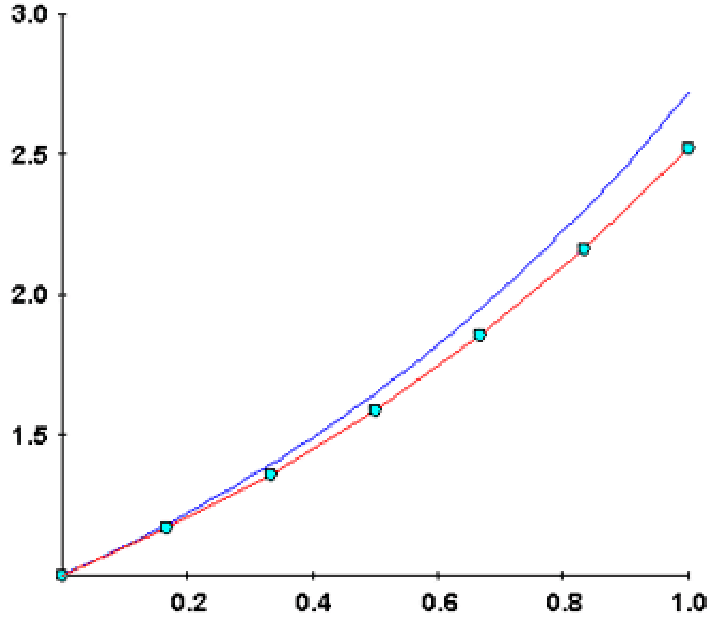


Figure 1.5: Euler's Method starts at some initial point (here $(x_0, y_0) = (0, 1)$), and proceeds for a distance h (in this plot $h = 1/6$.) at a slope that is equal to the slope at that point $y' = x_0 + y_0$. At the point $(x_1, y_1) = (x_0 + h, y_1)$, the slope is changed to what it is at (x_1, y_1), namely $x_1 + y + 1$, and proceeds for another distance h until it changes direction again.

Euler's method says to start at the point given by the initial value and proceed in the direction indicated by the direction field. Stop after a short time, look at the slope at the new location, and proceed in that direction. Keep stopping and changing direction according to the direction field. Euler's method does not produce an exact solution to the initial-value problem — it gives approximations. But by decreasing the step size (and therefore increasing the amount of corrections), we obtain successively better approximations to the correct solution.

For the general first-order initial-value problem $y' = F(x, y)$, $y(x_0) = y_0$, our aim is to find approximate values for the solution at equally spaced numbers x_0, $x_1 = x_0 + h$,$x_2 = x_0 + 2h = x_1 + h ...$, where h is the step size. The differential equation tells us that the slope at (x_0, y_0) is $y' = F(x_0, y_0)$:

This shows us that the approximate value of the solution when $x = x_1$ is

$$y_1 = y_0 + hF(x_0, y_0)$$

Similarly,

$$y_1 = y_0 + hF(x_0, y_0)$$

Euler's Method

If

$$\frac{dy}{dx} = F(x, y) \ , \ \ y(x_0) = y_0$$

is an initial value problem. If we are using Euler's method with step size h then

$$y(x_{n+1}) \approx y_{n+1} = y_n + hF(x_n, y_n) \tag{1.1}$$

for $n \geq 0$.

Example

Use Euler's method with step size $h = 0.1$ to approximate $y(1)$, where $y(1)$ is the solution of the initial value problem:

$$\frac{dy}{dx} = x + y \ , \ \ y(0) = 1$$

Solution: We are given that $h = 0.1$, $x_0 = 0$ and $y_0 = 1$, and $F(x, y) = x + y$. So we have

$$\begin{aligned} y_1 &= y_0 + F(x_0, y_0) = 1 + 0.1(0 + 1) = 1.1 \\ y_2 &= y_1 + F(x_1, y_1) = 1.1 + 0.1(0.1 + 1.1) = 1.22 \\ y_3 &= y_2 + F(x_2, y_2) = 1.22 + 0.1(0.2 + 1.22) = 1.362 \end{aligned}$$

Continue this process [Exercise] to get $y_{10} = 3.187485$, which approximates $y(x_{10}) = y(x_0 + 10(0.1)) = y(1)$, as required.

Exercises

1. *Use Euler's method with step size 0.5 to compute the approximate y-values* y_1*,* y_2*,* y_3 *and* y_4 *of the initial value problem* $y' = y - 2x$*,* $y(1) = 0$*.*

2. *Use Euler's method with step size to estimate* $y(1)$*, where* $y(x)$ *is the solution of the initial value problem* $y' = 1 - xy$*,* $y(0) = 0$*.*

3. *Use Euler's method with step size* 0.1 *to estimate* $y(0.5)$*, where* $y(x)$ *is the solution of the initial value problem* $y' - y = xy$*,* $y(0) = 1$*.*

4. *Use Euler's method with step size* 0.2 *to estimate* $y(1.4)$*, where* $y(x)$ *is the solution of the initial-value problem* $y' - x + xy = 0$*,* $y(1) = 0$*.*

1.3.3 The Three Term Taylor Method

1.4 Second Order Differential Equations

1.4.1 Separable Second Order Differential Equations

1.4.2 Second Order Linear Differential Equations

1.5 Step Functions

1.6 Applications to Beams & Beam Struts

Chapter 2

Probability

2.1 Introduction

Solution:

2.1.1 Conditional Probability

2.1.2 Independence

Solution:

Solution:

Solution:

Solution:

Solution:

2.1.3 Random Variables

2.2 Binomial Distribution

Solution:

Solution:

2.3 Poisson Distribution

Solution:

Solution:

2.4 Normal Distribution

Solution:

Solution:

Solution:

2.5 Applications to Engineering Problems

Chapter 3

Sampling Theory

3.1 The Central Limit Theorem

Solution:

Solution:

3.2 Confidence Intervals

3.3 Hypothesis Testing

3.3.1 Z-Test

3.3.2 T-Test

Solution:

Chapter 4

Quality Control

Chapter 5

Further Calculus

Awesome Calculus quote.

In this chapter we revise and expand on calculus.

5.1 Maclaurin and Taylor Series of a Function of a Single Variable

5.2 Review of Partial Differentiation

5.3 Taylor Series Expansion of Functions of Two Variables

Solution:

Solution:

5.4 Differentials

Further Remarks

Examples

1. ...

5.4.1 Definition

... ...

5.4.2 Theorem

Let... *Then ...*

Remark

...

Proof. ...•

Exercises
Evaluate each of the following.

1. ...

Chapter Checklist

1. ...