1. (a) Consider functions $f : [a, b] \to \mathbb{R}$ and $F : [a, b] \to \mathbb{R}$. What does it mean to say that F is an anti-derivative of f? 3

   (b) Suppose that $g : [1, 4] \to \mathbb{R}$ and $h : [1, 4] \to \mathbb{R}$ are continuous and positive; i.e. $g(x) \geq 0$ and $h(x) \geq 0$ for all $x \in [1, 4]$. Illustrate each of the following properties using diagrams:

   (i) $\int_1^4 g(x)\, dx \geq 0$ 2

   (ii) If $g(x) \leq h(x)$ for all $x \in [1, 4]$, then

   $$\int_1^4 g(x)\, dx \leq \int_1^4 h(x)\, dx.$$ 2

   (iii) For every $1 < c < 4$

   $$\int_1^4 g(x)\, dx = \int_1^c g(x)\, dx + \int_c^4 g(x)\, dx.$$ 2

   (iv) Where M is the maximum of $g(x)$ on $[1, 4]$:

   $$\int_1^4 g(x)\, dx \leq 3M.$$ 2

   (c) Evaluate each of the following

   (i) $\displaystyle\int_1^8 \left(\frac{\sqrt[3]{x^4} + 1}{x^{8/3}} \right) dx$ 14

   (ii) $\displaystyle\int_0^{\pi/2} \cos x\, e^{\sin x}\, dx$

2. (a) Define the natural logarithm function $\ln : (0, \infty) \to \mathbb{R}$ for $x > 0$ 2 and write down 1 the value of $\ln(1)$.

   Prove that $\ln(ab) = \ln(a) + \ln(b)$ 5 for all a, $b > 0$.

   Hence using induction, or otherwise, prove that $\ln(a^n) = n \ln(a)$ 4 for all $\underline{n \in \mathbb{N}}$ and $a > 0$.

   (b) Express

   $$\frac{x}{(x+1)^2(x-2)}$$

   in partial fractions and hence find 7

   $$\int \frac{x}{(x+1)^2(x-2)}\, dx$$

   (c) Use integration by parts twice to find

   $$\int x^2 e^x\, dx.$$ 6

3. Roughly sketch the region R bounded by the curves $y = 2x^2 - 5x + 4$ and $y = 5x - 2x^2$.
   (a) Find the area of R.
   (b) The region R is rotated about the x-axis. Compute the volume of the resulting solid.
   (c) The region R is rotated about the line $y = -1$. Compute the volume of the resulting solid.

4. (a) Let $k > 0$ be a real constant. Find the general solution of the differential equation
$$\frac{dy}{dx} = -ky.$$
   Write your answer $y(x)$ in terms of x, k and a constant of integration C.
   (b) Using an integrating factor, or otherwise, solve the following differential equation
$$\frac{dy}{dx} - xy = x,$$
   where $y(0) = 4$. Express your answer in the form $y = f(x)$.
   (c) Consider the integral $I = \int_0^2 x^2\, dx$. Show that the $n = 4$ trapezoidal rule approximation, T_4, overestimates the value of I by either appealing to a sketch of $y = x^2$ or by directly calculating I and T_4.

END OF EXAM

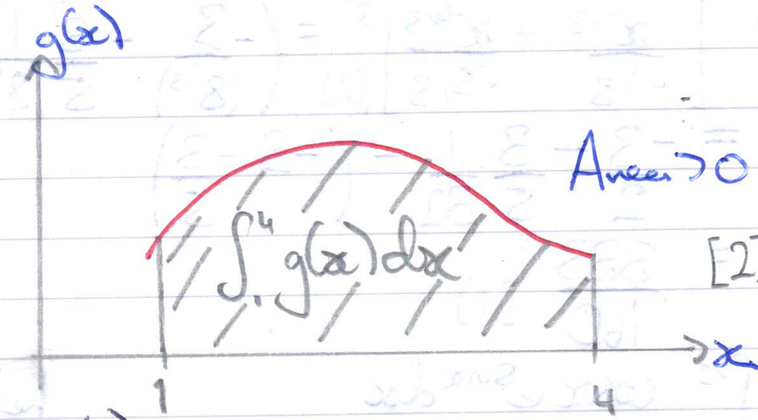
Solutions + Marking Scheme   MS2002

1. (a)

$$F'(x) = f(x) \quad \forall x \in [a,b] \qquad [3]$$

(b)

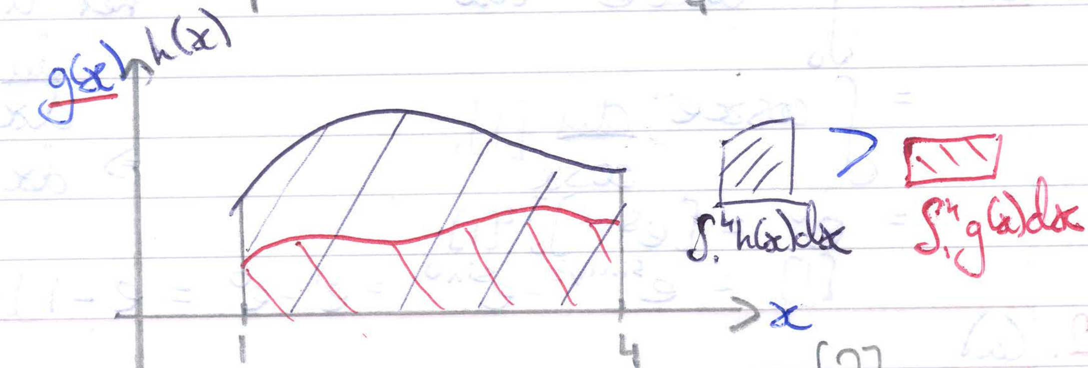
(i) –



Area > 0

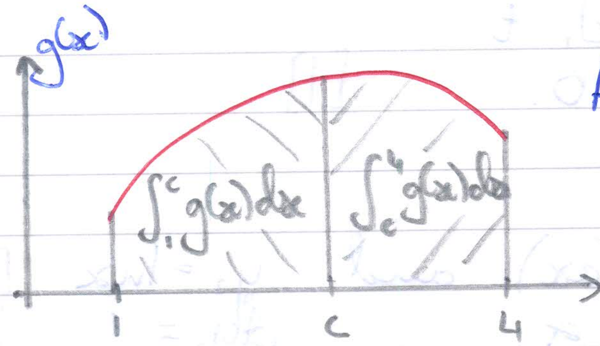
[2]

(ii) –



[2]

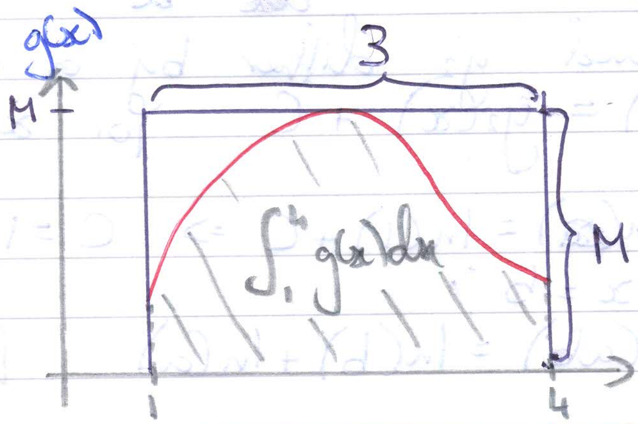
(iii) –



Area is additive

[2]

(iv) –



[2]

MS2002

Proof II

$$\ln(ab) = \int_1^{ab} \frac{1}{t}\,dt \ [1] = \int_1^{a} \frac{1}{t}\,dt + \int_a^{ab} \frac{1}{t}\,dt \ [1]$$

Let $u = \frac{t}{a}$ [1] $\Rightarrow \frac{du}{dt} = \frac{1}{a} \Rightarrow dt = a\,du$ and $a \to 1$, $ab \to b$ [1]

$$\Rightarrow \ln(ab) = \ln(a) + \int_1^{b} \frac{1}{au}\,a\,du \ [1]$$

$$= \ln(a) + \ln(b)$$

Induction

Let $P(n)$ be the proposition that $\ln(a^n) = n\ln(a)$. [1]

$P(1)$. $\ln(a^1) = \ln(a) = 1 \times \ln(a)$ ✓ [1]

Assume $P(k)$

$\ln(a^k) = k\ln(a)$ [1]

$P(k+1)$?

$$\ln(a^{k+1}) = \ln(a^k.a) = \ln(a^k) + \ln(a)$$

$$= k.\ln(a) + \ln(a) \ [1]$$

$$= (k+1)\ln(a).$$

$P(n)$ is true $\forall n \in \mathbb{N}$.

Note that straightforward $\ln(a^n) = \ln(a \times \cdots a) = \cdots$ is also acceptable [4]

Proof 2

$$\ln(a^n) = \int_1^{a^n} \frac{1}{t}\,dt = \int_1^{a} \frac{1}{t}\,dt + \int_a^{a^2} \frac{1}{t}\,dt + \cdots + \int_{a^{n-1}}^{a^n} \frac{1}{t}\,dt. \ [1]$$

$$= \ln(a) + \ln(a) + \cdots + \ln(a). \ [1]$$

Although must prove $\int_a^{ab} \frac{1}{t}\,dt = \ln(b)$ first. [2]

(c)

$$\int x^2 \cancel{\ln x}\, e^x \, dx$$

Let $u = \cancel{\ln x}\; x^2$ $dv = \cancel{e^x}\; e^x$

$\frac{du}{dx} = \cancel{\frac{1}{x}}\; 2x$ [1] $v = e^x$ [1]

$$I = x^2 e^x - \int e^x 2x\,dx$$

$$= x^2 e^x - 2\underbrace{\int x e^x dx}_{J}$$ [1]

$$J = \int x e^x \, dx.$$

$u = x$ [1] $dv = e^x dx$ [1]

$du = dx$ $v = e^x$

$$= x e^x - \int e^x dx$$

$$= x e^x - e^x$$

$\Rightarrow \; I = x^2 e^x - 2xe^x + 2e^x + C.$ [1]

3. Find the intersections:

$2x^2 - 5x + 4 = 5x - 2x^2$ [1]

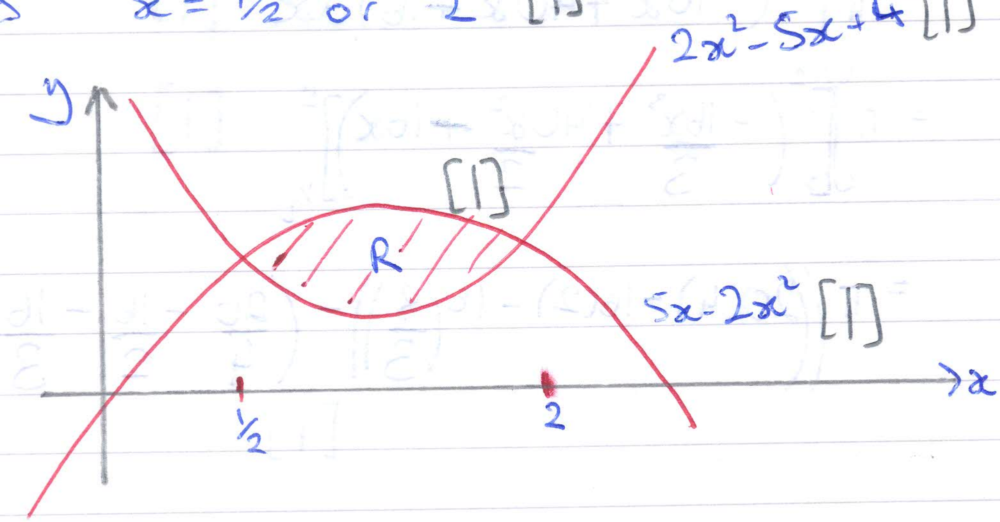
$\Rightarrow 4x^2 - 10x + 4 = 0$

$\Rightarrow 4x^2 - 8x - 2x + 4 = 0$

$\Rightarrow 4x(x-2) - 2(x-2) = 0$

$\Rightarrow (x-2)(4x-2) = 0$

$\Rightarrow x = 1/2$ or 2 [1]



(c)

$$\text{Ans} = \int_{1/2}^{2} \pi(5x-2x^2+1)^2\,dx - \int_{1/2}^{2} \pi(2x^2-5x+5)^2\,dx \quad [1][2]$$

$$= \pi\int_{1/2}^{2}(\cancel{4x^4}-\cancel{20x^3}+21x^2+10x+1)\,dx \; [1] - \pi\int_{1/2}^{2}(\cancel{4x^4}-\cancel{20x^3}+45x^2-50x+25)\,dx \; [1]$$

$$= \pi\int_{1/2}^{2}\left(-24x^2+60x-24\right)dx \quad [1]$$

$$= \pi\left[-\frac{24x^3}{3}+\frac{60x^2}{2}-24x\right]_{1/2}^{2}$$

$$= \pi\left[\left(-\frac{24}{3}(8)+30(2)^2-24(2)\right)-\left(-\frac{24}{3}\left(\frac{1}{8}\right)+30\left(\frac{1}{4}\right)-12\right)\right]$$

$$= \frac{27\pi}{2} \quad [1]$$

4. (a)

$$\frac{dy}{dx} = -ky$$

$$\Rightarrow \frac{dy}{y} = -k\,dx \quad [1]$$

$$\Rightarrow \int\frac{dy}{y} = -k\int dx \quad [2]$$

$$\Rightarrow \ln(y) = -kx + c \quad [1]$$

$$\Rightarrow y = e^{-kx+c} \; [3] \text{ or } Ce^{-kx} \; [2] \text{ with } C>0 \; [1]$$

(b)

B.C. $\ln(4+1) = C$

$\Rightarrow C = \ln(5)$ [1]

$\Rightarrow \ln(y+1) = \dfrac{x^2}{2} + \ln(5)$ [1]

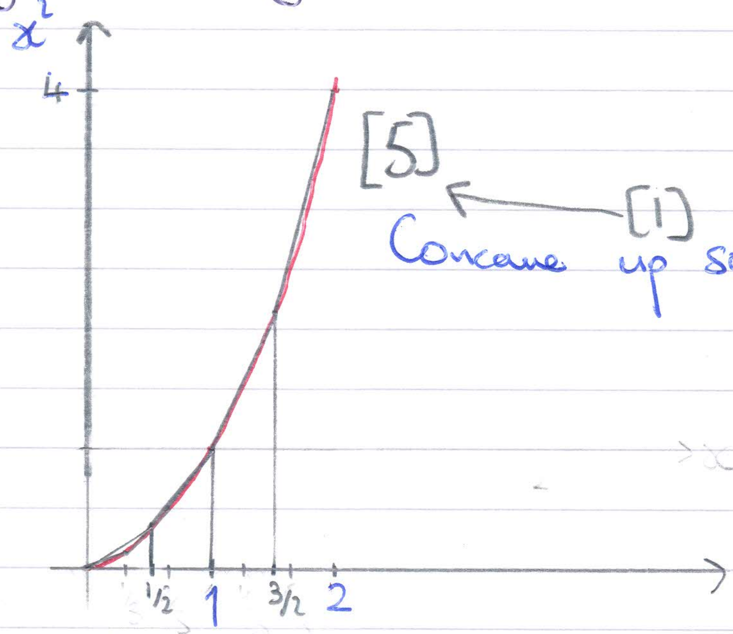
$\Rightarrow \ln\left(\dfrac{y+1}{5}\right) = \dfrac{x^2}{2}$ [2]

$\Rightarrow e^{x^2/2} = \dfrac{y+1}{5}$ [1]

$\Rightarrow y(x) = 5e^{x^2/2} - 1.$ [1]

(c)

Appealing to a diagram



[5]

[1] Concave up so overestimates

Direct Calculation

$$\int_0^2 x^2\,dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3}$$ [2]

$$T_4 = \frac{1}{4}\left[0 + 4 + 2\left(\tfrac{1}{4} + 1 + \tfrac{9}{4}\right)\right] = \frac{1}{4}[4+7] = \frac{11}{4} = \cancel{25}$$ [4]

And $\dfrac{11}{4} > \dfrac{8}{3}$