(c)

(i) - $\int_1^8 \left(\frac{\sqrt[3]{x^4}+1}{x^3}\right)dx = \int_1^8 \left(\frac{x^{4/3}+1}{x^{8/3}}\right)dx$ [1] $= \int_1^8 (x^{-5/3} + x^{-8/3})dx$ [2]

$= \left[\frac{x^{-1/3}}{-1/3} + \frac{x^{-5/3}}{-5/3}\right]_1^8$ [2] $= \left(\frac{-3}{(8^{1/3})} - \frac{3}{5}\frac{1}{(8^{1/3})^5}\right) - \left(-3(1)^{-1/3} - \frac{3}{5}(1)^{-5/3}\right)$ [1]

$= \frac{-3}{2} - \frac{3}{5}\cdot\frac{1}{32} - \left(-3 - \frac{3}{5}\right)$

$= \frac{333}{160}$ [1]

(ii) - $\int_0^{\pi/2} \cos x\, e^{\sin x}\, dx$

Let $u = \sin x$ [2]

$\frac{du}{dx} = \cos x$

$\Rightarrow dx = \frac{du}{\cos x}$ [1]

$= \int \cos x\, e^u \frac{du}{\cos x}$ [1],[1]

$= e^u = [e^{\sin x}]_0^{\pi/2}$ [1]

[1] $= e^{\sin \pi/2} - e^{\sin 0} = e^1 - e^0 = e - 1$ [1]

2. (a)

$\ln x = \int_1^x \frac{1}{t} dt$ [2]

$\ln(1) = 0.$ [1]

Proof I

Let $y_1 = \ln(ax)$ and $y_2 = \ln x$ [1]

$\frac{dy_1}{dx} = \frac{1}{ax} a$ $\frac{dy_2}{dx} = \frac{1}{x}$ [1]

So y_1 and y_2 differ by a constant

$y_1(x) = y_2(x) + C$ for $x > 0$ [1]

Let $x = 1$

$\Rightarrow \ln(a) = \ln(1) + C \Rightarrow C = \ln(a).$ [1]

Now let $x = b$

$\ln(ab) = \ln(b) + \ln(a)$ [1]

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(b)

$$\frac{x}{(x+1)^2(x-2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x-2)} \quad [1]$$

$$\Rightarrow \quad x = A(x+1)(x-2) + B(x-2) + C(x+1)^2 \quad [1]$$

Let $x = -1$

$\Rightarrow -1 = -3B \Rightarrow B = 1/3$

Let $x = 2$

$\Rightarrow 2 = 9C \Rightarrow C = 2/9.$ [1]

Let $x = 0$

$$0 = -2A - \frac{2}{3} + \frac{2}{9}$$

$$\Rightarrow A = \frac{1}{9} - \frac{1}{3} = -\frac{2}{9}.$$

$$\frac{x}{(x+1)^2(x-2)} = -\frac{2}{9}\frac{1}{(x+1)} + \frac{1}{3}\frac{1}{(x+1)^2} + \frac{2}{9}\frac{1}{x-2} \quad [1]$$

$$I = -\frac{2}{9}\int \frac{1}{x+1}dx + \frac{1}{3}\int \frac{1}{(x+1)^2}dx + \frac{2}{9}\int \frac{1}{x-2}dx \quad [1]$$

Let $u = x+1$, $v = x-2$

$$= -\frac{2}{9}\int \frac{du}{u} + \frac{1}{3}\int \frac{du}{u^2} + \frac{2}{9}\int \frac{1}{v}dv$$

$$= -\frac{2}{9}\ln|u| + \frac{1}{3}(-u^{-1}) + \frac{2}{9}\ln|v| \quad [1]$$

$$= \frac{2}{9}\ln|x-2| - \frac{2}{9}\ln|x+1| - \frac{1}{3(x+1)} + C \quad [1]$$

$$= \frac{2}{9}\ln\left|\frac{x-2}{x+1}\right| - \frac{1}{3}\frac{1}{x+1} + C.$$

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(a)

$$\text{Ans} = -\int_{1/2}^{2} (2x^2-5x+4-(5x-2x^2))\,dx \quad [1]$$

$$= -\int_{1/2}^{2} (4x^2-10x+4)\,dx = \left[\frac{4x^3}{3} - \frac{10x^2}{2} + 4x\right]_{1/2}^{2} \quad [1]$$

[1]

$$= -\left(\frac{4}{3}(8) - 5(4) + 4(2)\right) - \left(\frac{4}{3}\left(\frac{1}{8}\right) - 5\left(\frac{1}{4}\right) + 2\right) \quad [1]$$

$$= -\left(\frac{32}{3} - 20 + 8 - \frac{1}{6} + \frac{5}{4} - 2\right)$$

$$= \frac{9}{4} \quad [1]$$

(b)

$$\text{Ans} = \int_{1/2}^{2} \pi(5x-2x^2)^2\,dx - \pi\int_{1/2}^{2}(2x^2-5x+4)^2\,dx \quad [1]$$

$$= \pi\int(25x^2-20x^3+4x^4)\,dx - \pi\int_{1/2}^{2}\left(4x^4-20x^3+41x^2-40x+16\right)$$

[1] [2]

$$(2x^2-(5x-4))^2 = (2x^2)^2 - 2(2x^2)(5x-4) + (5x-4)^2$$
$$= 4x^4 - 20x^3 + 16x^2 + 25x^2 - 40x + 16$$
$$= 4x^4 - 20x^3 + 41x^2 - 40x + 16$$

$$= \pi\int_{1/2}^{2}(-16x^2+40x-16)\,dx \quad [1]$$

$$= \pi\left[\left(-\frac{16x^3}{3} + \frac{40x^2}{2} - 16x\right)\right]_{1/2}^{2} \quad [1]$$

$$= \pi\left(\left(20(4) - 16(2) - 16\left(\frac{8}{3}\right)\right) - \left(\frac{20}{4} - \frac{16}{2} - \frac{16}{3}\frac{1}{8}\right)\right) = 9\pi \quad [1]$$

[1]

$$\Rightarrow \underbrace{e^{-x^2/2}}_{u} \underbrace{\frac{dy}{dx}}_{v'} + \underbrace{(-x e^{-x^2/2})}_{u'} \underbrace{y}_{v} = x e^{-x^2/2} \quad [2]$$

$$\Rightarrow \frac{d}{dx}(e^{-x^2/2} y) = x e^{-x^2/2} \quad [1]+[1]$$

$$\Rightarrow \int d(e^{-x^2/2} \cdot y) = \int x e^{-x^2/2} dx \quad [1]$$

Let $u = \frac{-x^2}{2}$ [1] $\Rightarrow \frac{du}{dx} = -x \Rightarrow dx = -\frac{1}{x} du$ [1]

$$e^{-x^2/2} y = \int \cancel{x} e^u \left(\frac{-1}{\cancel{x}}\right) du$$

$$= -e^u \quad [1]$$

$$\Rightarrow e^{-x^2/2} y = -e^{-x^2/2} + C$$

$$\Rightarrow y(x) = Ce^{x^2/2} - 1$$

B.C.

$$4 = Ce^0 - 1 \quad [1]$$

$$\Rightarrow C = 5 \quad [1]$$

$$\Rightarrow y(x) = 5e^{x^2/2} - 1. \quad [1]$$

Otherwise

$$\frac{dy}{dx} = xy + x \ [2] \Rightarrow \frac{dy}{dx} = x(y+1) \ [2]$$

$$\Rightarrow \int \frac{dy}{y+1} = \int x dx \Rightarrow \ln(y+1) \ [1] = \frac{x^2}{2} + C \ [1]$$