

1. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a polynomial mapping.

- 6 (a) Suppose that x_0 is a prime-period- p point of f . Show that x_0 is a period- n point of f **if and only if** $n = pq$ for some $q \in \mathbb{N}$.
- 8 (b) Using the factor theorem, deduce that $f(x) - x$ is a factor of $f^2(x) - x$.
- 5 (c) Prove that if x is a period- n point of f , then each of the points in the orbit of x also has period- n .
- 4 (d) Use parts (b) and (c) to find the fixed points and prime-period-2 points of $g(x) = x^2 + x - 2$. Hence, describe separately the orbits of each of the period-2 points of g .
- 2 (e) Use part (a) to show that the prime-period-2 points of g are period-50 but not period-51.

2. Consider the *Doubling Mapping* given by

$$D(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1/2 \\ 2x - 1 & \text{if } 1/2 < x \leq 1 \end{cases}$$

- 6 (a) Where x has the binary representation

$$x = 0.a_1a_2a_3a_4a_5a_6a_7a_8 \dots,$$

find expressions for $D(x)$ and $D^5(x)$. Hence find points $y, z \in [0, 1]$ such that y and z agree to 5 binary digits but $D^N(y)$ and $D^N(z)$ differ in the first binary digit for some $N \in \mathbb{N}$.

- 3 (b) Describe the period-5 points of D . Let $w \in [0, 1]$ have a binary representation beginning $w = 0.01001\dots$. Find a period-5 point γ of D such that w and γ agree to five binary digits.
- 3 (c) Find a $\delta \in [0, 1]$ such that there are iterates of δ , $D^{n_1}(\delta)$, $D^{n_2}(\delta)$, $D^{n_3}(\delta)$, with $n_1, n_2, n_3 \in \mathbb{N}_0$, that agree with $0.111\dots$, $0.101\dots$ and $0.010\dots$ to three binary digits.

- 13 (d) Consider the statement:

These arguments could be generalised to show that $D(x)$ is a chaotic mapping.

Do you agree or disagree with this statement? Justify your answer.

$$x^2 + 0x - 2 \overline{\begin{array}{r} x+2 \\ x^3 + 2x^2 - 2x - 4 \\ x^3 + 0x^2 - 2x \\ \hline 2x^2 - 4 \end{array}} \quad [1]$$

$$\Rightarrow f^2(x) - x = (x^2 - 2)x(x+2) \quad [1]$$

The prime period-2 points are 0 and -2 [1] (4)
The fixed points are $\pm\sqrt{2}$. [1]

(c)

[Bookwork]

Let x be period n and have orbit
 $x_0, x_1, x_2, \dots, x_j, \dots, x_{n-1}, x_n, \dots$ [1]

We claim that $f^n(x_j) = x_j$. Write $n = (n-j) + j$ [1]
 $\Rightarrow f^n(x_j) = f^{(n-j)+j}(x_j) = f^j(f^{n-j}(x_j))$ [1]

$$= f^j(x_0) = x_j \quad [1] \quad \square$$

(d)

[Familiar]

$$\text{orb}(\sqrt{2}) = \{\sqrt{2}, \sqrt{2}, \sqrt{2}, \dots\} \quad [1]$$

$$\text{orb}(-\sqrt{2}) = \{-\sqrt{2}, -\sqrt{2}, -\sqrt{2}, \dots\} \quad [1]$$

$$\text{orb}(0) = \{0, 2, 0, -2, 0, -2, \dots\} \quad [1]$$

$$\text{orb}(-2) = \{-2, 0, -2, 0, -2, \dots\} \quad [1]$$

(e)

[Unseen]

Using part (a) $50 = 25 \times 2$ so the prime-period
2 points are period 50. [1]

$51 \neq 2q$ so " " "
" " " not period-51 [1]

(c) shows that there is an element whose orbit is dense. The element point S has an orbit which approximates the three points given to three binary places. [1]

This argument could be generalised by find an element which approximates any [1] point to an arbitrary number of binary digits. [1]

(b)

[Unseen]

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This is the orbit of 0 under the iterator function

$$f(x) = \sqrt{7-x} \quad [1]$$

We find the fixed points of $f(x)$

$$\sqrt{7-x} = x \quad [1]$$

$$\Rightarrow 7-x = x^2$$

$$\Rightarrow x^2 + x - 7 = 0$$

$$\Rightarrow x = \frac{-1 \pm \sqrt{1-4(1)(-7)}}{2}$$

$$= \frac{-1 + \sqrt{29}}{2} [1] \text{ and } \frac{-1 - \sqrt{29}}{2} [1] \quad x > 0$$

We examine $|f'(x)|$ at the fixed points [1]

$$f(x) = \sqrt{7-x} = (7-x)^{1/2}$$

$$\Rightarrow f'(x) = \frac{1}{2} (7-x)^{-1/2} (-1)$$

$$\Rightarrow |f'(x)| = \frac{1}{2\sqrt{7-x}} [1]$$

$$\text{Now } \frac{-1 + \sqrt{29}}{2} \approx 2 \text{ and } \frac{-1 - \sqrt{29}}{2} \approx$$

$$\text{so } |f'((-1 + \sqrt{29})/2)| \approx \frac{1}{4} < 1 \Rightarrow \text{attracting} [1]$$

Hence when we get close to $(\sqrt{29}-1)/2$ we converge to it. This must have happened in this example — one of the iterates $f^n(0)$ must have gotten close enough to ensure convergence to the attracting fixed point.

(e)

[familiar]

Consider

$$f^2(e^{2\pi i/2^2})$$

$$= \left(e^{2\pi i/2^2}\right)^{2^2} = e^{2\pi i \frac{2^2}{2^2}} = e^{2\pi i} = 1 \quad [1]$$

That is $f^2(e^{2\pi i/2^2})$ is eventually fixed at $z=1$. [1]

(f)

I disagree with this statement. Suppose it were true. [1] [Unseen]

$$f^N(e^{2\pi i}) = 1 \quad [1]$$

$$\Rightarrow e^{2\sqrt{2}\pi 2^N} = 1 \quad [1]$$

$$\Rightarrow 2\sqrt{2}\pi 2^N = K2\pi \quad [1]$$

$$\Rightarrow \sqrt{2} = \frac{K}{2^N} \quad \text{as } \sqrt{2} \notin \mathbb{Q} \quad [1]$$

The statement is not true