

# MATH6000 — Essential Mathematical Skills

J.P. McCarthy

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## 0.1 Introduction

### Lecturer

J.P. McCarthy

### Office

Meetings before class by appointment via email only.

### Email & Web:

jpmccarthymaths@gmail.com and <http://jpmccarthymaths.wordpress.com>

This page will comprise the webpage for this module and as such shall be the venue for course announcements including definitive dates for the tests. This page shall also house such resources as links (such as to exam papers), as well supplementary material. Please note that not all items here are relevant to MATH6000; only those in the category 'MATH6000'. Feel free to use the comment function therein as a point of contact.

### Module Objective

This module is about numeracy and basic algebraic competence. Its aim is to ensure that the first-year student acquires proficiency across the spectrum of numerical and algebraic skills needed for the study of science and engineering subjects. Continuous assessment is based on the principle of no compromise on minimal standards for essential skills.

### Module Content

#### The Fundamentals of Arithmetic with Applications

The arithmetic of fractions. Decimal notation and calculations. Instruction on how to use a calculator. Ratio and proportion. Percentages. Tax calculations, simple and compound interest. Mensuration to include problems involving basic trigonometry. Approximation, error estimation, absolute, relative and relative percentage error. The calculation of statistical measures of location and dispersion to include arithmetic mean, median, mode, range, quartiles and standard deviation.

#### Indices and Logarithms

Indices with a discussion of scientific notation and orders of magnitude. Conversion of units. Logarithms and their use in the solution of exponential equations. Discussion of the number  $e$  and natural logarithms.

#### Basic Algebra

The laws of algebra expressed literally and illustrated both numerically and geometrically. Algebraic manipulation and simplification to include the factorisation of reducible quadratics. Transposition of formulae. Function notation with particular emphasis on functions of one variable.

## Graphs

Graphs of quantities which are in direct proportion and indirect proportion. Graphs of simple linear, exponential and logarithmic functions. Reduction of non-linear relations to linear form to allow for the estimation of parameters.

## Assessment

This module is assessed entirely by coursework. The pass mark for this module is 60%. All questions in each assessment to be answered. The assessment of this module is inextricably linked to the delivery. The student must reattend the module in its entirety in the case of a failure.

## Continuous Assessment

There are five 20% short-answer question tests:

1. On the first portion of the indicative content described under heading one up to and including problems on mensuration.
2. On the second portion of the indicative content described under heading one to include approximation/error analysis and calculations of a statistical nature.
3. A general coverage of scientific notation, orders of magnitude and applied problems requiring a thorough knowledge of indices and logarithms for their solution.
4. Algebraic manipulation and simplification, transposition of formulae and effective use of function notation.
5. Plotting and analysis of graphs relating to quantities which are: in direct proportion and in inverse proportion; related linearly, exponentially or logarithmically.

## Lectures

It will be vital to attend all lectures as many of the examples, explanations, etc. will be completed by us in class.

## Tutorials

The aim of the tutorials will be to help you achieve your best performance in the tests and exam.

## Exercises

There are many ways to learn maths. Two methods which aren't going to work are

1. reading your notes and hoping it will all sink in
2. learning off a few key examples, solutions, etc.

By far and away the best way to learn maths is by doing exercises, and there are two main reasons for this. The best way to learn a mathematical fact/ theorem/ etc. is by using it in an exercise. Also the doing of maths is a skill as much as anything and requires practise. There are exercises in the notes for your consumption. The webpage may contain a link to a set of additional exercises. Past exam papers are fair game. Also during lectures there will be some things that will be *left as an exercise*. How much time you can or should devote to doing exercises is a matter of personal taste but be certain that effort is rewarded in maths.

In this set of notes, some exercises have been taken from John Bird, *Engineering Mathematics* Fifth Edition and *New Concise Maths 1* by George Humphrey.

## Reading

Your primary study material shall be the material presented in the lectures; i.e. the lecture notes. Exercises done in tutorials may comprise further worked examples. While the lectures will present everything you need to know about MATH6000, they will not detail all there is to know. Further references are to be found in the library. Good references include:

- John Bird 2010, *Basic Engineering Mathematics*, Fourth Ed., Elsevier Science Ltd England
- Stroud, K.A.; Booth, Dexter J. 2009, *Foundation Mathematics*, Palgrave MacMillan England

The webpage may contain supplementary material, and contains links and pieces about topics that are at or beyond the scope of the course. Finally the internet provides yet another resource. Even Wikipedia isn't too bad for this area of mathematics! You are encouraged to exploit these resources; they will also be useful for further maths modules.

## 0.2 Motivation: Eight Applications of Mathematics in Industry

*The essence of mathematics is not to make simple things complicated, but to make complicated things simple.*

S. Gudder



Figure 1: **Cryptography:** Internet security depends on *algebra and number theory*



Figure 2: **Aircraft and automobile design:** The behaviour of air flow and turbulence is modelled by *fluid dynamics* - a mathematical theory of fluids



Figure 3: **Finance:** Financial analysts attempt to predict trends in financial markets using *differential equations*.



Figure 4: **Scheduling Problems:** What is the best way to schedule a number of tasks? Who should a delivery company visit first, second, third,...? The answers to these questions and more like them are to be found using an area of maths called *discrete mathematics*.



Figure 5: **Card Shuffling:** If the casino doesn't shuffle the cards properly in Blackjack, tuned in players can gain a significant advantage against the house. Various branches of mathematics including *algebra*, *geometry* and *probability* can help answer the question: *how many shuffles to mix up a deck of cards?*



Figure 6: **Market Research:** Quite sophisticated mathematics underlies *statistics* - which are the basis on which many companies make very important decisions.





Figure 7: **Structural Design:** Materials technology is littered with equations, etc that then feeds into structural design. Flying buttresses were designed to balance these equations and hence the building itself.



Figure 8: **Medicine:** A load of modern medical equipment measures a patient indirectly by-way of measuring some signal that is emitted by the body. For example, a PET scan can tell a doctor how a sugar is going around your body: the patient takes ingests a mildly radioactive chemical that the body sees as sugar. As this chemical moves around the body it emits radiation which can be detected by the PET scanner. The mathematically-laden theory of *signal analysis* can be used to draw up a representation of what's going in the body.

# Chapter 1

## The Fundamentals of Arithmetic with Applications

*He who refuses to do arithmetic is doomed to talk nonsense.*

John McCarthy

### 1.1 Number

A *natural number* is an ordinary counting number 1,2,3,... Here the dots signify that this list goes on forever.

The *set* or collection of all natural numbers  $\{1, 2, 3, \dots\}$  we call  $\mathbb{N}$  for short:

We might use the following notation to signify the sentence “*n is a natural number*”:

said “*n in  $\mathbb{N}$* ” or “*n is an element of  $\mathbb{N}$* ”. The symbol ‘ $\in$ ’ means ‘in’ or ‘is an element of’.

The set of *integers* or *whole numbers* is the set of natural numbers together with 0 and all the negative numbers:  $\dots, -3, -2, -1, 0, 1, 2, 3, \dots$ . Once again the dots signify that this list continues indefinitely in both directions. We denote the set of “*integerZ*” by:

The set of *fractions* (or *Quotients*) is the set of all ratios of the kind

Examples include  $1/2, -4/3, 211/24$ . We call the ‘top’ number the *numerator* and the ‘bottom’ number the *denominator*. We will see in Chapter 3 why we don’t let the denominator equal zero here (it corresponds to division by zero, which is contradictory).

Like  $\mathbb{N}$  and  $\mathbb{Z}$  this is also an infinite set which we denote by:

$n \neq 0$  is shorthand for “ $n$  is not zero”.

A *real number* is *any* number that can be written as a decimal. Examples:

- $1 = 1.0$
- $3 = 3.0$
- $-5 = -5.0$
- $1/2 = 0.5$
- $2/3 = 0.6666\dots$
- $\sqrt{2} = 1.41421356\dots$
- $\pi = 3.14159265\dots$

In the context of this module a real number is just any number at all be it a natural number, negative number, fraction, square root, etc. For those of you looking to jump the gun *complex numbers* are not real numbers...

Again we write  $x \in \mathbb{R}$  to signify “ $x$  is a real number”.

## Evaluation of Formulae

The following question was asked on an Irish television quiz programm:

*Fill in the missing number:  $2 + 4 \times \square = 30$ .*

A contestant said the answer was 5. “Correct” said the quizmaster. But they were both wrong. The answer should have been 7.

In Mathematics, by convention alone, multiplications are done before additions, so that

$$2 + 4 \times 5 = 2 + 20 = 22, \text{ but } 2 + 4 \times 7 = 2 + 28 = 30.$$

So the correct answer is 7.

More generally, the order in which mathematical operations should be done — the *Hierarchy of Mathematics* — is as follows:

- 1.
- 2.
- 3.
- 4.

Hence if we want to write add two and four, then multiply this by five we write:

### Exercises

1. Calculate each of the following:

$$\begin{array}{lll}
 (i) 6 \times 2 + 3 & (ii) 3 + 12 \div 6 & (iii) 2(5) + 4 \\
 (iv) 2(2 + 3) & (v) 4 \times 5 - 12 & (vi) 36 \div 9 - 1 \\
 (vii) 5 + 2 \times 3 + 12 \div 4 & (viii) 4 - 18 \div 6 & (ix) 30 \div (6 - 1) + 2 \\
 (x) (8 - 2) \div (5 - 2)
 \end{array}$$

**Selected Answers: (iii) 14, (vi) 3, (ix) 8**

2. In the following problems, calculate the top and bottom separately before doing the division. The bar, —, acts exactly like a bracket.

$$\begin{array}{ll}
 (i) \frac{2(5-1)+2}{3 \times 6 - 4 \times 2} & (ii) \frac{5^2-7}{3^2-7} \\
 (iii) \frac{6(5-8)}{-3(7-4)} & (iv) \frac{5(-1)-3(3)^2}{4(3-1)}
 \end{array}$$

**Selected Answer: (iii) 2**

## 1.2 Fractions

### 1.2.1 Multiplication of Fractions

Ordinarily we would learn how to add numbers before multiplying them but this is not necessarily the case for fractions. We start here with so-called *Egyptian* fractions. When  $n \in \mathbb{N}$  a fraction of the form

is what we call an Egyptian fraction. We first encountered fractions as proportions of a whole. For example  $1/9$  is represented as the portion of area taken up by the slice as shown:

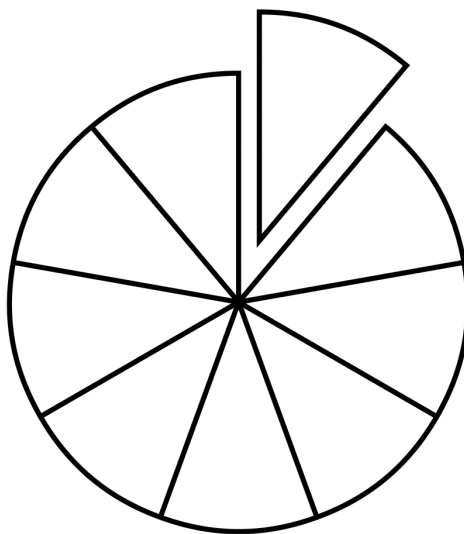


Figure 1.1:  $1/9$  can be represented as *one out of nine*.  $2/9$ ,  $3/9$  then is similarly easy to visualise.

There is a slightly more sophisticated way of looking at fractions and it might just help those of us who have had problems with the arithmetic of fractions before. We start with the integers  $\mathbb{Z}$ . We know that we can multiply them together no problem:

Also we can solve some simple equations such as  $4x = 20$  and  $3x = -18$  pretty much in our head.

Now what about the equations  $2x = 1$ ,  $3x = 1$ ,  $4x = 1$ , ...? Clearly there is no integer solution to these equations and we just define  $1/n$  as the number or ‘object’ that solves the equation

Or in other words,  $1/n$  is the number that when multiplied by  $n$  gives 1.

We could look at non-Egyptian fractions as solutions to these equations as well. For example,  $2/3$  is the solution to  $3x = 2$  but we won't do this exactly. We agree that we can multiply numbers? We will define  $m/n$  as

In a more technical setting we would say that  $1/3$  is the *multiplicative inverse* of 3 and even write  $1/3 = 3^{-1}$ . What happens when we multiply two Egyptian fractions? Well take say 3 and 7. What is

Now we note that  $5 \times 6 = 6 \times 5$ . That is the order of multiplication of numbers doesn't matter. Also recall that one times any number is the same number (e.g.  $1 \times 7 = 7$ ):

That means that  $\frac{1}{3} \cdot \frac{1}{7} = \frac{1}{21}$ ... This provides us a way of multiplying together two fractions:

$$\frac{5}{6} \times \frac{7}{4} =$$

This explains also how to properly cancel fractions. Take for example  $15/12$ . A lot of us here would say something along the lines of, *oh, divide the top and bottom by 3 to get  $\frac{5}{4}$* . Now why is this OK but say adding three above and below isn't? Well the actual calculations are

When the fraction cannot be simplified any more we might say that it is in 'reduced form'. Always try and write fractions in reduced form. More on this in Section 3. Note finally that we have  $\frac{7}{7} = 7 \times \frac{1}{7} = 1$  so that  $\frac{7}{7} = 1$ . We might encounter so-called 'mixed fractions' such as

I think that you are better off converting these to so-called<sup>1</sup> 'improper fractions' as this makes the arithmetic easier.

*Exercises:*

1. Convert to improper form:

$$(i) 1\frac{1}{4} \quad (ii) 2\frac{3}{4} \quad (iii) 1\frac{3}{11} \quad (iv) 2\frac{1}{4} \\ (v) 1\frac{5}{11} \quad (vi) 1\frac{1}{2}$$

**Selected Answers:** (iii)  $\frac{14}{11}$  (vi)  $\frac{3}{2}$ .

2. Write each of the following as a one fraction:

$$(i) \frac{3}{4} \times \frac{1}{5} \quad (ii) \frac{3}{5} \times \frac{5}{9} \quad (iii) \frac{3}{2} \times \frac{5}{8} \\ (iv) \frac{7}{2} \times \frac{7}{6} \quad (v) \frac{8}{3} \times \frac{15}{4} \quad (vi) \frac{7}{6} \times \frac{2}{3} \times \frac{5}{8}$$

**Selected Answers:** (iii)  $\frac{15}{16}$  (vi)  $\frac{35}{72}$

---

<sup>1</sup>crap terminology: there is nothing wrong with  $7/6$ . It just means seven sixths.

### 1.2.2 Division of Fractions

The way we looked at fractions in the last section conveniently allows us to redefine division:

#### Definition

To divide by a number simply multiply by one over that number.

#### Examples

Now how about say  $6 \div \frac{3}{7}$ ? Our new definition says that

But what the hell is  $\frac{1}{3/7}$ ? Well according to the last section we can either look at it as the solution to

or, equivalently, the number that when we multiply it by  $\frac{3}{7}$  we get one. It should be pretty clear at this point that

Hence  $6 \div \frac{3}{7}$



Another example:  $\frac{2}{3} \div \frac{6}{7}$

### A Slightly Different Way

There is a slightly different way to approach division of fractions which uses the fact that multiplying by one doesn't change anything — and we can express one in many different ways (e.g.  $121/121$ ). This approach invites you to write your division of fractions as one big fraction:

Now what you want to do is identify what is making this 'big' fraction awkward. How about that 3 on the bottom? What would get rid of that? Well you could multiply by 3 on the top... however we can't just do that. BUT if we also multiply by 3 on the bottom we shall have multiplied by  $3/3 = 1$ ...

Now the seven is in the way so repeat the trick:

Both methods are equivalent.

*Exercise:* Write each of the following as one fraction

$$\begin{array}{llll} (i) \frac{2}{3} \div \frac{5}{3} & (ii) \frac{7}{9} \div \frac{14}{3} & (iii) 1\frac{1}{5} \div 3\frac{3}{5} & (iv) 1\frac{1}{2} \div 1\frac{3}{4} \\ (v) 5\frac{1}{4} \div 10\frac{1}{2} & (vi) 4\frac{1}{3} \div 1\frac{6}{7} & (vii) 4 \div \frac{1}{5} & (viii) 6\frac{1}{4} \div 1\frac{7}{8} \end{array}$$

**Selected Answers:** (iii)  $1/3$  (vi)  $7/3$

### 1.2.3 Addition & of Fractions

Just like we can add together two natural numbers to produce another natural number we should be able to add together two fractions to produce another. In this era of the calculator most of you would input sums such as

into a calculator and be done with it. Although you might disagree with me the introduction of the calculator to the Junior Cert did a lot to restrict your mathematical ability at a later date. For example, where  $x$  is *some*, unspecified real number how do you add

Here we will look at a method of adding fractions that is not the traditional method of adding fractions but will help you later with adding algebraic fractions such as above.

#### Adding Fractions: Traditional Method Examples

1.  $\frac{3}{13} + \frac{7}{13}$

When the denominators are equal this should be very straightforward. Recall that  $\frac{3}{13}$  is just another way of writing

So if you are adding  $\frac{3}{13} + \frac{7}{13}$  all you are doing is adding three thirteenths to seven thirteenths:

2. When the denominators are unequal we cannot do this. Take for example  $\frac{1}{6} + \frac{2}{3}$ . Hence we find a way of writing the fractions over the same denominator as we know that  $\frac{2}{4} = \frac{1}{2}$  etc. The way we learnt how to do this in school was to first

(a) **Find the L.C.M. (lowest common multiple) of the two denominators**

(b) **Write both fractions over the L.C.M. and add**

Now there is nothing inherently bad about this method but finding the L.C.M. of the denominators is not really necessary as the next example might show.

3.  $\frac{3}{4} + \frac{4}{5}$

(a) **Find the L.C.M. (lowest common multiple) of the two denominators**

(b) **Write both fractions over the L.C.M. and add**

In this last example, the denominators 4 and 5 did not share any factors<sup>2</sup>, so the L.C.M. of 4 and 5 is just  $4 \times 5$ . What if we looked at  $\frac{1}{6} + \frac{2}{3}$  and just looked at the ‘L.C.M.’  $6 \times 3 = 18$ ?

This works perfectly well and this is the method that I would advocate. Now to go one step further I will show how to implement this method easily in a way which allows us to add algebraic fractions easily. The problem is that we can’t really put, say  $\frac{7}{41}$  in terms of 1/1763s very easily. What we are going to do is simply multiply by one! Subtraction works in the obvious way.

---

<sup>2</sup>we say that 4 is *co-prime* to 5

**Examples**

1. Write as a single fraction  $\frac{3}{4} + \frac{5}{8}$ .

*Solution:*

2. Write as a single fraction  $4\frac{1}{5} + 1\frac{1}{4}$ .

*Solution:*

3. Write as a single fraction  $\frac{2}{7} - \frac{3}{5}$

*Solution:*

*Exercises:*

$(i) \frac{3}{4} + \frac{5}{8}$	$(ii) \frac{3}{5} + \frac{2}{3}$	$(iii) \frac{2}{3} - \frac{1}{2}$
$(iv) \frac{5}{8} - \frac{3}{4}$	$(v) \frac{2}{9} - \frac{3}{5}$	$(vi) 5\frac{1}{4} + 4\frac{1}{3}$
$(vii) 3\frac{1}{3} + 2\frac{4}{5}$	$(viii) 3\frac{1}{3} - 2\frac{5}{6}$	$(ix) 2\frac{1}{3} - 5\frac{3}{4}$
$(x) 3\frac{1}{4} - 2\frac{1}{8} + 2\frac{1}{2}$	$(xi) -2\frac{1}{5} + 3\frac{7}{10} - 4\frac{1}{15}$	$(xii) 1 - \frac{2}{3}$
$(xiii) 4 - 1\frac{1}{5}$	$(xiv) 2\frac{1}{2} - \frac{29}{8} - 3 + 1\frac{1}{2}$	

**Selected Answers:** (iii)  $\frac{1}{6}$  (vi)  $\frac{115}{12}$  (ix)  $-\frac{41}{12}$  (xii)  $\frac{1}{3}$

### 1.3 Decimals: The Real Number System

In a lot of applied sciences the numbers we use tend to be decimals. The main advantage of decimals is that we have a good idea of their order of magnitude. For example, which is larger 4.6 or  $\frac{943}{205}$ ...

One thing that we might not appreciate is what does say 5.67891 mean:

One fact that we might have been told in school is that every fraction can be written as a decimal:

A fact that we can probably appreciate right here is that any *terminating* decimal, e.g. 0.127 can also be written as a fraction:

A less well known fact is that every *repeating* decimal, for example 0.123211232112321..... can also be written as a decimal<sup>3</sup>. However there are decimals, real numbers, which cannot be written as a fraction. What is  $\pi$ ? What is  $\sqrt{2}$ ? Neither of these can be written as a fraction<sup>4</sup>.

All the rules of arithmetic we learnt already (such as the Hierarchy of Mathematics) also apply to decimals. I would actually prefer ye to do a lot of arithmetic with fractions by hand but calculators are of course very useful when doing arithmetic with decimals. Let this exercise be an exercise in the Hierarchy of Mathematics and your use of the calculator also.

---

<sup>3</sup>using *infinite geometric series*. 0.123211232112321..... is in fact equal to  $\frac{1369}{11111}$  for example

<sup>4</sup>there is a way to show that the infinity of fractions is in a certain sense smaller than the infinity of decimals.

*Exercise:*

Find each of the following:

part	number	answer
(i)	$3(4.5) + 8.4$	23.7
(ii)	$12.88 \div 4.6 + 1.2$	4
(iii)	$3(8.2) + 2(5.4)$	35.4
(iv)	$2.7(7.3 - 5.1)$	5.94
(v)	$5.5 \div 0.25 + 4.5 \div 0.9$	27
(vi)	$50.46 \div (10.1 - 1.4)$	5.8
(vii)	$4.5(0.036 \div 0.06 + .4)$	20.7
(viii)	$(2.4)^2 + (0.3)^2$	5.85
(ix)	$6.4(1.2 + 1.3)^2$	40
(x)	$\frac{10(7.168+2.832)}{8(8.762-5.637)}$	4
(xi)	$\frac{(2.7-0.3)(2.7+0.3)}{(1.2)^2}$	5
(xii)	$\sqrt{22.4 \div 3.5 + 2.01}$	2.9
(xiii)	$\frac{3(5.6)+1.6}{\sqrt{37.21}+\sqrt{3.61}}$	2.3
(xiv)	$\sqrt{\frac{9}{25}} \times \sqrt{\frac{25}{16}}$	0.75
(xv)	$\frac{1}{(0.4)^2}$	6.25

## 1.4 Ratio & Proportion

Consider the following problem. Three individuals *Abe*, *Barney* and *Carl* are playing a dice game in an illegal casino for a total of E3,000 and the game has got to the point where there is only one throw of a die left such that

1. if the die shows 1, 2 or 3 then Abe wins
2. if the die shows 4 or 5 then Barney wins
3. if the die shows a 6 then Carl wins.

Suddenly, before the die is thrown, the Gardaí raid the gambling den and grab all the dice and equipment. Luckily for the gamblers they escape with the E3,000. How should they divide up the E3,000 now that they game has been finished prematurely?

A ratio is a comparison between two or more similar quantities measured in the same units. In this example the ratio was 3 : 2 : 1. Notice however how natural this problem was in comparison to the problem

*Divide 3,000 in the ratio 3:2:1.*

We will use this problem as a reminder of how to do ratios. First note that 2 : 1 is the same as 4 : 2... multiplying across does not change the comparison<sup>5</sup>. We will thus endeavour to write our ratios in the *proportional form*:

Such that all the fractions add up to a whole. All we have to do is divide across by the sum of the numbers!

### Examples

Express each of the following ratios in proportional form:

1. 12 : 15
2. 14 : 28 : 35

---

<sup>5</sup>hence we can divide also!

3.  $2 : 1\frac{1}{2}$

4.  $0.25 : 0.75$

5. 800 m to 2 km

Writing ratios in this form makes it very straightforward to divide or share quantities as we have seen above.

**Example**

Divide

1. E28 in the ratio  $2 : 5$

2. 300 kg in the ratio  $2 : 5 : 8$



In some questions the ratios are given in disguise.

**Example**

£560 is shared between  $A$ ,  $B$  and  $C$  so that  $A$  gets twice as much as  $B$  and  $B$  gets twice as much as  $C$ . How much does each receive?

The best approach is let the smallest share be one part. Therefore  $C$  receives one part,  $B$  receives two parts and  $A$  four parts. Now the requirement is divide £560 in the ratio  $1 : 2 : 4$

Sometimes we are given the value of some of the parts.

**Example**

$A$  and  $B$  share a sum of money in the ratio  $4 : 3$ . If  $B$ 's share is £15, calculate:

1. the total amount shared between  $A$  and  $B$
2. the amount that  $A$  received

*Solution:*

1. First we deal with the ratio:

Now £15 represents  $\frac{3}{7}$  so that  $\frac{1}{7}$  is £5 and thus the whole is £35.

2. £20...

**Applied Example**

An alloy is made up of metals  $A$  and  $B$  in the ratio  $2.5 : 1$  by mass. How much of  $A$  has to be added to 6 kg of  $B$  to make the alloy?

*Solution:* First of all deal with the ratio:

So we can say that 6 kg is  $\frac{2}{7}$  of the mass so that  $\frac{1}{7}$  of the mass is 3 kg so the total mass is 21 kg. Therefore there is 15 kg of metal  $A$ .

*Exercises: Starred exercises should be of test standard.*

1. Express each of the following ratios in proportional form:

$$\begin{array}{lll}
 (i) \ 3 : 9 & (ii) \ 14 : 21 & (iii) \ 30 : 25 \\
 (iv) \ 20 : 100 & (v) \ 88 : 77 & (vi) \ 15 : 20 : 25 \\
 (vii) \ 1\frac{1}{4} : 2 & (viii) \ 1\frac{3}{4} : 2\frac{1}{4} & (ix) \ 2\frac{1}{2} : 7\frac{1}{2} \\
 (x) \ 0.25 : 1.25
 \end{array}$$

**Selected Solutions:** (iii)  $\frac{6}{5} : \frac{1}{5}$  (vi)  $\frac{1}{4} : \frac{1}{3} : \frac{5}{12}$  (ix)  $\frac{1}{4} : \frac{3}{4}$

2. In a class of 30 pupils there are 20 girls. Find the ratio of
- (a) the number of boys to the number of girls
  - (b) the number of boys to the number of pupils in the whole class
3. A football pitch has a length of 120 m and its width is 80 m. Calculate:
- (a) the ratio of its length to its width
  - (b) the ratio of its length to its perimeter
  - (c) the ratio its perimeter to its width
4. Divide each of the following quantities in the given ratio:

	Quantity	Ratio
(i)	E24	1 : 3
(ii)	50 cm	3 : 2
(iii)	88	7 : 4
(iv)	260 g	6 : 7
(v)	E15.30	2 : 3
(vi)	E4,000	5 : 7 : 8
(vii)	28 kg	$1\frac{1}{2} : 2$
(viii)	24 kg	0 : 25 : 1.25
(ix)	E44	$1\frac{1}{3} : 2\frac{1}{3}$

**Selected Solutions** (iii) 56 : 32 (vi) E1,000 : E1,400 : E1,600 (ix) E16 : E28.

5. Divide E24 into three equal parts. Hence, or otherwise, divide E24 in the ratio 1:2.

6. E1,040 was divided in the ratio 6:7. The larger amount was given to charity. How much was this?
7. A man has two children, a boy aged eight years and a girl aged 12 years. If he divides E400 in the ratio of their ages, how much does each child get?
8. Two schools are to receive a grant from the Department of Education in proportion to their number of pupils. If one school has 450 pupils and the other has 720, how would a grant of E56,160 be divided between them.
9. Divide 60 in the ratio 1.5 : 2.5 : 3.5.
10. E120 is shared between  $P$ ,  $Q$  and  $R$  so that  $P$  gets twice as much as  $Q$  and  $Q$  and  $R$  get equal shares. How much does  $P$  receive.
11. E300 is shared between  $X$ ,  $Y$  and  $Z$  so that  $Y$  gets three times as much as  $X$  and  $Z$  gets twice as much as  $Y$ .
  - (a) Who received the smallest share?
  - (b) How much does each receive?
12. A prize fund was divided between two people in the ratio 2:3. If the larger prize was E120, calculate the total prize fund.
13. The ages of a father and daughter are in the ratio 8:3. If the father is 48 years old, how old is the daughter?
14. Two lengths are in the ratio 8 : 5. If the larger length is 120 cm, find the other length.
15.  $A$ ,  $B$  and  $C$  share a sum of money in the ratio 2 : 3 : 4. If  $C$ 's share is E48, find
  - (a) the total sum of money shared and
  - (b) the amount  $A$  and  $B$  received.
16. A woman gave some money to her four children in the ratio 2 : 3 : 5 : 9. If the difference between the largest and smallest share is E11.76, how much money did she give altogether?
17. \* When mixing a quantity of paints, dyes of four different colours are used in the ratio of 7 : 3 : 19 : 5. If the mass of the first dye used is  $3\frac{1}{2}$  g, determine the total mass of the dyes used.
18. \* Determine how much copper and how much zinc is needed to make a 99 kg brass ingot if they have to be in the proportions copper : zinc to be :8 : 3 by mass.

## 1.5 Percentages

The word *per cent* means ‘per 100’ but we will be better off thinking of 1% as one hundredth of a whole. So what we can actually do is think

This allows us to pass easily from percentage to fraction to decimal:

### Examples

1. Write as a fraction in its simplest form:

$$(a) \ 15\% = 15 \frac{1}{100} = \frac{15}{100} = \frac{3}{20}$$

$$(b) \ 37\frac{1}{2}\% = 37.5 \frac{1}{100} = \frac{37.5}{100} = \frac{75}{200} = \frac{3}{8}$$

2. Write as decimals:

$$(a) \ 32\% = 32 \frac{1}{100} = 0.32$$

$$(b) \ 27\frac{1}{2}\% = 27.5 \frac{1}{100} = 275 \frac{1}{1000} = 0.275$$

*Exercise:*

Write each of the following percentages as: (i) a fraction in reduced form and (ii) a decimal:

$$(i) \ 50\% \quad (ii) \ 5\% \quad (iii) \ 80\% \quad (iv) \ 45\% \quad (v) \ 55\%$$

$$(vi) \ 15\% \quad (vii) \ 120\% \quad (viii) \ 180\% \quad (ix) \ 62\frac{1}{2}\% \quad (x) \ 18.4\%$$

$$(xi) \ 16\frac{2}{3}\% \quad (xii) \ \frac{1}{2}\%$$

**Selected Solutions:** (iii)  $\frac{4}{5}$  and 0.8 (vi)  $\frac{3}{20}$  and 0.15 (ix)  $\frac{5}{8}$  and 0.625

We have already seen the power of multiplying by one. Converting from fractions and decimals to percentages is another case of these. If  $\% = 1/100$  then

**Example**

Write as percentages:

1.  $0.42 = 0.42(100\%) = 42\%$
2.  $\frac{3}{5} = \frac{3}{5}(100\%) = 3(20)\% = 60\%$
3.  $\frac{3}{40} = \frac{3}{40}(100\%) = 3\frac{5}{2}\% = 7.5\%$

*Exercise*

1. Write each of the following as a percentage

$$(i) 0.18 \quad (ii) \frac{1}{2} \quad (iii) \frac{2}{5} \quad (iv) \frac{33}{50}$$

$$(v) \frac{5}{6} \quad (vi) \frac{27}{25} \quad (vii) \frac{1}{3} \quad (viii) \frac{2}{3}$$

2. Seán scored 63 out of 90 in a maths test and 18 out of 30 in an English test. In which test did he achieve the higher percentage?
3. A dealer bought a machine for E7,040 and immediately sold it for E8,000. Calculate his profit, as a percentage of the selling price.
4. \* When 1600 bolts are manufactured, 36 are unsatisfactory. Determine the percentage unsatisfactory.

**Selected Solutions: (iii) 40% (vi) 108%**

We know that on a journey of, say 10 km, that after 2 km we have completed one fifth, or 0.2 or indeed 20% of the journey. All we did here is say

This means that we can quite easily express one quantity as a percentage of another.

**Example**

Express

1. 80 c as a percentage of E2.40
2. 400 m as a percentage of 2 km

*Exercises*

1. In each of the following, express the first quantity as a percentage of the second:

$$\begin{array}{lll} (i) 3, 5 & (ii) 36, 300 & (iii) 15 \text{ cm}, 60 \text{ cm} \\ (iv) E1.50, E7.50 & (v) E.126, E8.40 & (vi) 63 \text{ c}, E1.80 \end{array}$$

**Selected Answers: (iii) 25% (vi) 35%**

2. There are 40 red, 60 green, 100 blue and 50 white marbles in a bag. What percentage of the total number of marbles is each colour?
3. An article that costs E240 is increased in price by E60. What is the increase as a percentage of the old price? What is the increase as a percentage of the new price?
4. \* Two kilograms of a compound contains 30% of element  $A$ , 45% of element  $B$  and 25% of element  $C$ . Determine the masses of the three elements present.
5. \* A concrete mixture contains seven parts by volume of ballast, four parts by volume of sand and two parts by volume of cement. Determine the percentage of each of these three constituents correct to the nearest 1% and the mass of cement in a two tonne dry mix, correct to 1 significant figure.

Often like the last exercise there we have to express a change in a quantity as a percentage change. We use the convention that we express the change in the quantity as a percentage of the original quantity. In terms of a formula

**Examples**

1. A woman's salary was increase from E40,000 to E44,800. Calculate the percentage increase in her salary.

*Solution:*

2. A man went on a diet and reduced his weight from 120 kg to 111 kg. Calculate his percentage decrease in weight.

*Solution:*

*Exercises*

1. Calculate the percentage increase or decrease in each of the following quantities:

Part	Original Quantity	New Quantity
(i)	60 marks	45 marks
(ii)	25 litres	22 litres
(iii)	250 m	215 m
(iv)	80 km/h	84 km/h
(v)	72 marks	81 marks
(vi)	120 litres	40 litres

2. A man's salary was increased from E30,900 to E33,063. Calculate his percentage increase in salary.
3. \* A drilling machine should be set to 250 rev/min. The nearest speed available on the machine is 268 rev/min. Calculate the percentage 'over-speed'.

Our definition of  $\% = \frac{1}{100}$  also makes finding a given percentage of a quantity straightforward because instead of 24% we can talk about  $\frac{24}{100}$  of... all we have to do is multiply by  $\frac{24}{100}$ .

**Example**

Find 12% of E284.

If we use percentages the following become quite familiar

$$50\% = \frac{1}{2}$$

$$25\% =$$

$$10\% =$$

$$33\frac{1}{3}\% =$$

$$20\% =$$

$$75\% =$$

$$12\frac{1}{2}\% =$$

$$5\% =$$

$$66\frac{2}{3}\% =$$

$$2\frac{1}{2}\% =$$

### Exercise

Find

- (i) 25% of E150    (ii) 50% of 184 m    (iii) 35% of E520.20  
 (iv) 13% of  $2\frac{1}{2}$  km    (v) 8% of E50    (vi)  $2\frac{1}{2}\%$  of E80.40

We can now increase or decrease a quantity by a percentage. To increase by say 24% find 24% of the quantity and add it to the original. Similarly for decreases:

Here  $\Delta Q$  means a change in the quantity  $Q$ .

### Examples

1. In 1997 the population of a town was 5,400. In 1998 it increased by 12%. What was the population in 1998?

*Solution:*

Alternatively we can see that the '98 population is 112% (or 1.12) of the '97 population:



2. A machine was bought for E15,000. During the year its value depreciated by 15%. Calculate the value of the machine at the end of the year.

*Solution:*

Alternatively we can see that the end of year price is 85% (or 0.85) of the start of year price:

### *Exercises*

1. John weighs 125 kg before going on a diet. He sets himself a target of losing 8% of his original weight. What is his target weight?
2. Apples cost 60 c each. The reduces the price by 5%. Calculate the new price.
3. Because of low sales, a firm reduces its work force by (a) 18% of the factory workers and (b) 24% of the office staff. If 350 people work in the factory and 50 people work in the office, how many people will work in the firm after the reductions.
4. Express  $\frac{2}{3}$  of 0.48 as a percentage of 2.56.
5. \* A screws dimension is  $12.5 \pm 8\%$  mm. Calculate the possible maximum and minimum length of the screw.

Sometimes we will be given a percentage of a quantity and have to find the whole.

### **Examples**

1. 85% of a number is 153. Find the number.

Alternatively using  $\% = 1/100$

The first method is probably easier.

2.  $4\frac{1}{2}\%$  of a number is 36. Find 16% of the number.

3. The cost of a train ticket has risen by 5% to E15.96. What was the original price of the ticket.

*Solution:* Now E15.96 is 105% of the original price:

4. A bicycle was sold for E384 at a loss of 20%. Find the original cost of the bicycle.

*Solution:* Now E384 is 80% of the original cost:

### *Exercises.*

1. 15% of a number is 90. Find the number.
2.  $12\frac{1}{2}\%$  of a certain sum of money is E72. Find the sum of money.
3. 125% of a sum of money is E60. Find this sum
4. A bicycle was sold for E272 at a loss of 15%. Find the original cost and the amount of loss.
5. A piece of elastic was stretched by 15% to a length of 16.1 cm. Calculate its unstretched original length.
6. By selling a car for E8,840, the owner incurs a loss of 35% on the purchase price. Calculate the purchase price.
7. *A* sold an article to *B* for a gain of 10%. *B* sold it on to *C* for a gain of 15%. If *C* paid E23.50, how much did *A* and *B* pay for it?
8. 13% of a number is 156. Find 8% of the number.
9. When 9% of the pupils in a school are absent, 637 are present. How many students are on the school roll?
10. A youth club collects the same membership subscription from each of its 140 members. 15% of the club's members have still to pay their subscription. How many members have not paid their subscription? If the subscriptions not yet paid amount to E50.40, calculate the subscription for each member.
11. \* A block of monel alloy consists of 70% nickel and 30% copper. If it contains 88.2 g of nickel, determine the mass of copper in the block.
12. \* In a sample of iron ore, 18% is iron. How much ore is needed to produce 3600 kg of iron?
13. \* The output power of an engine is 450 kW. If the efficiency of the engine is 75%, determine the power input.

## 1.6 Tax & Interest

### 1.6.1 VAT

Value-added tax (VAT) is a tax on goods and services. The rate of VAT is given in the form of a percentage, for example 23% VAT on alcohol. We need to be careful as customers, as some prices are inclusive of VAT, while other prices are exclusive of VAT.

#### Examples

1. VAT at 23% is added to a bill of E160. Calculate the total bill, including the VAT.

*Solution:* The bill is now

2. A garage bill came to E120. When VAT was added to the bill it amounted to E138. Calculate the rate of VAT.

*Solution:* Clearly the VAT is E18.

#### Exercises

1. Calculate the VAT payable on a garage bill of E380 if the rate of VAT is 18%.
2. A man bought a coat for E150 + VAT an electric razor for E80 + VAT. If VAT on clothes is 12% and VAT on electronic equipment is 21%, find how much he paid in total including the VAT,
3. A tanker delivered oil to a school. Before the delivery the meter showed 11,360 litres of oil in the tanker. After the delivery, the meter reading was 7,160 litres.
  - (a) How many litres of oil were delivered to the school?
  - (b) Calculate the cost of oil delivered if 1 litre of oil costs 41c.
  - (c) When VAT was added to the price of the oil delivered, the bill to the school amounted to E2,083.62. Calculate the rate of VAT added.

Often a price includes VAT, and we have to work in reverse to calculate the VAT or the price before VAT was added on.

#### Example

A garage bill for repairs came to E295.20, including VAT at 23%. Calculate the bill before VAT was added.

*Solution:* Now E295.20 is 123% so we divide by 123 to find 1%:

*Exercises:*

1. Complete the following table:

	Price including VAT	VAT Rate	Price without VAT	VAT
(i)	E252	5%		
(ii)	E89.60	12%		
(iii)	E56.16	8%		

**Selected Answer: (iii) E52, E4.16**

2. A table is priced at E767, which includes VAT at 18%. How much does the table cost without VAT?
3. A motorbike is priced at E1,086.40 which includes VAT at 12%. Calculate (i) the price without VAT and (ii) the VAT.
4. A boy bought a calculator for E73.80, which included VAT at 23%. Find the price of the calculator if VAT is reduced to 15%.

### 1.6.2 Income Tax

Here we present the PAYE tax system in Ireland<sup>6</sup>. In reality it doesn't happen exactly

1. For all your work over the year you are paid. This is your *gross income*. Depending on your personal circumstances, you will be given a *standard rate cut-off point*. Income below this is considered low earnings and income above this considered high earnings.

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<sup>6</sup>we're not going to include the Universal Social Charge here

2. Now to run public services the government comes along and takes taxation of you. They don't want to take too much off low earners so what they do is take 20% of your low earnings. As high earners have more ability to pay they take 41% of your high earnings. 20% is called the *standard rate* and 41% is called the *higher rate*. What they take off you is your *gross tax*.
3. However is this a just system? Should a person supporting children have to pay as much tax as someone who doesn't have any? Should somebody who is working in retirement pay as much tax as a young person who has never paid a cent of tax? If we tax a family so much that they can't afford the mortgage social services might have to intervene etc. So depending on your circumstances they give you some of your gross tax back. The money they give you back are called your *tax credits* and the taxation they hold onto is called your *nett tax* or *tax payable*.

What you have left over is your *nett income* or your *take home pay*.

In summary,

**Examples**

1. A man has a gross yearly income of E25,000. He has a standard rate cut-off point of E28,000 and a tax credit of E1,200. The standard rate of tax is 18% of income up to the standard rate cut-off point. Calculate:
  - (a) the amount of gross tax for the year.
  - (b) the amount of tax paid for the year.
  - (c) the nett income for the year.

*Solution:*

- (a) O.K. 18% on low earnings and what rate on high earnings...

- (b) We know the gross tax but we get tax credits back:

- (c) Nett income is just gross income less nett tax

2. A woman has a gross yearly income of E47,000. She has a standard rate cut-off point of E25,000 and a tax credit of E3,800. The standard rate of tax is 18% of income up to the standard rate cut-off point and 35% on all income above the standard rate cut-off point. Calculate:
  - (a) the amount of gross tax for the year.
  - (b) the amount of tax paid for the year.

*Solution:*

- (a) Recall 18% on low earnings and 35% on the high earnings. Here we have E25k of low earnings and E22k of high earnings:

This is her gross tax.

- (b) To find the tax payable we just give her back her tax credits:

3. A man has a gross yearly income of E24,000. He has a standard rate cut-off point of E27,200 and a tax credit of E2,850. If he pays tax of E1,470, calculate the standard rate of tax.

*Solution:* If his nett tax is E1,470 and his tax credits are E2,850 then we know that

What percentage is this of his gross income?

*Exercises:*

1. Complete the following table. Note G.I. is gross income, C.O.P. is standard rate cut-off point, S.R. is standard rate, G.T. is gross tax, T.C. is tax credits, N.T. is nett tax and N.I. is nett income.

	G.I.	C.O.P.	S.R.	G.T.	T.C.	N.T.	N.I.
(i)	E17,500	E21,200	20%		E2,600		
(ii)	E19,400	E20,300	17%		E2,728		
(iii)	E19,570	E22,450	21%		E2,082		

**Selected Answer: (iii) E4,109.70, E2,027.70 E17,542.30**

2. A man has a gross yearly income of E19,000. He has a standard rate cut-off point of E23,400 and a tax credit of E2,090. The standard rate of tax is 18% of income up to the standard rate cut-off point. Calculate:
- (a) the gross tax for the year.
  - (b) the nett tax for the year.
  - (c) the nett income for the year.

Express nett tax for the year as a percentage of gross income for the year.

3. A man has a gross yearly income of E38,000. He has a standard rate cut-off point of E23,000 and a tax credit of E3,300. The standard rate of tax is 20% of income up to the standard rate cut-off point and 38% on all income above the standard rate cut-off point. Calculate:
- (a) the gross tax for the year.
  - (b) the tax payable for the year.
4. A woman has a gross yearly income of E28,500. She has a standard rate cut-off point of E30,100 and a tax credit of E2,240. If she pays tax of E2,890, calculate the standard rate of tax.

### 1.6.3 Interest

How do banks make money?

This E100 is called the principal. We will denote it by  $P$  later when we want to write down a formula. The E1 that the saver gets back is interest on the savings/investment and the extra E2 that the borrower pays back is interest on the loan. The E1 leftover is the bank's profit. Interest is a payment made that is a percentage of the principal saved or borrowed. In this example the interest on savings/investment is 1% on the interest on borrowings is 2%. The interest rate or percentage is usually yearly but could be quarterly, monthly, weekly, daily, etc.

#### Examples

1. E250 was invested for a year at 2.5% per annum. Calculate the interest earned.
2. E540 is borrowed at 8% per annum. Calculate the amount owed after one year.  
*Solution:* The amount after one year will be

So we simply find the interest

#### Exercises:

1. E540 was invested for a year at 6% per annum. Calculate the interest.
2. E580 is borrowed at  $3\frac{1}{2}\%$  per annum. Calculate the amount owed after one year.
3. A man borrowed E5,000 from a bank at a rate of 4% per annum. He agreed to pay off the loan at the end of one year with one payment. How much did he need to pay to clear the loan?



Usually when people save they don't save for just one year but longer. If we want we can analyse this year by year. For example, 1.2% on E1,000 over five years will accrue as

$$E1012, E1024.14, E1,036.43, E1,048.87, E1061.46.$$

If we were naïve we might think that because the interest earned in the first year is E12 the interest after five years should be E60...

Having to calculate the interest year by year is a bit tedious so we come up with a formula/equation instead. Formulae express a relationship between various quantities no matter what their values are; for example  $E = mc^2$ . We use letters to stand for quantities. Here  $E$  is rest-energy,  $m$  is mass and  $c$  is the speed of light. If we want to calculate the rest-energy of 3 kg of a substance we simply substitute in  $m = 3$  in this equation. So we need notation.

Let  $i$  be the interest rate. This will be a percentage, for example 3%, so will usually be a number between 0 and 1. For example  $3\% = 3\frac{1}{100} = 0.03$ . Now let  $P_n$  be the value of the investment after  $n$  years. We will call  $P_0 = P$  the principal. Now we want analyse how  $P_1$  is related to  $P$ . Just for now we will denote interest by  $I$ . Well

So to get  $P_1$  from  $P$  we simply multiply by  $(1 + i)$ . How about how do we get from  $P_1$  to  $P_2$ ?

There is clearly nothing special about years zero, one or two here. Clearly for any year  $n$

Now we can ask what is  $P_n$ :

$$\begin{aligned} P_n &= (1 + i)P_{n-1} \\ &= (1 + i)[(1 + i)P_{n-2}] = (1 + i)^2 P_{n-2} \\ &= (1 + i)[(1 + i)P_{n-3}] = (1 + i)^3 P_{n-3} \\ &= \vdots \\ &= (1 + i)^n P \end{aligned}$$

This is our first formula.

### Formula: Compound Interest

The value of an investment of  $P$  after  $n$  years at an annual interest rate of  $i$  is given by

$$P_n = P(1 + i)^n \tag{1.1}$$

Note that this formula does not apply when the interest rate changes or where repayments are made.

**Example**

Calculate the compound interest on E2,500 for three years at 6% per annum.

*Solution:* We have a principal  $P = 2,500$ , invested at a rate of  $i = 6\% = 0.06$  for  $n = 3$  years:

*Exercises:*

1. Calculate the compound interest on each of the following investments.

- (i) E2,500 for 2 years at 8% per annum      (ii) E1,200 for 8 years at 3% per annum
- (iii) E350 for 12 years at 10% per annum      (iv) E10,000 for 15 years at 6% per annum
- (v) E15,000 for 18 years at 12% per annum

**Selected Answer: (iii) E1098.45**

2. A man borrowed E7,500 at 8% per annum compound interest. He agreed to pay back the entire loan including interest after three years. How much did he pay back?
3. Post Office savings certificates pay 3% per annum compound interest if the money is invested for three years. How much would E20,000 amount to if invested with the Post Office for three years?

## 1.7 Mensuration

In this section we are interested in geometric problems of length, area and volume.

### 1.7.1 Area

A way to start thinking about area is in terms of tiling. We take a unit of length, form a square and call this one unit of area. Now we can find the area of a rectangle:

If we accept the axiom (assumption) that areas are *additive* and are preserved under translations then we can find the area of a triangle:

This means that, theoretically, we can find the area of *polygons*<sup>7</sup> by using triangulation:

One particular class of polygons are *quadrilaterals* — which are polygons with four sides. A parallelogram is an example of a quadrilateral:

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<sup>7</sup>a figure formed by three or more points in the plane joined by line segments

We can see that its area is its base by its perpendicular height. We can find the area of a circle by using integration — basically forming an infinite triangulation of very small triangles:

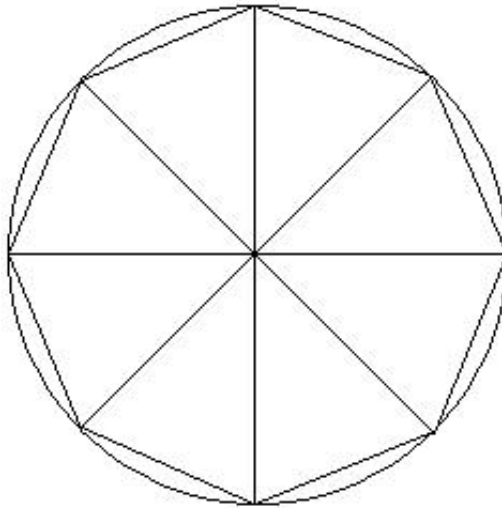


Figure 1.2: By following out this procedure we can show that the area of a circle is given by  $\pi r^2$  — where  $r$  is the radius. We define  $\pi$  here by length/diameter: this gives the circumference as  $2\pi r$ .

**Examples**

1. Find the cross-sectional area of the girder (a) and the area of the perimeter path in (b).

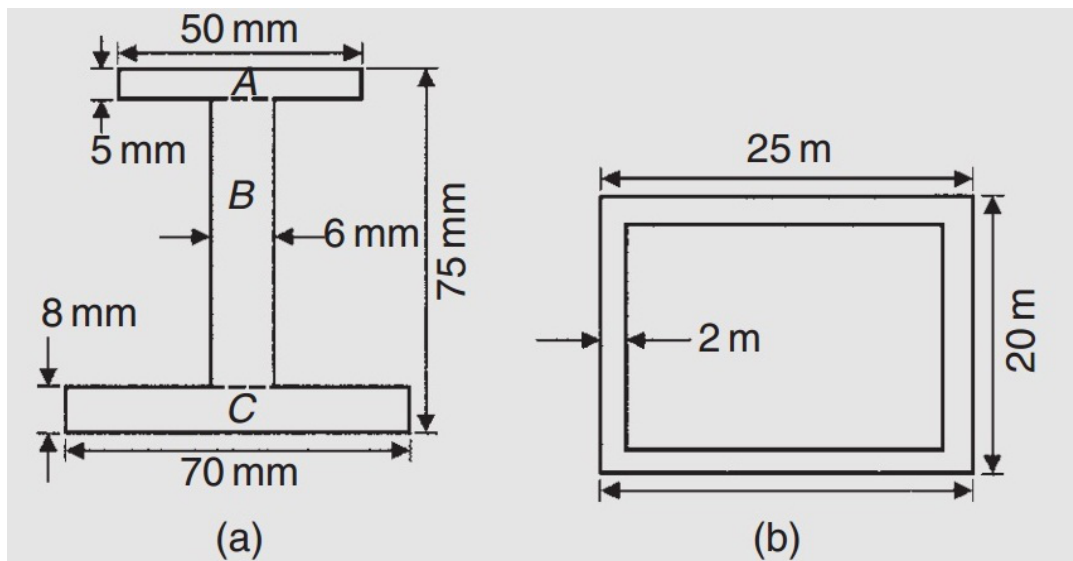


Figure 1.3: Diagram taken from Engineering Mathematics by Bird.

*Solution:*

- (a) Using the additivity of area we can find the areas of the three rectangles  $A$ ,  $B$  and  $C$  separately and add them up:

- (b) We can use additivity of area to get the answer by looking at

2. This diagram shows the gable end of a building. Determine the area of brickwork in the gable end.

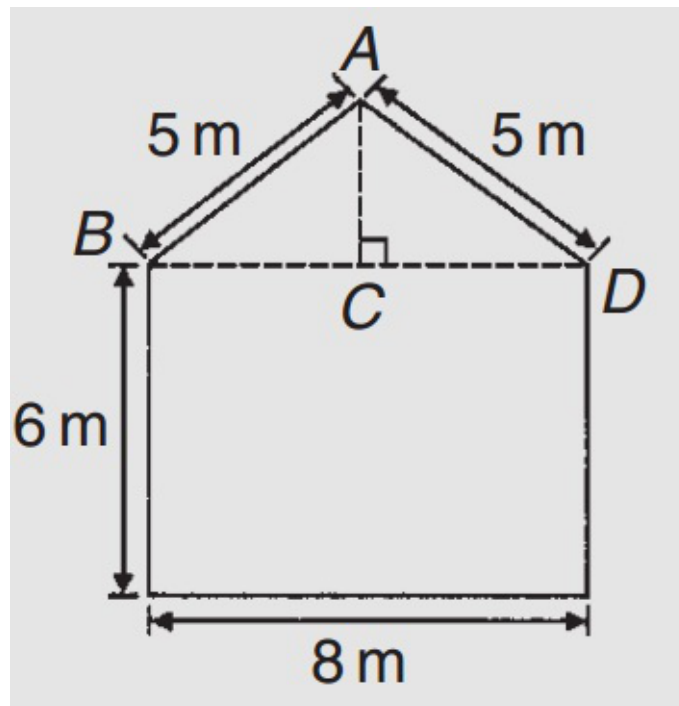


Figure 1.4: Diagram taken from Engineering Mathematics by Bird.

### Exercises

1. A rectangular plate is 85 mm long and 42 mm wide. Find its area in square centimetres.  
**Answer 35.7 cm<sup>2</sup>**
2. \* A rectangular garden measures 40 m by 15 m. A 1 m flower border is made round the two shorter sides and one long side. A circular swimming pool of diameter 8 m is constructed in the middle of the garden. Find, correct to the nearest square metre, the area remaining. **Answer 482 m<sup>2</sup>**

3. Determine the area of each of the angle iron sections shown in the diagram.

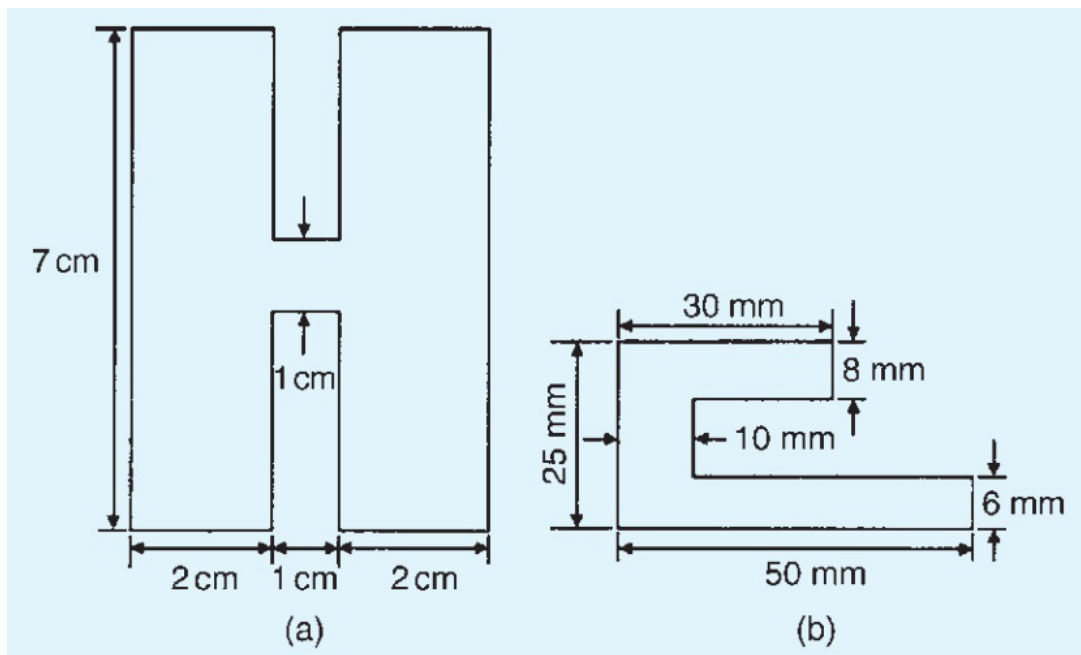


Figure 1.5: Diagram taken from Engineering Mathematics by Bird.

**Answers (a)  $29 \text{ cm}^2$  (b)  $650 \text{ mm}^2$**

### Examples

1. Find the areas of the circles having

- (a) a radius of 5 cm
- (b) a diameter of 15 mm
- (c) a circumference of 70 mm

*Solution:*

- (a) Just using the formula  $A = \pi r^2$ :

- (b) If the diameter is 15 mm then the radius is 7.5 mm:

- (c) The circumference is given by  $2\pi r$ . So  $70 = 2\pi r$ . If we divide by  $2\pi$  we get the radius:

2. A metal pipe has an outside diameter of 5.45 cm and an inside diameter of 2.25 cm. Calculate the cross-sectional area of the shaft

*Solution:* First a diagram:

This shape is known as an *annulus* Now we use the additivity of area:

*Exercises:*

1. Determine the area of circles having a (a) radius of 4 cm (b) diameter of 30 mm (c) circumference of 200 mm. **(Answers: (a) 50.27 cm<sup>2</sup>) (b) 706.9 mm<sup>2</sup> (c) 3183 mm<sup>2</sup>)**
2. An annulus has an outside diameter of 60 mm and an inside diameter of 20 mm. Determine its area. **(2513 mm<sup>2</sup>)**
3. \* If the area of a circle is 320 mm<sup>2</sup> find (a) its diameter, and (b) its circumference. **((a) 20.19 mm (b) 63.41 mm)**
4. Determine the area of the shaded area **(5773 mm<sup>2</sup>)**:

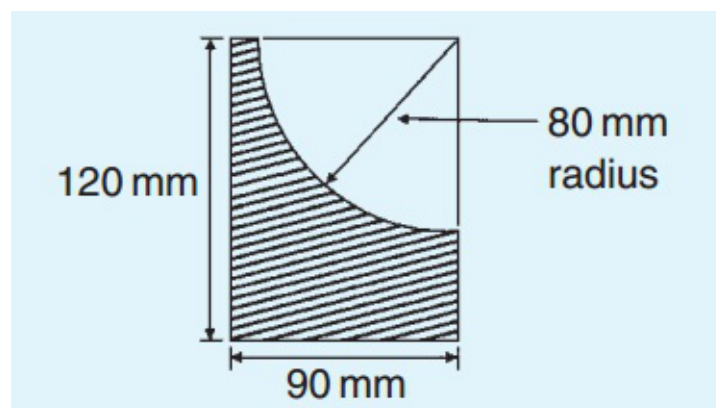


Figure 1.6: Diagram taken from Engineering Mathematics by Bird.



5. An archway consists of a rectangular opening topped by a semi-circular arch as shown in the Fig 1.7. Determine the area of the opening if the width is 1 m and the greatest height is 2 m. (**1.89 m<sup>2</sup>**)

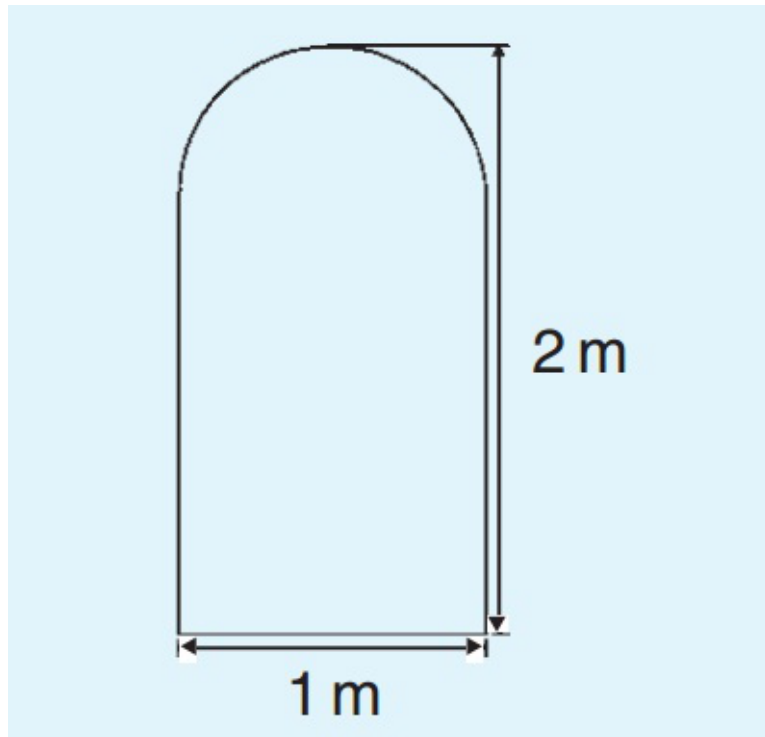


Figure 1.7: Diagram taken from Engineering Mathematics by Bird.

## 1.7.2 Trigonometric Ratios

### Triangle Facts

1. We usually refer to a triangle with ‘points’  $A$ ,  $B$  and  $C$ . In this case the angle ‘at’  $C$  is denoted by  $\angle C$  and the side opposite  $C$  is called  $AB$ :

Although we might also refer to a triangle with angles  $A$ ,  $B$  and  $C$ ; and, perhaps, opposite sides  $a$ ,  $b$  and  $c$ :

2. The angles in a triangle add up to  $180^\circ$ .
3. If the triangle is right-angled we have Pythagoras Theorem:

4. A triangle with two equal sides will also have their opposite angles equal. The converse statement is also true: if two angles in a triangle have equal measure then the length of the sides opposite them will also be equal. Such a triangle is called *isosceles*:

### Example

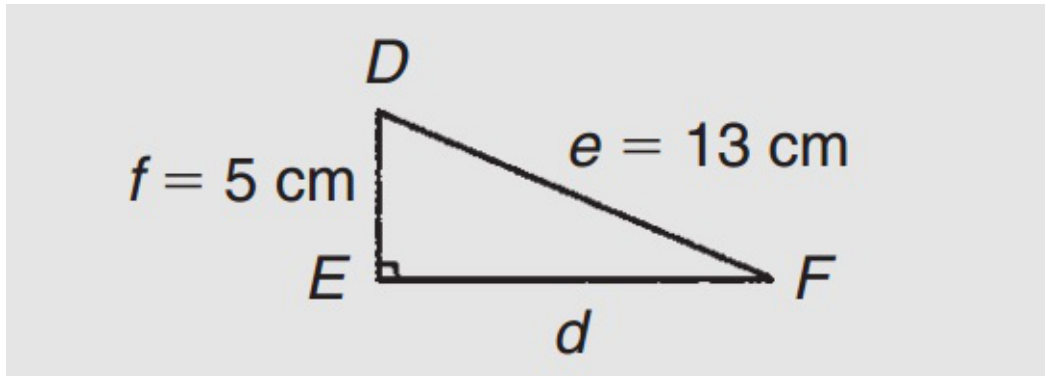


Figure 1.8: Find the length of  $EF$  (Diagram taken from Engineering Mathematics by Bird).

*Solution:* As we have a right-angled triangle (R.A.T.), Pythagoras Theorem applies:

### Exercises:

- \* In a triangle  $CDE$ ,  $D = 90^\circ$ ,  $CD = 14.83$  mm and  $CE = 28.31$  mm. Determine the length of  $DE$ . [24.11 mm]
- Triangle  $PQR$  is isosceles,  $Q$  being a right angle. If the hypotenuse is 38.47 cm find (a) the lengths of sides  $PQ$  and  $QR$ , and (b) the value of  $\angle QPR$ . [(a) 27.20 cm each (b)  $45^\circ$ ]
- A man cycles 24 km due south and then 20 km due east. Another man, starting at the same time as the first man, cycles 32 km due east and then 7 km due south. Find the distance between the two men. [20.81 km]
- \* A ladder 3.5 m long is placed against a perpendicular wall with its foot 1.0 m from the wall. How far up the wall (to the nearest centimetre) does the ladder reach? If the foot of the ladder is now moved 30 cm further away from the wall, how far does the top of the ladder fall? [3.35 m, 10 cm]

5. Two ships leave a port at the same time. One travels due west at 18.4 km/h and the other due south at 27.6 km/h. Calculate how far apart the two ships are after 4 hours.  
[132.7 km]

In a right-angled triangle, special ratios exist between the angles and the lengths of the sides. We look at three of these ratios. Consider the right-angled triangle below:

We can define three trigonometric ratios for this triangle in terms of the side lengths  $a$ ,  $b$  and  $c$ :

We can show that the trigonometric ratios are independent of the size of the triangle<sup>8</sup>

### Example

If  $\cos X = \frac{9}{41}$ , determine the value of the other trigonometric ratios.

*Solution:* First we draw the situation in a *model triangle*<sup>9</sup>:

Now we use Pythagoras to calculate the length of the other side:

---

<sup>8</sup>briefly, *similar pairs triangles* — that is those which have the same angles — have sides that are in proportion. For example, if  $\triangle ABC$  is similar to  $\triangle PQR$  then  $|AB|/|BC| = |PQ|/|QR|$  if the angle opposite  $AB$  is equal to the angle opposite  $PQ$ , etc.

<sup>9</sup>the trigonometric ratios are equal for similar triangles so we can choose whatever size triangle we want to figure out the other ratios.

Now we have

### Exercises

1. In triangle  $ABC$  shown below, find  $\sin A$ ,  $\cos A$ ,  $\tan A$ ,  $\sin B$ ,  $\cos B$  and  $\tan B$

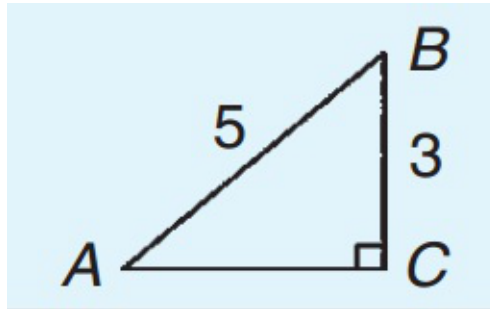


Figure 1.9: Diagram taken from Engineering Mathematics by Bird.

2. For the R.A.T. shown below please find  $\sin \alpha$ ,  $\cos \theta$  and  $\tan \theta$ :

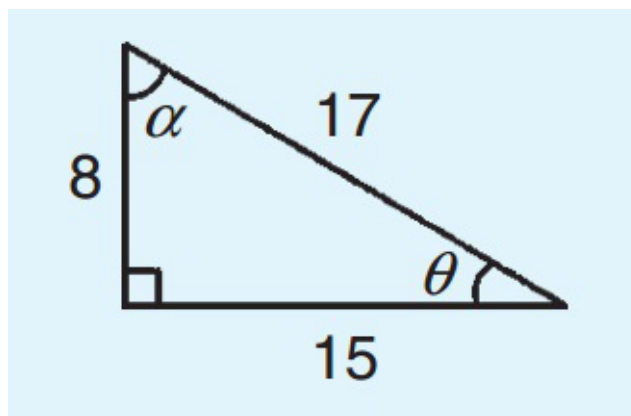


Figure 1.10: Diagram taken from Engineering Mathematics by Bird.

3. \* If  $\cos A = 12/13$  please find  $\sin A$  and  $\tan A$ .

### Three Special Triangles

There are three particular triangles that occur very often:

*Exercises*

1. Using a 30-60 triangle, calculate  $\cos 60^\circ$ .
2. Show that  $\tan 80 \neq 2 \tan 40$  and that  $\cos 50 \neq 2 \cos 25$ .
3. Find a  $\theta$  in each case: (i)  $\tan \theta = \sqrt{3}$  (ii)  $\sin \theta = 1/\sqrt{2}$ .

**‘Solving Right-Angled Triangles’**

In the same way as giving two distinct points defines a line, there are a number of ways of ‘defining’ a triangle.

- (SSS) The lengths of the three sides
- (SAS) The lengths of two sides and the angle between them
- (ASA) The length of one side and two angles
- (RSS) In a R.A.T., any two sides.

This means that if we have enough information about a triangle, then there is only one triangle with those properties, and we can use both Pythagoras Theorem and the trigonometric ratios to find *all* the sides and angles. We will have to have enough information to define a triangle however. Note that *AAA* and *SS* is *not* enough to define a triangle:

**Examples**

1. Suppose  $\triangle$  is a right-angled triangle with an angle  $A = 32^\circ$  and a hypotenuse of length 8 cm. How long is the side opposite  $32^\circ$ ?

*Solution:* First draw a drawing of the situation:

Now choose the trigonometric ratio that links the required side with the known angle and known side and write down this equation and solve.

2. In  $\triangle abc$ ,  $|\angle bca| = 90^\circ$ ,  $|\angle abc| = 34^\circ$ , and  $|ac| = 20$  m. Calculate  $|bc|$ , correct to two decimal places.
3. Calculate the length of the hypotenuse of a triangle with an acute angle  $31^\circ$  with adjacent side of length 10.

*Solution:*

### Practical Applications

Many practical problems in navigation, surveying, engineering and geography involve solving a triangle. Mark on your triangle the angles and lengths you know, and label what you need to calculate, using the correct ratio to link the angle or length required with the known angle or length. Angles of elevation occur quite often in problems that can be solved with trigonometry.

### Examples

1. A ladder, of length 5 m, rests against a vertical wall so that the base of the ladder is 1.5 m from the foot wall. Calculate the angle between the ladder and the ground, to the nearest degree.

*Solution:*

2. When the angle of elevation of the sun is  $28^\circ$ , an upright flagpole casts a shadow of length 6 m. Calculate the height of the pole, correct to one decimal place.

*Solution:*

### Exercises

- \* In  $\triangle xyz$ ,  $|\angle xyz| = 90^\circ$ ,  $|xz| = 200$  cm, and  $|yz| = 150$  cm. Write down the ratio  $\sin \angle yxz$ , in its simplest form. Calculate, to the nearest degree  $|\angle yxz|$  and  $|\angle yzx|$ .
- Solve the following triangles:

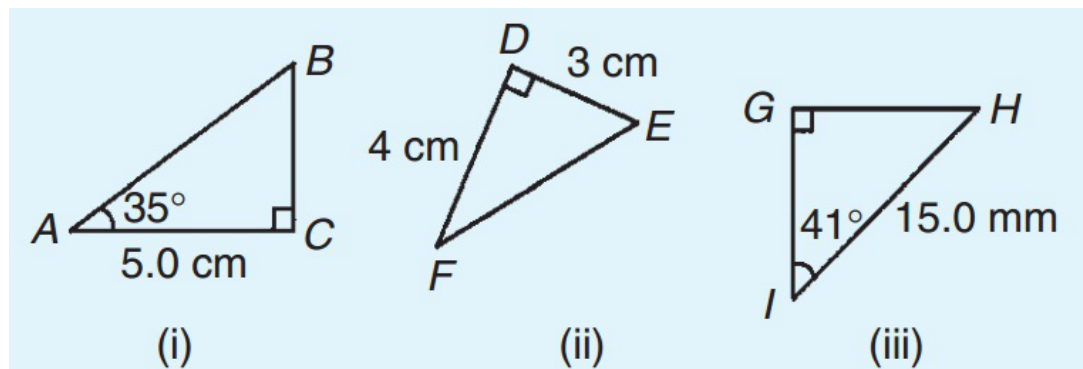


Figure 1.11: Diagram taken from Engineering Mathematics by Bird.

- (i)  $BC = 3.50$  cm,  $AB = 6.10$  cm,  $\angle B = 55^\circ$  (ii)  $FE = 5$  cm,  $\angle E = 53.13^\circ$ ,  $\angle F = 36.87^\circ$  (iii)  $GH = 9.841$  mm,  $GI = 11.32$  mm,  $\angle H = 49^\circ$
- \* A ladder rests against the top of the perpendicular wall of a building and makes an angle of  $73^\circ$  with the ground. If the foot of the ladder is 2 m from the wall, calculate the height of the building.
  - \* An electricity pylon stands on horizontal ground. At a point 80 m from the base of the pylon, the angle of elevation of the top of the pylon is  $23^\circ$ . Calculate the height of the pylon to the nearest metre.
  - \* If the angle of elevation of the top of a vertical 30 m high aerial is  $32^\circ$ , how far is it to the aerial?

## 1.8 Approximation\*

Being able to estimate an answer to a problem before making an exact calculation can be an important skill. One example where an estimate can help is in eliminating arithmetic slips. For example if you hope to find 13% of 801 and your answer is 10.413 then you have clearly made a mistake because

Another reason why we might make an approximation would be to make a forecast. For example, suppose that you spend 5 minutes every night writing a novel. Can you write a 1,000 page novel done in ten years in this fashion? Well, approximately:

It happens quite frequently in both pure and applied mathematics that there are problems which can't be solved exactly. An example from pure mathematics is *Abel's Impossibility Theorem* which states that

but we still might want to solve such an equation. We have numerical techniques to find approximate solutions.

A very simple example from 'applied mathematics' is the question

Theoretically we could answer this but...

In practise we will be approximating quantities by numbers and we will write

read *the quantity  $Q$  is approximated by  $x$* , or *the quantity  $Q$  is approximately equal to  $x$* .



**Examples**

1. We can approximate quantities such as

$$\frac{48.27 + 12.146}{14.82 - 3.02}.$$

by replacing the numbers by whole numbers:

We can compare this with the exact answer which turns out to be 5.12. Here  $5 \approx 5.12$ .

2. The average height of the front row  $\bar{x}$  approximates the average height of the entire class  $\mu$ .
3.  $\pi \approx 3.14$  and  $\pi \approx \frac{22}{7}$ .
4. When an angle  $A$  is measured in *radians* we have

Now what are radians?

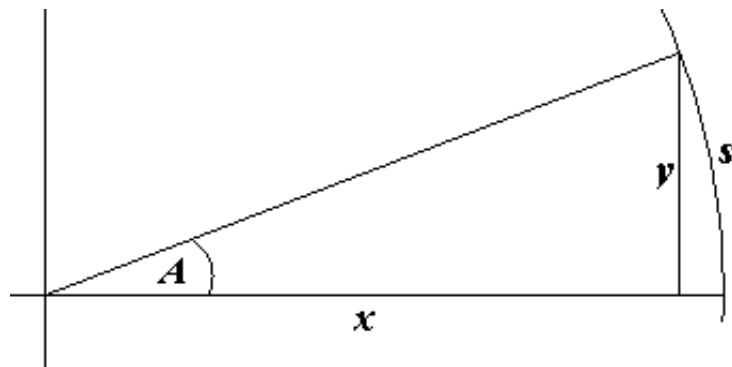


Figure 1.12: An angle  $A$  at the centre of the unit circle

$A$  measured in radians is given by:

$\sin A$  is given by:

Now suppose  $A$  is small, that is  $A \approx 0$ ;

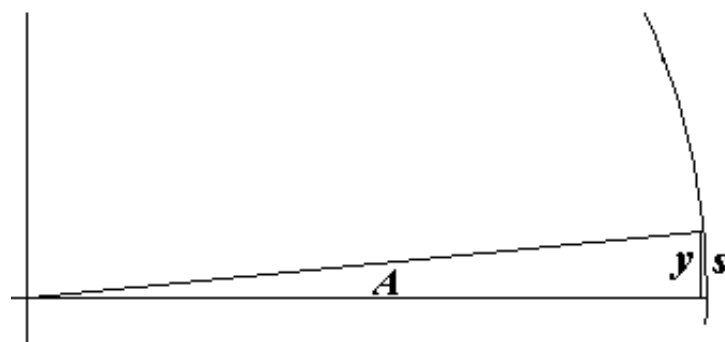


Figure 1.13: When  $A \approx 0$ ,  $s \approx y$  hence  $A \approx \sin A$

5. Given a curve, what is the slope of the tangent at a point?

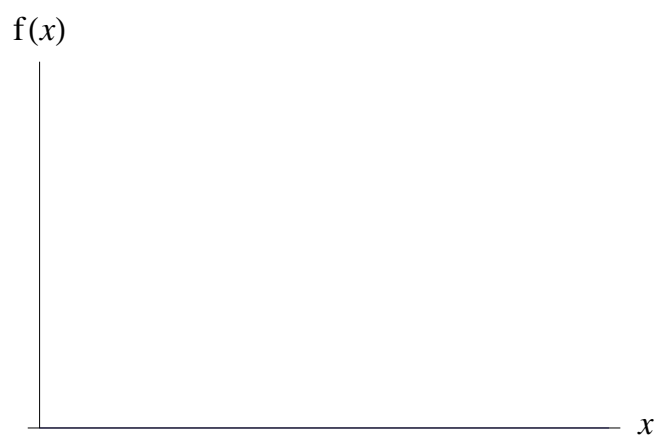


Figure 1.14: We approximate the slope of the tangent to the curve at the point  $(3, f(3))$  by the slope of the line passing through  $(3, f(3))$  and a point close to  $(3, f(3))$ .

6. Recall our approximation of the area of a circle by the area of a polygon.

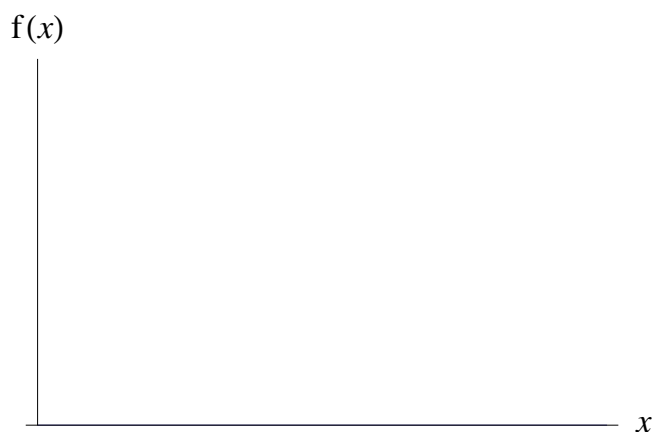


Figure 1.15: We approximate the area under the curve using rectangles.

### 1.8.1 Error

A good question is when we have made an approximation is

Usually a branch of mathematics called *numerical methods* comes up with methods of approximation and the branch of mathematics called *numerical methods* answers the above question. Remarkably in a lot of cases we can know in advance that an approximation can't be *too bad*. All of this

#### Examples

1. We can approximate the area below by using ‘midpoint’ rectangles.

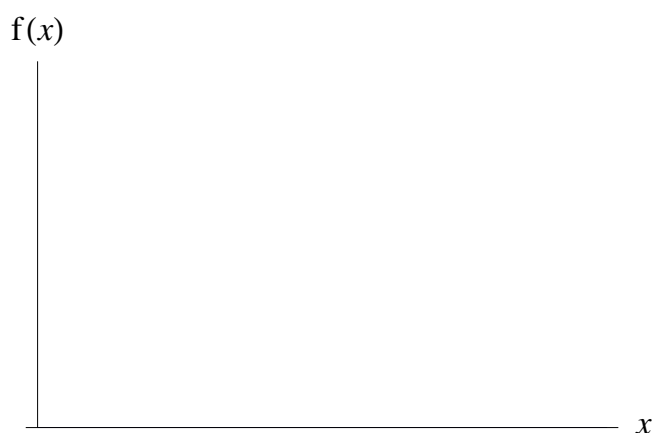


Figure 1.16: This approximation submits readily to numerical analysis.

We can show (and know in advance in fact) that if 29 rectangles are used, then the difference between the area and the approximation is at least less than 0.0001.

2. When doing mortgage payment calculations quantities such as

occur. By rounding off the  $i/12$  term, say using the approximation

$$0.006 \approx 0.0058\dot{3} = \frac{0.07}{12}$$

the errors can ‘propagate’ in significant ways. For example

$$\left(1 + \frac{0.07}{12}\right)^{360} \approx 8.1165$$

$$(1 + 0.006)^{360} \approx 8.61535$$

It doesn’t sound too much but for the example of a thirty year mortgage for E300,000 this little but manifests as a difference of more than E40 a month, that is over E485 per year and more than E14,500 over the lifetime of the mortgage. All over a fourth decimal place!! Beware of this so called ‘round-off error’ and where possible use exact numbers.

For us here in Essential Maths we are just going to look at round-off error. To talk about this we need a number of definitions.

### Absolute Error

Suppose that  $A$  is an approximation to a quantity  $Q$ . Then the *absolute error* of the approximation is given by

### Example

Find the absolute error when we round-off to approximate  $4.81 + 7.25$ .

*Solution:* Our round-off gives the approximation  $5 + 7 = 12$ . The exact value of  $4.81 + 7.25 = 12.06$  so the absolute error is

Suppose that you are approximating the heights of a statue and a cliff:

In both cases the absolute error is the same but clearly we feel that the statue approximation is worse. We believe this because our statue approximation overshoot the true value by 50% and the cliff approximation was under by only 2%.

As another example, note that whenever you use a ruler that there is a possible error of 1 mm in your measurement as the ruler is only graduated to millimetres. Using a ruler to measure the width of a iPhone is clearly worse than using it to measure the dimension of a television. To capture this we define the relative error (which is usually converted to a percentage).

### Relative Error

Suppose that  $A$  is an approximation to a quantity  $Q$  with absolute error  $\Delta Q$ . Then the *relative error* of the approximation is given by

### Example

Use rounding to approximate

$$\frac{2.9 \times 11.4 + 2.196}{21.018 \div 3.1}.$$

Also calculate the absolute and relative error.

*Solution:* First off we implement the rounding:

Now using a calculator we can see that the exact value is 5.2. Therefore the absolute error is 0.2. We find the relative error:

which is approximately (!) 3.846%.

*Exercise:* In each case, use rounding to calculate an approximate answer, find the absolute error and the relative percentage error.

$$(i) 8.73 - 5.82 \qquad (ii) 3.8 \times 5.3$$

$$(iii) 48.36 \div 7.8 \qquad (iv) 4.15(11.1 - 4.3)$$

$$(v) 2.9 \times 4.1 + 3.04 \qquad (vi) \frac{15.332 + 8.94}{9.1 - 3.18}$$

$$(vii) \frac{3.95 \times 8.42 + 3.953}{1.8 \times 4.3 + 1.2} \qquad (viii) \frac{30.317}{\sqrt{24.7009}}$$

**Selected Answers:** (iii) 6, 0.2, 3.23% (vi) 4, 0.1, 2.44%

## 1.9 Statistics

Statistics is the science of collecting, studying, analysing and making judgements based on data. The subject divides broadly into two branches: *descriptive* and *inferential statistics*.

*Descriptive statistics* involves describing the main features of a collection of data. Descriptive statistics are distinguished from inferential statistics in that descriptive statistics aim to summarize a data set, rather than use the data to learn about the population that the data are thought to represent. Activities include graphing the data (putting a spin on things) and calculating key summary statistics such as the *average* or the *standard deviation*. The aim of descriptive statistics is to summarise the data. We could also include methods of collection of data.

*Inferential statistics* is the process of drawing conclusions from recorded data. Typically the data is not complete in that there are measurement errors or the data is just a sample from a much larger population. The outcome of statistical inference may be an answer to the question “what should be done next?”, where this might be a decision about making further experiments or surveys, or about drawing a conclusion before implementing some organizational or governmental policy. Note that descriptive statistics precedes inferential statistics in the sense that data is necessary for inferential statistics and it is descriptive statistics that provides this. Therefore, for MATH6000, we are interested in Descriptive statistics.

There are many different types of data and it is useful to be aware of this. As a quick example, the heights of the MATH6000 class is numerical and hence ordered data. However, the hometowns of the MATH6000 class is not numerical and not ordered so that these are fundamentally different types of data that require different presentations and summary statistics in order to summarise them.

*Nominal Data* — is data that cannot be ordered, for example the eye colours of ten children or the marital status of groups of individuals as single, married, widowed or divorced.

*Ordinal Data* — is data that can be ordered, for example satisfaction levels in a consumer survey: *very happy, happy, indifferent, unhappy, very unhappy*.

*Numerical Data* — is exactly what you think it is. Numerical data has the advantage of having a natural ordering. More ‘mathematical’ methods of data presentation can be used to demonstrate numerical data. An example: the heights of the MATH6038 class.

*Whole Number Data* — again, exactly what you think it is. Numerical data where the only possibilities are whole numbers; for example, the number of employees in a business.

*Continuous Data* — data that can take on a infinite number of values that are arbitrarily close to each other; for example, the time taken to serve customers could be anything from zero to an infinite number of seconds.

### 1.9.1 Average

When you hear the word average we often think of the mean. For example, if Ann is 28, Betty is 31 and Carolone is 31 we say that their average age is 30. However this is not the only ‘average’, To be more careful, an average is any ‘measure of central tendency’ — or if you will the ‘middle value’. We might want to talk about the average of nominal data: for example who does the average Irish male support in soccer?

#### Mean

The mean is the ‘usual’ average that we usually use. It is used for numerical data only. It is calculated by adding up the all the data points and dividing by the number of data points:

Let  $x = \{x_1, \dots, x_n\}$  be a collection of data. The *mean-average* of  $x$ ,  $\bar{x}$ , is given by

$$\begin{aligned}\bar{x} &= \frac{x_1 + \dots + x_n}{n} \\ &= \frac{\sum_{i=1}^n x_i}{n}.\end{aligned}$$

Consider the expression  $x_1 + x_2 + \dots + x_n$ . In statistics and in mathematics more generally we will often like to add up a number of of numbers; quite often in fact. To handle this we use what is called *sigma*-notation. The symbol ‘ $\Sigma$ ’ means add up. So for example

means add up all the ages in this class. Sometimes there will be a natural order on what we want to count up. For example the sum of the first five square numbers:

If the  $i$ -th thing that we want up can be given in terms of  $i$ , say  $a_i$ , then for the sum we can write something like

This means ‘add up the pattern  $a_i$ , starting at  $i = 1$  and up to  $i = n$ . For example the sum of the first five square numbers could be denoted by

In the case where we want to find the mean-average of the numbers  $x = \{x_1, x_2, \dots, x_n\}$ , we add them up and divide by the number of numbers — in this case  $n$ . Now we could write this as

$$\bar{x} = \frac{\sum_{i=1}^n x_i}{n}, \tag{1.2}$$

where  $\sum_{i=1}^n x_i$  means add up all the  $x_i$  from  $x_1$  to  $x_n$ .

**Mode**

The mode(s) of a data set is (are) those data points which occur most often and is often the best average to use for nominal data. What does ‘a la mode’ mean? As an example which county do you ‘live in’:

**Median**

The median is an average that is used for ordinal data. It is primarily used for a finite set of discrete data, it is that data point which divides the data into a ‘bottom half’ and ‘lower half’.

**Example**

Find the median of the following data:

15, 4, 46, 23, 57, 3, 5, 34, 57, 243, 5.

*Solution:* First we order the data:

Hence 23 is the median.

When the number of data points is even, the median is given as the midpoint of the two ‘middle’ elements.

**Example**

Find the median of

2, 344, 23, 555, 643, 2, 542, 57

*Solution:*



**Example**

Determine the mean, median and mode for the set  $\{2, 3, 7, 5, 5, 13, 1, 7, 4, 8, 3, 4, 3\}$ .

*Solution:* The mean is found by adding the numbers together and dividing by the number of numbers:

To find the median first write the numbers in ascending order. The middle number is the median.

The mode is the element which occurs most often:

Which average should you use? By and large you have a choice but hard and fast rules are:

1. Use the mean unless there are large *outliers* in the data set.
2. If there are large outliers in the data set use the median.

When the data is symmetric about its mode all three will agree.

**Example: Average Industrial Wage**

The distribution of income looks like

A few millionaires will skew the mean to the right — so that more than half the population don't then make “the average industrial wage”. So depending on what your vested interest is, you may use the mean or the median (or the mode). This is a demonstration of the famous quip “*There are only three kinds of lies. Lies, damned lies and statistics*”. If you are disingenuous with statistics you can support a lot of marginal arguments.

*Exercises:* In Problems 1 to 4, determine the mean, median and modal values for the sets given.

1.  $\{3, 8, 10, 7, 5, 14, 2, 9, 8\}$  [**mean 7.33, median 8, mode 8**]
2.  $\{26, 31, 21, 29, 32, 26, 25, 28\}$  [**mean 27.25, median 27, mode 26**]
3.  $\{4.72, 4.71, 4.74, 4.73, 4.72, 4.71, 4.73, 4.72\}$  [**mean 4.7225, median 4.72, mode 4.72**]
4.  $\{73.8, 126.4, 40.7, 141.7, 28.5, 237.4, 157.9\}$  [**mean 115.2, median 126.4, no mode**]

### 1.9.2 Deviation

Note that data can be spread out or quite concentrated (around the average). For example, consider soccer players vs rugby players:

#### Range

The *range* is simply the difference between the largest and smallest elements in the data set. For example, Munster have Paul O'Connell (198 cm) and Peter Stringer (170 cm) — a range of at least 28 cm.

#### Interquartile Range

In the same way that the median divides the data into the bottom 50% and the top 50%, the interquartile range is the *middle* 50%. To find the interquartile range we first divide the data into four parts. For example:  $\{2, 3, 4, 5, 5, 7, 9, 11, 13, 14, 17\}$  is divided into four by 4, 7 and 13. We call 4 the first quartile,  $Q_1$ , 7 the second quartile AKA the median,  $Q_2$  and 14 the third quartile,  $Q_3$ . The interquartile range is  $Q_3 - Q_1$ . Alternatively specify  $Q_1 \rightarrow Q_3$ . There are slightly different conventions for choosing the quartiles so we will just introduce the idea of the 'middle 50%' and leave it at that.

#### Standard Deviation

The *standard deviation* is a measure the of spread of the data. Standard Deviation is a kind of average deviation/distance from the mean. A small standard deviation means that the data is concentrated around the 'average' (soccer), while a large standard deviation means that the data is more spread out (rugby).

Let  $S = \{x_1, \dots, x_n\}$  be numerical data with a mean-average of  $\bar{x}$ . Then the *standard deviation*,  $\sigma$ , is given by:

$$\sigma = \sqrt{\frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n}} \quad (1.3)$$

The Greek letter is pronounced *sigma*.  $\Sigma$  is the capital Greek S and  $\sigma$  is the small s. We haven't actually said what the square root of a number is even though a lot of people have a good idea. Similarly to how we defined  $1/n$  as the solution of the equation  $nx = 1$ , we could define  $\sqrt{n}$  as the *positive* solution to the equation  $x^2 = n$  — that is the square root of  $n$  is the positive number that when squared equals  $n$ ; e.g.  $\sqrt{49} = 7$  as  $7^2 = 49$ .

In terms of actually calculating this object by hand let us dissect this formula

$$\sigma = \sqrt{\frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n}}.$$

It looks like a bit of a mouthful but let's take our time.

1. If we know that the data is  $\{x_1, x_2, \dots, x_n\}$  we can at least calculate the mean-average,  $\bar{x}$ . It appears in the formula so we will need it.
2. Next we could calculate the *deviations*,  $(x_i - \bar{x})$  from the mean
3. We are going to add up all the  $(x_i - \bar{x})^2$  so we may as well square these deviations:
4. Now we need to add them up —  $\sum (x_i - \bar{x})^2$ :

Note  $\sum (x_i - \bar{x})^2$  does *not* mean add up all the deviations  $(x_i - \bar{x})$  and *then* square them. If we wanted to do that we would have written  $\left(\sum (x_i - \bar{x})\right)^2$ .

5. Now divide by the number of elements in the data set to get the  $\frac{\sum (x_i - \bar{x})^2}{n}$ .
6. Finally take the square root of this using a calculator.

We don't have to follow these exact six steps but when we use the formula they should happen naturally.

**Example**

Determine the standard deviation from the mean of the set of numbers:  $\{5, 6, 8, 4, 10, 3\}$ , correct to 4 significant figures.

*Solution:* First we must calculate the mean-average

Now we calculate the deviations... and square them

Now we can use the formula:

*Exercises:* Questions 1-3 taken from Engineering Math by Bird.

1. Determine the standard deviation from the mean of the set of numbers:  $\{35, 22, 25, 23, 28, 33, 30\}$  correct to 3 significant figures. [**4.60**].
2. The values of capacitances, in microfarads, of ten capacitors selected at random from a large batch of similar capacitors are:  $\{34.3, 25.0, 30.4, 34.6, 29.6, 28.7, 33.4, 32.7, 29.0, 31.3\}$ . Determine the standard deviation from the mean for these capacitors, correct to 3 significant figures. [**2.83 microfarads**].
3. The tensile strength in megapascals for 15 samples of tin were determined and found to be  $\{34.61, 34.57, 34.40, 34.63, 34.63, 34.51, 34.49, 34.61, 34.52, 34.55, 34.58, 34.53, 34.44, 34.48, 34.40\}$ . Calculate the mean and standard deviation from the mean for these 15 values, correct to 4 significant figures. [**mean 34.53 MPa, standard deviation 0.07474 MPa**]

# Chapter 2

## Indices and Logarithms

*By relieving the brain of all unnecessary work, a good notation sets it free to concentrate on more advanced problems, and in effect increases the mental power of the race.*

Alfred North Whitehead

### 2.1 Indices

#### Definition

Let  $a \in \mathbb{R}$  and  $n \in \mathbb{N}$ . Then

$$a^n = \underbrace{a \times a \times \cdots \times a}_{n \text{ multiplicands}} \quad (2.1)$$

We can refer to  $a^n$  as “ $a$  to the power of  $n$ ” and say  $a^n$  is “ $a$  power of  $a$ ”. Finally  $a^2$  is a *square* ( $a$ -squared),  $a^3$  is a *cube* ( $a$ -cubed) and  $a^n$  is an  $n$ -th power. For example,  $3^7$  is an example of a power of 3.

### Basic Properties of Powers

Now that we have a shorthand way of writing repeated multiplication we must investigate how they combine together? What happens when we multiply together two powers? If we divide one power by another? What about a power of a power? What if we took two numbers, multiplied them together and took a power of that? We will see that answering these questions will raise more questions.

Before we start our investigation I want you to be aware of the following:

#### Definition

Let  $a \in \mathbb{R}$  and  $m, n \in \mathbb{N}$  (both  $m$  and  $n$  are natural numbers). Then  $a^m$  and  $a^n$  are *like powers*. Now nobody apart from me in a second calls this the Golden Rule of Powers, but it'll become clear very soon why it is indeed the case:

(The Golden Rule of Powers)

**Multiplication of Like Powers**

Consider  $7^2$  and  $7^3$ . What is really happening when we multiply them together?

There is nothing special about  $7^2$  and  $7^3$  here. Indeed any real number  $a$  and natural numbers  $m, n$  could replace the roles of 7, 2 and 3:

*Proof.*

Try to get any kind of similar rule for unlike powers and you'll see what I mean by the Golden Rule.

**Example**

Write  $4 \times 4 \times 16$  as a power of 4.

*Solution:*

**Division of Like Powers**

Consider  $10^5$  and  $10^2$ . What is really happening when we divide, say,  $10^5$  by  $10^2$ :

Again there is nothing special about  $10^5$  and  $10^2$  here. Again let  $a \in \mathbb{R}$  and  $m, n \in \mathbb{N}$ :

*Proof.*

**Example**

Write

$$\frac{10^2 \times 10^5}{100}$$

in the form  $10^n$  for  $n \in \mathbb{N}$ .

*Solution:*

**Repeated Powers?**

What is  $(9^3)^4$ ?

Again this is a general result. Let  $a \in \mathbb{R}$  and  $m, n \in \mathbb{N}$ :

*Proof.*

**Example**

Write  $(4^3)^5$  in the form  $2^n$  for  $n \in \mathbb{N}$ .

*Solution:* First write 4 as a power of 2:

Now we can write  $(4^3)^5$  as a power of two:

**Powers of a Product**

What about  $(2 \times 5)^3$ ?

Again this is a general result. Let  $a \in \mathbb{R}$  and  $m, n \in \mathbb{N}$ :

*Proof.*

### Example

Let  $x \in \mathbb{R}$ . Write  $4x^2$  as a square.

*Solution:*

Another similar case is that of a power of a fraction. Now our work from the first chapter should pay dividends. We know how to multiply fractions together so we have

## Zero Powers, Negative Powers & Fractional Powers

You may have noticed that our definition for indices/ powers has only defined powers when the exponent is a natural number. What about the following:

(P1)  $2^0?$

(P2)  $3^{-8}??$

(P3)  $4^{1/2}???$

(P4)  $5^{\sqrt{2}}?????$

Note that every natural number is an integer and that every integer is a fraction (e.g.  $-11 = -11/1$ ). What we will do is construct definitions for zero, negative numbers and fractions that *extend the definition for natural numbers and make the laws of indices consistent*.



In other words we will choose a definition for fractional powers<sup>1</sup> such that for all  $a \in \mathbb{R}$ ,  $m, n \in \mathbb{Q}$  the following make sense:

If it makes sense for fractions it will also make sense for integers and natural numbers as these are fractions also. Note at this point we will not define all real powers. To define, for example,  $5^{\sqrt{2}}$ , in such a way that the definition extends that of natural numbers and the laws of indices still hold, is a bit trickier. The first odd thing about this is that  $\sqrt{2} \notin \mathbb{Q}$  - that is  $\sqrt{2}$  *cannot be written as a ratio of whole numbers*. The two ways that I know how to define  $5^{\sqrt{2}}$  properly involve quite a bit of calculus or quite a bit of sequences and series. We don't worry too much about this - we know it can be done - you can work away with any real exponent as long as you follow the laws of indices!

## Zero Powers

We have already stated the 'fact' about all non-zero numbers that:

Now consider  $2^5/2^5$ , using the second law of indices:

Once again this is a result independent of 2 (but note  $0^n/0^n = 0/0$  which doesn't make sense... we don't define  $0^0$  - it won't come up anywhere either). Let  $a \in \mathbb{R}$ ,  $a \neq 0$ :

## Example

Suppose that  $a, b \in \mathbb{R}$  such that  $a = b^3$ . Simplify

$$\frac{a^2}{(b^2)^3}$$

*Solution:* First we replace  $a = b^3$ :

---

<sup>1</sup>will be for  $a > 0$  only

## Negative Powers

What is  $3^{-8}$ ? Using the first law of indices and the definition of  $3^0$ :

We know the story by now. Let  $a \in \mathbb{R}$ ,  $a \neq 0$ . Let  $n \in \mathbb{N}$ :

### Example

Evaluate  $10^{-1} \times 100$ . What does this tell you about  $x^{-1}$ ?

*Solution:*

This tells us a few things. Firstly if you want to divide by  $x$  you can multiply by  $x^{-1}$ . We already saw that if you want to divide by  $x$  multiply by  $\frac{1}{x}$ ... but then of course  $x^{-1} = \frac{1}{x^1} = \frac{1}{x}$ .

There is actually another way of explaining why zero powers are one that might also shed light on how we do negative powers. We start with the fact that multiplying by one doesn't change anything. So given any number you want, say 8 we can write

In fact we can think of *always starting at one*. Now we can make products or powers of eight as follows:

So to increase powers you multiply by 8. Now coming back in the other direction from  $8^5$ .

To go decrease powers you divide by 8. Hence if we want to go from  $8^1 \rightarrow 8^0$  we divide by eight a final time:

If we go back to the idea that we should *always starting at one* we see that  $8^0$  starts at 1 and... well we don't do anything there are no eights to multiply it and we are stuck at one! This is called an *empty product*. Utterly bizarre examples include

*Let  $S$  be the set of all students who are at least ten foot tall. Now take the ages of all these students and multiply them together. The answer is one.*

Such is the mystery of the so-called *empty-set*!

If we continue this policy of dividing reduces powers we can see why the negative powers make sense:

### Fractional Powers

What is  $4^{1/2}$ ? Now using the third law of indices:

So we see that  $4^{1/2}$  is the number that when squared gives 4... Now to keep things easy for us, we will only consider fractional powers of *positive real numbers* (i.e. numbers bigger than zero. We write  $a > 0$  for “*a is bigger than zero*”). Try and figure out what  $(-1)^{1/2}$  if you want to see what I mean. Hence let  $a \in \mathbb{R}$ ,  $a > 0$ :

What about  $5^{1/6}$ ? Similar story, using the third law of indices:

So we see that  $5^{1/6}$  is the number when brought to the power of 6 gives you 5. This is known as the *sixth root* of 5 which we denote by  $\sqrt[6]{5}$ , and extends naturally from “*the*” square root (e.g. Let  $x \in \mathbb{R}$ ,  $x > 0$ . Then  $\sqrt{x} = \sqrt[2]{x}$ ). Again, let  $a \in \mathbb{R}$ ,  $a > 0$  and  $n \in \mathbb{N}$ :

What about  $6^{3/5}$ ? If we agree that whatever it is, it has to agree with, say the third law of indices, so that:

A subtlety of this is that we have  $3 \times (1/5) = (1/5) \times 3$ :

Finally, let  $a \in \mathbb{R}$ ,  $a > 0$  and  $m/n \in \mathbb{Q}$ :

**Example**

Putting our skills together!! Evaluate

$$\sqrt{\left(\frac{9^2 9^3}{3^4}\right)^5 3^{-6}}$$

*Solution:*

*Exercises*

1. Evaluate:

$$\begin{array}{lllll} (i) 36^{1/2} & (ii) 125^{1/3} & (iii) 16^{1/4} & (iv) 1000^{1/3} & (v) 1000^{2/3} \\ (vi) 2^{-5} & (vii) 5^{-2} & (viii) 8^{2/3} & (ix) 4^{-1/2} & (x) 4^{-1} \end{array}$$

2. Write these as  $a/b$ , where  $a, b \in \mathbb{Z}$ :

$$(i) 2^{-2} \quad (ii) (1/4)^{1/2} \quad (iii) 32^{-3/5} \quad (iv) 16^{-1/4} \quad (v) 27^{-2/3}$$

3. Which of each pair is greater?:

$$\begin{array}{lll} (i) 2^5 \text{ or } 5^2 & (ii) 4^{1/2} \text{ or } (1/2)^4 & (iii) 2^{-1/2} \text{ or } (-1/2)^2 \\ (iv) (1/2)^6 \text{ or } (1/2)^7 & (v) 7^2 \times 7^3 \text{ or } (7^2)^3 & \end{array}$$

4. Solve for  $k$ :

$$\begin{array}{ll} (i) 2^k = 4 & (ii) 4^k = 64 \\ (iii) 8^k = 64 & (iv) 2^k = 128 \\ (v) 4^k = 2 & (vi) 25^k = 5 \\ (vii) 8^k = 4 & (viii) 1000^k = 100 \\ (ix) 32^k = 16 & (x) 8^k = 1/2 \end{array}$$

5. Write each of these in the form  $a^p$ , where  $p \in \mathbb{Q}$ :

$$\begin{array}{ll} (i) a^7 \div a^2 & (ii) a^7 \times a^2 \\ (iii) (a^7)^2 & (iv) \sqrt{a} \\ (v) \sqrt[3]{a} & (vi) \sqrt{a^7} \\ (vii) 1/a^3 & (viii) 1/\sqrt{a} \\ (ix) (\sqrt{a})^3 & (x) 1/(a\sqrt{a}) \end{array}$$

6. Find the value(s) of  $k$  in each case.

$$\begin{array}{lll} (i) 2^{3k} \cdot 2^k = 16 & (ii) 2^{2k+1} = 8^2 & (iii) 16^4 \cdot 8 = 2^k \\ (iv) 5^{2k+1} = 125 & (v) 3^{k+1} = 9^{k-1} & (vi) 10^k \cdot 10^{2k} = 1,000,000 \\ (vii) 2^{(k^2)} \cdot 2^k = 64 & (viii) 10^{(k^2)} \div 10^{2k} = 1,000 & \\ (ix) 5^k = 1/125 & (x) 2^k = 1/(4\sqrt{2}) & \end{array}$$

7. Write these as  $10^p$ :

$$(i) (10^9)^2 \quad (ii) 10^9 \times 10^2 \quad (iii) (10^5 \cdot 10^3)^2 \quad (iv) (\sqrt{10})^{100}$$

8. State if these are true or false:

$$\begin{array}{lll} (i) 2^3 \cdot 5^3 = 10^3 & (ii) (x^6)^7 = x^{13} & (iii) (3\sqrt{3})^3 = 3^{4.5} \\ (iv) (10\sqrt{10})^4 = 10^6 & (v) 2^7 \times 3^7 = 5^7 & (vi) (-3)^4 = -3^4 \end{array}$$

9. Write  $\sqrt[3]{\sqrt{2}}$  as  $2^p$  for  $p \in \mathbb{Q}$ .

10. If

$$\frac{((2\sqrt{2})^2)^3}{(2\sqrt{2})^2(2\sqrt{2})^3} = 2^n,$$

find the value of  $n$ .

### 2.1.1 Indices: Harder Examples

Here we present some more problems on indices to get us ready for the more difficult problems on the exercise sheet and test 3. In reality this will be the first time that we do algebra. These are *not* letters. We don't specify whether they are real numbers, fractions, integers or just natural numbers. To avoid problems with division by zero and finding roots of negative numbers we will declare at this point that

#### Algebraic Notation

We often use  $x$  and  $y$  to stand for some real numbers ( $x, y \in \mathbb{R}$ ). We are inclined to multiply real numbers together. How do we represent  $x$  multiplied by  $y$ :

As you can imagine this can lead to all kinds of confusion. Instead we use  $xy$  to signify ' $x$  multiplied by  $y$ '. This is exactly why I don't like so-called 'mixed fractions'.

1. Simplify  $a^3b^2c \times ab^3$ .

*Solution:* The first thing to remark here is that we have a number of numbers multiplied together:

We know that the order of multiplication of numbers is not important ( $6 \times 5 = 5 \times 6$ ), therefore we may write:

How many  $as$ ,  $bs$ ,  $cs$ ?

2. Simplify

$$a^{1/3}b^2c^{-2} \times a_{1/2}b^{1/2}c^3.$$

*Solution:* Once again we must recognise this as

Now we have seen that two multiply like powers we add the powers. For the  $c^{-2}c^3$  we can either think in terms of adding the powers or else thinking  $c^{-2} = \frac{1}{c^2}$ :

3. Simplify

$$\frac{p^3 q^2 r^5}{p q r^{-1}}.$$

Also evaluate for  $p = 3$ ,  $q = \frac{1}{8}$  and  $r = 2$ .

*Solution:* Now recalling how we defined fraction multiplication:

Handling the  $ps$  and  $qs$  isn't too hard. However the  $\frac{r^5}{r^{-1}}$  isn't very natural so we go back to thinking that to divide powers you subtract:

Now, as the question requests, evaluate at  $p = 3$ ,  $q = \frac{1}{8}$  and  $r = 2$ :

4. Simplify

$$\frac{a^{1/2} b^3 c^{2/3}}{a^{1/4} b^{3/2} c^{1/6}}.$$

Also evaluate when  $a = 16$ ,  $b = 9$  and  $c = 4$ .

*Solution:* Again recalling fraction multiplication:

Now none of these are natural so we go back the abstraction  $\frac{x^m}{x^n} = x^{m-n}$ :

Now we know that  $x^{1/2} = \sqrt{x}$  but what about  $b^{3/2}$ :

5. Simplify

$$(a^6)^{1/2} (b^3)^3.$$

*Solution:* Now we know that  $(b^3)^3$  is three copies of three copies of  $bs$  multiplied together so equal to  $b^9$ . We will have to use the abstraction of this  $(x^m)^n = x^{mn}$  to calculate  $(a^6)^{1/2} = a^{6 \times \frac{1}{2}} = a^3$ . So our answer is  $a^3 b^9$ .

6. Simplify

$$\frac{(at^2)^3}{(at^{1/4})^4}.$$

*Solution:* For this we should look at the top and bottom separately first like we do whenever we have a fraction of complicated numbers. The  $(at^2)^3$  is

Similarly the  $(at^{1/4})^4$  is

Now  $t^{1/4} = \sqrt[4]{t}$  so by definition  $(t^{1/4})^4 = t$ . Putting this together we have

7. Simplify

$$\frac{x^2y^2z^{1/2}}{(x^{3/2}yz^{5/2})^2},$$

but express in positive powers only.

*Solution:* Firstly we deal with the denominator:

Now separate using the multiplication of fractions:

However the question asks for positive powers only

8. Simplify

$$\frac{(a^3b^{1/3})(\sqrt{a}\sqrt[3]{b^2})}{(a^5b^3)^{1/2}}.$$

*Solution:* Again simplifying and below separately:



Now putting the  $a$ s with the  $a$ s and the  $b$ s with the  $b$ s:

Now looking at the bottom, and using the abstraction  $(ab)^n = a^n b^n$ :

Now putting everything back together:

Of course there is nothing stopping us simplifying the top and the bottom at the same time.

*Exercises:* These exercises are taken from page 41 of Engineering Mathematics (Fifth Ed.) by John Bird.

1. Simplify  $(x^2 y^3 z)(x^3 y z^2)$  and evaluate when  $x = \frac{1}{2}$ ,  $y = 2$  and  $z = 3$ .

**Answer:**  $x^5 y^4 z^3$  and  $\frac{27}{2} = 13.5 = 13\frac{1}{2}$ .

2. Simplify  $(a^{1.5} b^1 c^{-3})(\sqrt{abc}^{-1/2} c)$  and evaluate when  $a = 3$ ,  $b = 4$  and  $c = 2$ .

**Answer:**  $(a^2 b^{1/2} c^{-2})$  and  $\frac{9}{2} = 4.5 = 4\frac{1}{2}$

3. Simplify

$$\frac{a^5 b c^3}{a^2 b^3 c^2},$$

and evaluate when  $a = 1.5$ ,  $b = 0.5$  and  $c = \frac{2}{3}$ .

**Answer:**  $a^3 b^{-2} c$  and **9**

4. Simplify

$$\frac{x^{1/5} y^{1/2} z^{1/3}}{x^{-1/2} y^{1/3} z^{-1/6}}.$$

**Answer:**  $x^{7/10} y^{1/6} z^{1/2}$

5. Simplify

$$(a^2)^{1/2} (b^2)^3 (c^{1/2})^3.$$

**Answer:**  $ab^6 c^{3/2}$

6. Simplify

$$\frac{(abc)^2}{(a^2b^{-1}c^{-3})^3}.$$

**Answer:**  $a^{-4}b^5c^{11}$

7. Simplify

$$(\sqrt{x}\sqrt{y^3}\sqrt[3]{z^2})(\sqrt{x}\sqrt{y^3}\sqrt{z^3}).$$

**Answer:**  $xy^3\sqrt[6]{z^{13}}$

8. Simplify

$$\frac{(a^3b^{1/2}c^{-1/2})(ab)^{1/3}}{(\sqrt{a^3}\sqrt[3]{bc})}.$$

**Answer:**  $a^{11/6}b^{1/3}c^{-3/2} = \frac{\sqrt[6]{a^{11}}\sqrt[3]{b}}{\sqrt{c^3}}$

## 2.2 Scientific Notation

### 2.2.1 Powers of Ten

Consider the number 3,487. What *is* this?

Three thousand, four hundred, and eighty seven

How about 0.123?

Any real number<sup>2</sup> at all may be written in what's called *scientific notation*. This notation handles extremely large and small numbers and negates the need for too many decimal points. For science and engineering at least, scientific notation greatly eases calculations and data can be presented in a much nicer manner. In scientific notation, real numbers are written in the form:

---

<sup>2</sup>except 0

with  $a \in [1, 10)$  and  $n \in \mathbb{Z}$ . Writing numbers in scientific notation is also called writing them in *standard form*.

### Examples

1.  $300 = 3 \times 10^2$ .
2.  $4,000 = 4 \times 10^4$ .
3.  $5,720,000,000 = 5.72 \times 10^9$ .
4.  $0.0000000061 = 61 \times 10^{-9}$ .

Scientific notation allows us to get a rough order on the magnitude of very small and very large numbers. How do we write a decimal  $x$  in scientific notation? We'll break it up into three cases.

### Numbers Between 1 and 10

Write 4.345 in scientific notation.

**Abstraction:** If  $1 \leq x < 10$ :

### Numbers Bigger than 10

What about 3187.2?

$$3,187.2 = 3.1872 \times 10^3$$

Why is the case? Well to go from 3,187.2 we can go in three steps:

Now to go from 3,187.2 to 318.72 we divide by ten. To go to 31.872 we divide by ten again and finally we divide by ten a final time. To balance the books as it were we must account for these three divisions by ten. If we multiply now multiply by ten three times:  $10^3$  we haven't changed the original number and we are happy to write  $3,187.2 = 3.1872 \times 10^3$ .

### Examples

1. 4342.43
2. 35.76
3. 1423344.243

## Numbers Less than 1

What about 0.022?

$$0.022 \rightarrow 0.022 \times 10^2$$

Moving the decimal spot one to the right is equivalent to multiplying by ten — for every spot you move to the right you must multiply by ten — and if you multiply by 10  $n$  times then you must divide by 10  $n$  times:  $10^{-n}$ .

## Examples

1. 0.00243
2. 0.0076
3. 0.000000243

## Examples

1. An electron's mass is about 0.0000000000000000000000000091093822 kg. In scientific notation, this is
2. The Earth's mass is about 5973600000000000000000 kg. In scientific notation, this is
3. The Earth's circumference is approximately 40000000 m. In scientific notation, this is

## Examples

1. Express the following numbers, which are in standard form, as decimal numbers:

$$(i) 2.67 \times 10^{-2} \quad (ii) 5.487 \times 10^5 \quad (iii) 3.44 \times 10^0.$$

*Solution:* Firstly

Also



Hence

$$\frac{a \times 10^n}{b \times 10^m} = \frac{a}{b} \times 10^{n-m} \quad (2.3)$$

### Powers

What about  $(4.55 \times 10^{-13})^5$ ?

Hence

$$(a \times 10^n)^m = a^m 10^{nm} \quad (2.4)$$

### Example

Write the following in scientific notation:

$$(i) (3.75 \times 10^4)(5.6 \times 10^5) \quad (ii) \frac{3.5 \times 10^6}{7 \times 3}.$$

*Solution:* For the first one we have

And the second

## 2.2.4 Engineering Notation

Engineering notation is exactly the same as scientific notation except we always write our power of ten as a multiple of three.

*Exercises:*

1. By converting each element to scientific notation, calculate each of the following and write your answer in the form  $a \times 10^n$ , where  $1 \leq a < 10$  and  $n \in \mathbb{Z}$ :
  - (a)  $400,000 \times 2,000$ .
  - (b)  $6,000 \times 1,400$ .
  - (c)  $25,000 \times 0.0018$ .
  - (d)  $4,500 \times 1.5 \times 10^{-4}$ .
  - (e)  $6,000,100 / (3 \times 10^5)$ .
  - (f)  $8,888 \times 10^{-4} / 0.000432$ .
  - (g)  $(10,000)^4$ .

(h)  $(1.8 \times 10^{-6})^{-2}$ .

(i)  $(2.4 \times 10^{-31}) \times (4.123 \times 10^3)/(10^{-18})^3$ .

2. Multiply 3,700 by 0.2 and express your answer in the form  $a \times 10^n$ , where  $1 \leq a < 10$  and  $n \in \mathbb{Z}$ .
3. Write  $2.8 \times 10^3$  as a natural number.
4. Express  $(10^5) \times (1.8 \times 10^{-4})^{-4}$  in scientific notation.

## 2.3 Conversion of Units\*

[RATIOS OF UNITS] [EXPRESSING ONE QUANTITY AS A PERCENTAGE OF ANOTHER] [PERCENTAGES][CURRENCY]

### SI Units

The central concept of a physics theory is a physical quantity. A physical quantity is any property of matter, time and space that can be measured. Everyday physical quantities include:

time	length	area
volume	distance	speed
acceleration	mass	temperature

When a physical quantity is measured, it is compared with a standard amount, or *unit*, of the same quantity. For example, to say that a length of wire is twelve metres means that the piece of wire is twelve times longer than the metre. The metre is the unit of length. The result of a measurement is always a multiple of a unit. To say that a length is twelve is meaningless.

There are many different units of length: there is the centimetre, the inch, the foot, the yard, the metre, the furlong, the kilometre, the mile, the parsec, etc. In 1960 physicists agreed to use a particular system of units; the *Système Internationale*, or the SI.

To relate physical quantities easily, the SI defines *seven* basic quantities and associated units. To ease presentation of laws, for example, a symbol or letter is assigned to stand for physical quantities. Furthermore, the associated units have associated letter symbols. For example, the physical quantity is length, denoted by  $l$  (or  $s$  often); and the associated SI unit is the metre, denoted by m. Hence the physical quantity twelve metres may be denoted  $l = 12$  m. The five basic quantities, their symbol, associated unit and unit symbol are as follows:

Basic Quantity	Symbol	SI Unit	Symbol
length			
time			
mass			
electric current			
temperature			

The unit of every other quantity is called a *derived unit* because it can be expressed as the product or quotient of one or more of the basic units. For example, density is kilogram per cubic metre. The symbol for density is  $\rho$  (the Greek letter *rho*). The unit is the kilogram per cubic metre:  $\text{kg m}^{-3}$ . A few examples of derived units are:

Physical Quantity	Symbol	SI Unit	Symbol
area			
volume			
speed			



Sometimes a derived unit can become quite complex. For example the unit of energy,  $E$ , in basic units is  $\text{kg m}^2 \text{s}^{-2}$ . This is given the special name to simplify matters, namely the *joule*,  $J$ . Here are a number of common quantities with derived units with their own name:

Physical Quantity	Symbol	SI Unit	Derived Unit	Symbol
force				
pressure	$P$	$\text{kg m}^{-1} \text{s}^{-2}$		
power	$P$	$\text{kg m}^2 \text{s}^{-3}$		W
frequency				
magnetic flux density	$B$		tesla	T
magnetic flux	$\Phi$		weber	Wb
activity of a radioactive source	$A$		becquerel	Bq

## Conventions

- When writing a unit in terms of basic units, a space is left between the symbol of each basic unit. For example, the unit of density is abbreviated  $\text{kg m}^{-3}$  not  $\text{kgm}^{-3}$ .
- Sometimes the standard SI units are too large or too small to be used easily. Therefore multiples of the standard units are often used. The most common multiples used are:

Multiple	Prefix	Symbol
$10^9$	giga-	G
$10^6$		
$10^3$		
$10^{-2}$		
$10^{-3}$		
$10^{-6}$		$\mu$
$10^{-9}$	nano-	n
$10^{-12}$	pico-	p

- The name of the unit has the prefix written before it; e.g. 10 kilometres, 5 millinewtons.
- There is no space between the prefix and the symbol for the unit: e.g. 60 millimetres is written as 60 mm; 20 kilowatts is written as 20 kW.
- The kilogram *is* the SI unit of mass - not the gram.

## Converting Units

What is 100 kilometres an hour in metres per second? The way to do this is to not worry about the '100' but just change the kilometres and the hour.

So now:

*To convert from one unit to another, substitute each unit on the left-hand-side into terms of another on the right-hand side.*

### Example

Express 100 kilometres in furlongs per fortnight!

### Exercises

1. Show that the SI unit of area is the square metre.
2. If a body of mass  $m$  kilograms is moving with a velocity of  $v$  metres per second its momentum  $P$  is defined by the equation  $P = mv$ . Find the SI unit of momentum in terms of basic units.
3. When the velocity of a body changes its average acceleration  $a$  is given by:

$$a = \frac{\text{change in velocity}}{\text{time taken for change}} \quad (2.5)$$

Find the SI unit of acceleration.

4. What is the SI unit of density?
5. Pressure  $P$  is defined as force per unit area, i.e.  $P = F/A$ . If the SI unit of force is the newton (N), find the SI unit of pressure in terms of the newton and the metre.
6. How many square centimetres in one square metre?  
How many cubic centimetres in one cubic metre?  
How many grams in a kilogram?

7. Convert each of the following to standard SI units:

$$\begin{array}{lll} (i) 5 \text{ cm}^2 & (ii) 40 \text{ cm}^2 & (iii) 1 \text{ cm}^3 \\ (iv) 456 \text{ cm}^3 & (v) 1,000,000,000 \text{ cm}^3 & \end{array}$$

8. Convert each of the following to standard SI units. Please write your final answer in scientific notation:

$$\begin{array}{lll} (i) 105 \text{ km} & (ii) 57 \text{ mm} & (iii) 6.67 \times 10^{-11} \text{ cm} \\ (iv) 6 \times 10^{27} \text{ grams} & (v) 9 \text{ grams per cubic centimetre} & (vi) 100 \text{ km h}^{-1} \\ (vii) 5 \text{ nN} & (viii) 10 \mu\text{W} & (ix) 5 \text{ Gm} \end{array}$$

9. The unit of acceleration  $a$  is the metre per second squared ( $\text{m s}^{-2}$ ). Force is equal to mass by acceleration, i.e.  $F = ma$ . The unit of force is the newton. Express the newton in terms of basic units.
10. By definition work  $W$  is equal to force  $F$  multiplied by distance  $s$  travelled, i.e.  $W = Fs$ . The unit of work is the joule (J). Express the Joule in terms of the newton.
11. Use the results of the previous two questions to write the unit of work, the joule, in terms of basic units.
12. By definition power  $P$  is equal to work done divided by the time taken, i.e.  $P = W/t$ . The unit of power is called the watt (W). Express the watt in terms of the joule and the second. Then express the watt in basic units.

## 2.4 Functions

### Functions and Graphs

A *function* is an object that takes real numbers as an input and gives a (unique) real number output:

Most if not all of the functions that we will encounter will be of the form:

What this means is that whatever number we feed into the function,  $x$ , the expression on the right gives the recipe for calculating  $f(x)$ .

#### Examples

1.  $f(x) = 3x + 2$ .
2.  $f(x) = x^2$ .

3.  $f(x) = 1/(3x + 5)$ .

4.  $f(x) = \sqrt{x^2 + 1}$ .

5.  $f(x) = \sin x$ .

6.  $f(x) = 2^x$ .

In general, if we want to say that  $f$  is a function we write:

Occasionally we will only take some of the real numbers to be our input. For example:

defines a function on the degree-measure of the acute angles in a right-angled triangle. Now consider the function  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = 10^x$ . We can tabulate some of the values that  $f(x)$  takes:

Input	-4	-3	-2	-1	-1/2	0	1/2	1	4/3
Output					0.316...		3.162...		21.554...

A *graph* of a function is a picture of a function constructed as follows. Take a pair of axes labeled  $x$  and  $f(x)$  (or  $x$  and  $y$ ). Now  $x$  is the input while  $f(x)$  is the output:

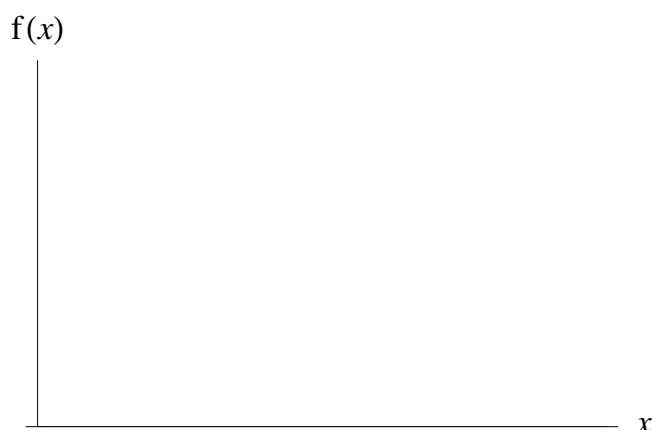


Figure 2.1: The graph of a function  $f(x)$ .

We could plot the points of  $f(x) = 10^x$  on a graph:

Now a feature of the graph of  $10^x$  is that it is always increasing and always positive (this will be true for any  $a^x$  as long as  $a > 1$ ):

This suggests that we could define a function going in the opposite direction, where “ $y$ ” would be the input. Now this other function, which we call ‘ $g$ ’ only takes *positive* numbers as input. So

where  $\mathbb{R}^+$  is the set of positive real numbers. We define  $g$  as:

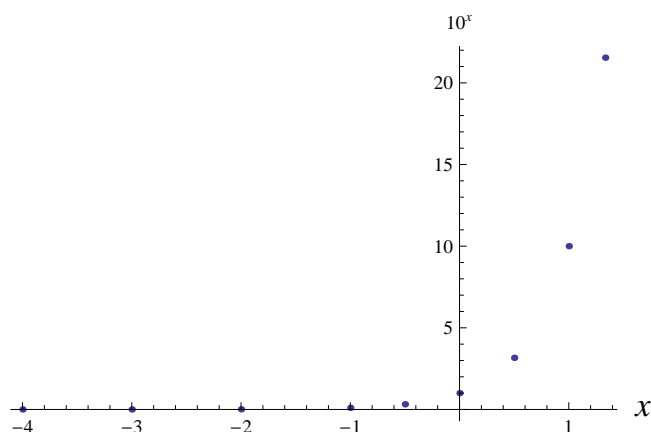
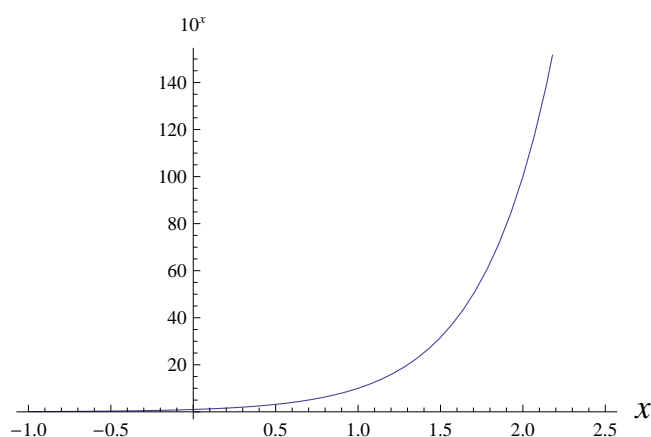
Figure 2.2: The graph of  $f(x) = 10^x$ .

Figure 2.3: To every output we can associate a unique input.

The terminology here is that  $g$  is the *inverse* of  $f(x) = 10^x$ . Again as we did in the last chapter, we can generalise this discussion - i.e there is nothing special here about 10.

## 2.5 Logarithms\*

### Definition

Let  $a > 1$  and let  $f(x) = a^x$ . The *logarithm to base  $a$*  is a function  $\log_a : \mathbb{R}^+ \rightarrow \mathbb{R}$  that is the inverse function of  $f(x)$  so that

In other words,  $\log_a y$  asks the question:

**Examples***Evaluate*

1.  $\log_2 8$

2.  $\log_3 81$

3.  $\log_{25} 5$

4.  $\log_{10} 0.001$

**Properties of Logarithms**

Now that we have defined what a logarithm is we must investigate common properties of them. What happens when we take the log of a product? If we take the log of a fraction? What about the log of a power?

**The Log of a Product?**

What is the log of  $8 \times 16$  to the base 2:

$$\log_2(8 \times 16) = ?$$

As we've seen time and again, a process like this can be abstracted to prove a general rule for *all* numbers:

**Proof:** Let  $L = \log_a(xy)$ :

Now there exist numbers  $p, q \in \mathbb{R}$  such that  $a^p = x$  and  $a^q = y$ ...

Well  $p = \log_a x$  and  $q = \log_a y$ , so

### Example

*Evaluate*  $\log_4 2 + \log_4 32$ .

### The Log of a Power

What is  $\log_3 8^4$ ?

So...

(Note we've only stated this for  $n \in \mathbb{N}$  but in fact this is true for all  $n \in \mathbb{R}$ .)

**Proof:**

### **Example**

*Find  $\log_5 25^{60}$*

### **The Log of a Fraction**

What about  $\log_{12}(34/56)$ ?

So

**Proof:**



**Example**

*Evaluate  $\log_2 80 - \log_2 5$ .*

**The Change of Base Rule - Why Can't I Find  $\log_3 12$  on My Calculator?**

What is  $\log_3 12$  actually equal to as a decimal...

So if we want to change from base  $a$  to base  $b$ :

**Proof:****Example**

*Evaluate  $\log_8 5$  correct to 5 decimal places:*

**Prominent Results**

What is  $\log_3 3^x$ ?

What is  $4^{\log_4 69}$ ?

What is  $\log_a 1$ ?

What is  $\log_a \sqrt{x}$ ?

What is  $\log_a (1/x)$ ?

**Example**

*Express as a single logarithm:*

$$\log_a x + \frac{1}{2} \log_a y$$

## 2.6 The Natural Exponential & Logarithm Functions\*

**Two Distinguished Bases**

For good reason the two bases that ‘turn-up’ most frequently are 10 and the special real number  $e \approx 2.718$ . As we will see in the next chapter, any real number at all may be expressed in the form:

Here  $a \in [1, 10)$  means “ $a$  is a (real) number between 1 and 10, possibly equal to 1 but not equal to 10. For example,

Now taking the log to the base 10 of both sides:

Hence to find the base-10 logarithm of any number at all we must simply know all the logarithms (to base 10) of all the numbers between 1 and 10... When we write log with no base (cf  $\log_8$ ) then we usually mean  $\log_{10}$ .

The importance of the base  $e$  comes from following observation:

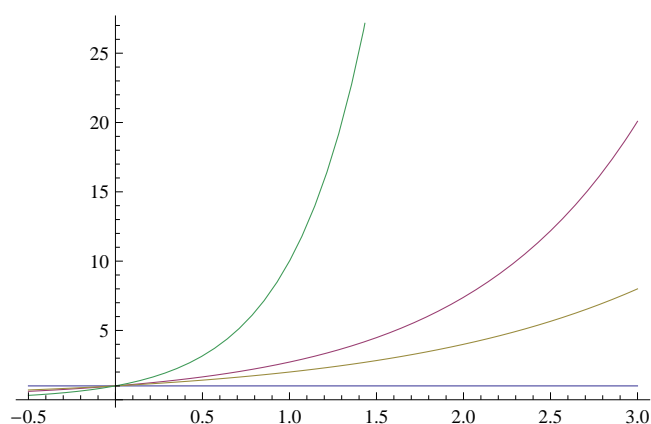


Figure 2.4: The slope of the curve of  $a^x$  is similar to  $a^x$  itself.

As it turns out,  $e^x$  is the unique function of the form  $f(x) = a^x$  for which the slope of the curve of  $a^x$  is equal to  $a^x$ . We will write  $\ln x$  for  $\log_e x$ , where ‘ln’ is somehow meant to stand for natural logarithm.

## 2.6.1 Exponential Equations

### Examples

If  $4^n = 2000$ , find  $n$  correct to 4 significant figures

Expand:

$$\log_a \sqrt{\frac{ab}{c}}$$

## Exercises

1. Solve for  $x$ :

$$(i) \log_2 8 = x \quad (ii) \log_{10} x = 2 \quad (iii) \log_3 81 = x$$

$$(iv) \log_5 x = 3 \quad (v) \log_2 x = 10 \quad (vi) \log_x 64 = 3$$

$$(vii) \log_{10} 10 = x \quad (viii) \log_{10} 1 = x \quad (ix) \log_{25} x = 1/2$$

2. Evaluate  $\log_7 35 - \log_7 5$ .

3. Solve for  $n$  (to 2 decimal places):  $2^n = 20$ .

4. Use a calculator to evaluate to two decimal places:

$$(i) \log_{10} 3 \quad (ii) \log_{10} 7 \quad (iii) \log_3 7 \quad (iv) \log_7 3$$

5. Using the fact that  $3^n$  is increasing, find the least value of  $n \in \mathbb{N}$  such that  $3^n > 1,000,000$ .

6. If  $p = \log_3 10$  and  $q = \log_3 7$ , write  $\log_3 700$  in terms of  $p$  and  $q$ .

7. (a) Find the least value of  $n \in \mathbb{N}$  such that  $1.03^n > 2$ .

(b) Suppose that the population of a town increases by 3% every year. Using the previous result, write down how many years it takes to double.

8. If  $p = \log_6 5$  and  $q = \log_6 10$ , write  $\log_6(2\sqrt{6})$  in terms of  $p$  and  $q$ .

# Chapter 3

## Basic Algebra

*Algebra is the metaphysics of arithmetic.*

John Ray

### 3.1 Introduction

#### 3.1.1 Laws of Algebra

Five facts about numbers<sup>1</sup>:

1. **Subtraction:** For all real numbers  $x$ :

**Example:**  $5 - 5 = 0$ .

2. **Identity:** For all real numbers  $x$ :

**Example:**  $\pi \times 1 = \pi$

3. **Distributivity** For any three real numbers  $x, y, z$ :

This is called the *distributive law*. Note that this works both ways and explains how to group terms and factorise. One way says how to write a product as a sum; the other how to write a sum as a product.

4. **Division:** For all  $x \neq 0$ ,

**Example:**  $-123 / -123 = 1$ .

---

<sup>1</sup>four axioms and one theorem

5. **No Zero Divisors:** Suppose that  $a, b \in \mathbb{R}$  such that  $a \times b = 0$ :

**Example:** Which number multiplied by 5 gives you 0?

## Why Bother?

Consider the following two problems:

1. What real number, when added to its square, equals 0?
2.  $5^2 - 4^2 = (5 - 4)(5 + 4)$ . Is this formula correct for all pairs of numbers?

Here there are two types of problem. One is a puzzle — what is  $x$ ? Another is asking us to *prove* that an identity holds.

### Example 1

*What real number, when added to its square, equals 0?*

Now first off, can anyone guess... How do you think there are no more solutions??? This is how we do algebra:

Now we use some of our facts in a logical way. Well  $x^2 = x \times x$  and  $x = x \times 1$ :

Now using distributivity we can write this sum as a product:

Now using ‘No Zero Divisors’:

Now considering  $x + 1 = 0$  separately. I’m happy enough that  $x = -1$  but if we still weren’t convinced we could add  $-1$  to both sides:

We set up  $x$  as the fall guy — and by a series of logical deductions, show what  $x$  must be.

### Example 2

We want to investigate whether or not the statement:

*The difference of two real numbers is the product of their sum and their difference.*

is true or false. Now let  $x, y$  stand for any two numbers at all. Now an equivalent statement is:

for all  $x, y \in \mathbb{R}$ . How can we show this? Well by distributivity (twice):

Now  $xy = yx...$

The statement is true for any pair  $x, y \in \mathbb{R}$ .

## Some More Facts (!)

### Multiplication by Zero

For all  $x \in \mathbb{R}$ ,  $0 \times x = 0$ .

#### Proof

$0 = 0 + 0$ . Therefore  $x \times 0 = x \times (0 + 0)$ . By distributivity,  $x \times (0 + 0) = x \times 0 + x \times 0$ . Also  $x \times 0$  has an additive inverse  $-(x \times 0)$ :

### Division by Zero is Contradictory

*There does not equal a real number equal to  $1/0$ . (Division by any number  $y$  may be realised as multiplication by  $1/y$ .)*

#### Proof

Assume there exists a real number  $1/0$ . Now by the previous theorem:

However, by division:

Hence if  $1/0$  exists  $1 = 0$ . This contradicts the assumption that  $1 \neq 0$ . Hence  $1/0$  does not exist. (i.e. division by zero is not allowed) •

## 3.2 Algebraic Manipulation & Simplification

### Canceling above and below

Suppose  $a, b, c \in \mathbb{R}$ ,  $b, c \neq 0$ . Then

$$\frac{ac}{bc} = \frac{a}{b} \quad (3.1)$$

### Proof

Now  $c \neq 0$  so from division there exists a number  $1/c$  such that  $c \times 1/c = 1$ .

### Remark

Careful!! Let  $a, b, c, d \in \mathbb{R}$ ,  $b, c \neq 0$ . The following move is nonsense (unless  $d = 0$ ):

$$\frac{ac + d}{bc} = \frac{a\cancel{c} + d}{b\cancel{c}} = \frac{a + d}{b}$$

Nowhere does it say that this should make sense. The above theorem is very precise — it only allows cancelations as above.

### Four More:

1. For all numbers  $-1 \times x = -x$ .
2.  $-1 \times -1 = +1$ .
3.  $(-a) \times (-b) = ab$

$$4. ax + bx = (a + b)x$$

### Examples

Unless stated otherwise, all letters are real numbers.

1. *Multiply out:*

$$(x + 1)(3x^2 + x + 1)$$



2. If

$$x = \frac{2t}{1+t^2}, \text{ and } y = \frac{1-t^2}{1+t^2},$$

write

$$\frac{x}{1-y}$$

in terms of  $t$ .

3. Express as a single fraction:

$$\frac{x+2}{3} + \frac{x+5}{4}$$

4. Simplify

$$\frac{10ab}{2b}$$

5. Factorise (i)  $5ab + 10ac$  (ii)  $6xy - 3y^2$  (iii)  $pq - pr + p$

6. Factorise (i)  $ax + bx + ay + by$  (ii)  $pq + pr - q - r$

7. Simplify

$$\frac{4x - 8}{x^2 - 4}$$

8. Simplify

$$\left[ \frac{1+x}{1-x} - 1 \right] \div \frac{1}{1-x}$$

9. Solve for  $a^x$ :  $a^x - 3 = -2a^{-x}$ .

Suppose that  $a^x$  solves the equation...

10. Find the value of  $x$ :

$$\log_3(2x + 5) = \log_3(x - 8) + 1$$

Suppose  $x$  solves the equation...

## Exercises

1. When  $a = 1$ ,  $b = 2$ ,  $c = 3$ ,  $x = 4$ ,  $y = 5$  and  $z = 6$ , find the value of:

$$\begin{array}{ll} (i) 2a + 3b + 4c & (ii) abc \\ (iii) 3xz + 5bx & (iv) 3(2b + a) + 2(3x + 2y) \\ (v) (a + b)^b & (vi) (2abc/z)^c \\ (vii) (5(y + a))/(3(a^2 + b^2)) & (viii) 2(b^2 + c + bc^3) \end{array}$$

**Selected Answers:** (iii) 112 (vi) 8

2. Remove the brackets in each of the following:

$$\begin{array}{ll} (i) 2(x + 4) & (ii) 5(2x^2 + 3x + 4) \\ (iii) -3(x - 4) & (iv) -4(x^2 - 3x + 4) \end{array}$$

**Selected Answers:** (iii)  $12 - 3x$ .

3. Remove the brackets and then simplify each of the following:

$$\begin{array}{ll} (i) 3(2x + 4) + 2(5x + 3) & (ii) 4(2x + 3) + 2(4x + 6) \\ (iii) 3(2x + 4) + 7 - 3(x + 5) - 2x - 4 & (iv) 11x - 3(6 - x) + 13 - 5(2x - 1) \\ (v) 7(6 - x) + 21 + 5(x - 7) - 3(8 - x) & (vi) 3p + 2(4 - p) + 3(p - 5) - 2p + 6 \\ (vii) 5(q + 4) - 3q - 29 + 3(q + 3) & (viii) 5(2x^2 - 3x + 2) - 3(3x^2 - 6x + 2) \\ (ix) 5(1 - x + 2x^2) - 5(1 - x - 2x^2 - 18x^2) & (x) 2(x + y) + 3(2x + 3y) \\ (xi) 5(2x - y) - 4(x - 3) + 3(y - 5) & (xii) 4x - [4x - 2(2x - 2)] \end{array}$$

**Selected Answers:** (iii)  $x$  (vi)  $2p - 1$  (ix)  $2x^2$  (xii)  $4x - 4$

4. Simplify each of the following:

$$\begin{array}{lll} (i) 2x.5x & (ii) x.x^2 & (iii) 4x^2(2x) \\ (iv) (5a^2)(-3a) & (v) 2p.2p.2p & (vi) (-x)(-x) \\ (vii) 4ab(3a^2b^2) & (viii) (3xy)(xy) & (ix) p.3p(-3p^2) \\ (x) (8xy)(xy)(4x) & & \end{array}$$

**Selected Answers:** (iii)  $8x^3$  (vi)  $x^2$  (ix)  $-9p^4$

5. If  $6x(4x^2) = kx^3$ , find the value of  $k$ .
6. Remove the brackets and then simplify each of the following:

$$(i) 2x(x + 5) + 3(x + 5)$$

$$(ii) 2x(x - 1) - 3(x - 1)$$

$$(iii) 2x(2x + 3) + 3(2x + 3)$$

$$(iv) 2x(x^2 + 5x + 3) + 3(x^2 + 5x + 3)$$

$$(v) 4x(2x^2 - 3x - 6) + 6(2x^2 - 4x) \quad (vi) a(a + 1) + 2a(a - 3) + 6a - 3a^2$$

$$(vii) 5x(2x - y) - 2y(2x - y) \quad (viii) 3x(2x + 4y) + 6y^2 - 6y(x + y)$$

$$(ix) a(b + c) - b(c - a) - c(a - b) \quad (x) 2[a(a + b) + b(b - a)]$$

**Selected Answers:** (iii)  $4x^2 + 12x + 9$  (vi)  $a$  (ix)  $2ab$

7. Multiply  $2x^2 - 2x + 1$  by  $x + 1$ .
8. Write as a single fraction:

$$(i) \frac{1}{5} + \frac{3}{4}$$

$$(ii) \frac{7}{5} - \frac{9}{10} + \frac{11}{15}$$

$$(iii) \frac{x}{5} + \frac{x}{4}$$

$$(iv) \frac{x}{2} + \frac{3x}{4} - \frac{5x}{3}$$

$$(v) \frac{x+2}{5} + \frac{x+7}{10}$$

$$(vi) \frac{2x+3}{7} + \frac{x+1}{3}$$

$$(vii) \frac{5x-3}{2} - \frac{3x-4}{3}$$

$$(viii) \frac{2x+5}{3} - \frac{4x-3}{2}$$

$$(ix) \frac{5x-1}{4} + \frac{x}{3} - \frac{5}{6}$$

$$(x) \frac{5x}{3} - \frac{1}{6} + \frac{2-3x}{2}$$

$$(xi) \frac{4x-3}{5} - \frac{x}{2} + \frac{1}{10}$$

**Selected Answers:**

$$(iii) \frac{9x}{20} \quad (vi) \frac{13x + 16}{21} \quad (ix) \frac{19x - 13}{12}$$

9. Simplify each of the following:

$$(i) 20ab/5b \quad (ii) 21pq/7q \quad (iii) 28p/7$$

$$(iv) x^2/x \quad (v) 3pq/3pq \quad (vi) 6x^2y/3xy$$

$$(vii) 30p^2q/15p^2$$

Note: When we write  $a/b$  we mean:

$$\frac{a}{b} = a \times \frac{1}{b} = a \div b = ab^{-1}$$

**Selected Answers:** (iii)  $4p$  (vi)  $2x$

10. Simplify (i)  $(x - 1)^2 - (x + 1)^2$  (ii)  $2(x + 2)^2 - 8(x + 1)$ .

11. Simplify:

$$\left(1 - \frac{1}{y}\right) \left(\frac{y}{y-1}\right)$$

12. Factorise

$$\begin{aligned} (i) \quad & 4a^2 - 2a & (ii) \quad & 2pq - 5r - 5q + 2pr \\ (iii) \quad & 25a^2 - 16b^2 & (iv) \quad & 3x^2 - 12x \\ (v) \quad & 16a^2 - 81b^4 \end{aligned}$$

13. By factorising, simplify

$$\frac{4x^2 - 64}{x - 4}$$

14. Simplify

$$\left(\frac{1}{x} - \frac{1}{x+h}\right) \div h$$

15. Simplify

$$\frac{(2x+3)^2 - 6(2x+3)}{2x-3}.$$

16. Simplify

$$\left(p - \frac{q^2}{p}\right) \div \left(1 + \frac{q}{p}\right)$$

17. If  $3 \log_a y - \log_a(x+1) = \log_a 2$ , write  $y$  in terms of  $x$ .

18. The sides of a triangle have lengths  $m^2 + n^2$ ,  $m^2 - n^2$  and  $2mn$ . Prove that the triangle is right-angled.

19. These statements are either true or false. If true, provide a proof. If false provide a counter-example (i.e. an example where the left-hand side is not equal to the right-hand side). Ignore or account for division by zero.

(a)

$$\frac{1}{a+b} = \frac{1}{a} + \frac{1}{b}$$

(b)

$$\frac{a+b}{a} = 1 + b$$

(c)

$$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$$

(d)

$$\frac{a^2 + a}{a} = a + 1$$

(e)

$$\frac{a^2 + 2a}{2a} = a^2 + 1$$

$$\begin{array}{ll}
 \text{(f)} & \frac{a(b+c)+3a}{a} = b+c+3a \\
 \text{(g)} & \frac{a(b+c)+3a}{a} = b+c+3 \\
 \text{(h)} & \frac{m^2-n^2}{m-n} = m+n \\
 \text{(i)} & \frac{m^2+n^2}{m+n} = m+n \\
 \text{(j)} & \frac{m^2-n^2}{n-m} = -n-m
 \end{array}$$

### 3.2.1 Quadratic Functions

Equations which can be written in the form:

where  $a, b, c \in \mathbb{R}$  are real constants with  $a \neq 0$  are called *quadratic equations*. Examples:

Take the equation  $x^2 - 5x + 6 = 0$ . When  $x = 2$ :

When  $x = 3$ :

When  $x = 2, 3$  both sides of the equation are zero. When this happens we say that  $x = 2, 3$  are *roots* of the equation. Solving a quadratic equation involves finding the values of  $x$  which satisfy the equation. An analogous problem is finding the *roots* of the quadratic function:

### Finding the Roots

The following is *the* crucial theorem in this section:

#### No Zero Divisors

*Suppose  $a$  and  $b$  are real numbers and*

$$a.b = 0.$$

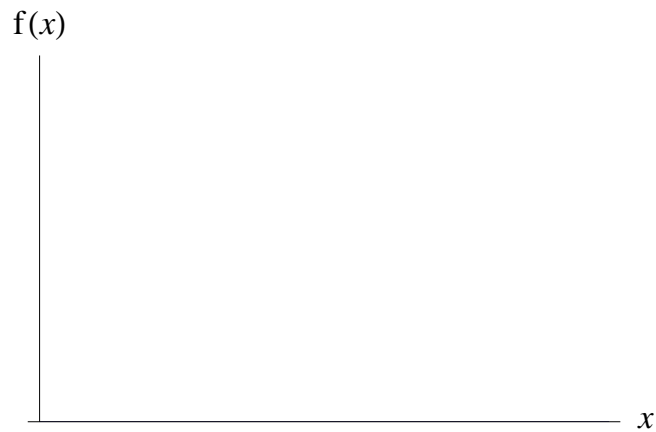


Figure 3.1: The *roots* of  $f(x)$  occur where the graph cuts the  $x$ -axis.

Hence what we can do is *factorise the quadratic* — i.e. write it as a product. The simplest recipe for factorisation uses the following theorem:

**Theorem**

*Let  $f(x) = x^2 + bx + c$  be a quadratic function. Suppose  $f(x)$  can be written in the form*

*Then*

*If this is the case, then  $\alpha$  and  $\beta$  are the roots of  $f(x)$ .*

**Proof:**

**Remark**

This means that a quadratic has form

Hence suppose given a quadratic  $f$ :

If numbers  $\alpha$  and  $\beta$  can be found such that

then

which has roots  $\alpha$  and  $\beta$  according to the No Zero Divisors Theorem.

### Examples

*Factorise the following quadratic functions and hence find the roots:*

1.  $x^2 - 7x + 10$ .

2.  $x^2 - 5x - 14$ .

3.  $x^2 + 12x + 32$ .



**Remark**

The first complication here is when the coefficient of  $x^2$  is not 1. That is

where  $a \neq 1$ . Recall the aim of our endeavours is to find roots. The roots of  $f(x)$  are solutions to the equation

If we multiply both sides of an equation, the solution to the equation is unchanged. Multiply both sides by  $1/a$  ((M2): equivalently divide by  $a$ ):

Therefore the solutions of this equation are the roots of  $f$ . Solve this in the above way.

**Examples**

*Factorise the quadratic function  $f(x) = 3x^2 - 15x - 42$ . Hence find the roots.*

*Factorise the quadratic function  $f(x) = 2x^2 - x - 6$ . Hence find the roots.*

**Remark**

An equivalent (and in my opinion far superior — this method handles cases with  $a \neq 1$  much better) method for finding factors of a quadratic  $f$  is as follows. Find a re-writing of

$$f(x) = ax^2 + bx + c$$

as

such that  $b = k + m$  (so as not to change the quadratic) and  $km = ac$ . Then

and the roots of  $f(x)$  are apparent. In practise this method is known as *doing the cross*.

**Examples**

1. Solve the equation  $3x^2 + 10x = 8$ .

2. Solve the equation  $12x^2 - 4x - 5$ .

3. Solve the equation

$$\frac{6}{x} - \frac{5}{2x-1} = 1.$$

### Example

*Factorise  $5x^2 + 7x - 3$ ...*

Sometimes the polynomial will not have simple roots and then the two techniques for finding the roots will not suffice. There is a formula for these cases. Of course the formula may always be used to extract the two roots but the above methods are preferable.

**Theorem**

*A polynomial*

*has roots*

**Example**

*Find the roots of  $f(x) = 5x^2 + 7x - 3$  correct to two decimal places.*

**Only Two Roots???**

In each case we have seen that there are two roots — is this always going to be the case?

**Proposition**

*Suppose  $f(x) = ax^2 + bx + c$ . Then  $f(x)$  may be re-written as:*

$$f(x) = a \left( x + \frac{b}{2a} \right)^2 + \left( c - \frac{b^2}{4a} \right). \quad (3.2)$$

**Remark**

It can be shown that because of this, the graph of  $f(x)$  is similar to the graph of  $f(x) = x^2$  (it is got by translating and stretching the axes.):

**Remark**

Examining (\*), note that if  $b^2 - 4ac < 0$  then there is no (real) number equal to  $\sqrt{b^2 - 4ac}$  as a real number squared is always positive. The roots are said to be *unreal* and later it will be seen that they are *complex*. In this case the graph of  $f$  does not cut the  $x$ -axis at any point:

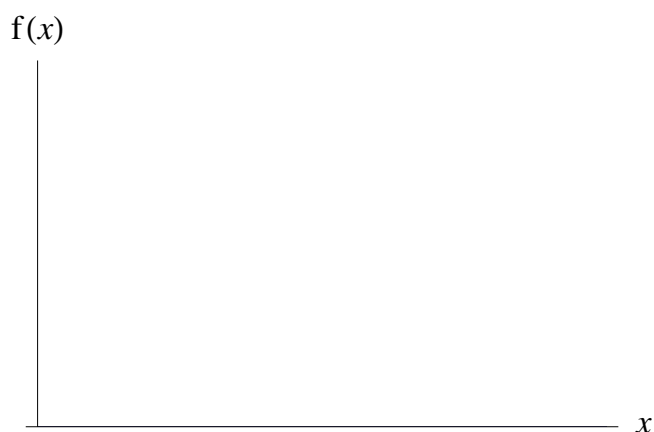


Figure 3.2: Quadratic functions look like  $x^2$  — hence they can only cut the  $x$ -axis at most once.

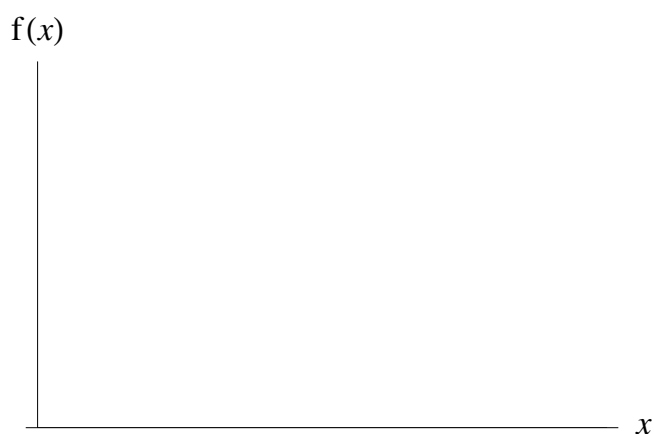


Figure 3.3: Quadratic functions with  $b^2 - 4ac < 0$  have no real roots.

If  $b^2 - 4ac = 0$  then the roots are real and equal,

In this case the graph has as a tangent the  $x$ -axis:

It is only in the case that  $b^2 - 4ac > 0$  do we have two distinct, real roots.

## Some Harder Examples

*Solve the equation*

$$(y - 2)^2 - 7(y - 2) + 10 = 0.$$

*Solve the equation*

$$6x^4 - 11x^2 - 10.$$

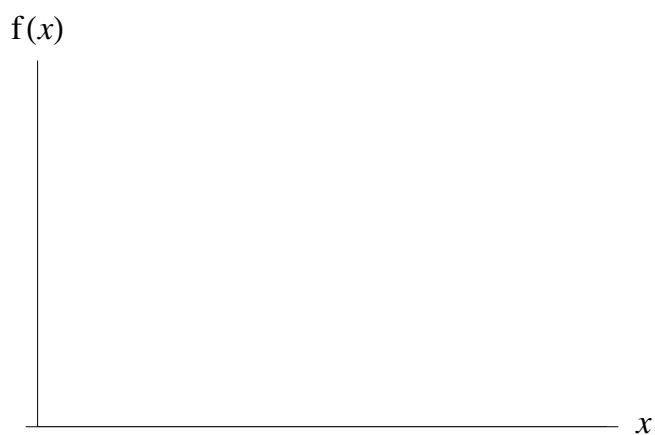


Figure 3.4: Quadratic functions with  $b^2 - 4ac = 0$  have two real, repeated roots.

*Solve the equation*

$$\left(t - \frac{6}{t}\right)^2 - 6\left(t - \frac{6}{t}\right) + 5 = 0.$$

*Solve the equation*

$$1 + \sqrt{x+1} = x.$$

## Exercises

*To check your answer, substitute in for  $x$ . Please use the  $\alpha$ - $\beta$  method or the cross method. You can use your own method in tests & exams.*

1. Find the roots of the following functions:

$$\begin{array}{lll} (i) (2x - 5)(x + 3) & (ii) 2x(3x - 1) & (iii) x^2 - 8x + 12 \\ (iv) x^2 - 7x - 18 & (v) x^2 - 9x - 10 & (vi) x^2 + 5x - 36 \\ (vii) 2x^2 - x - 6 & (viii) 3x^2 + x - 10 & (ix) 3x^2 + 19x - 14 \\ (x) 5x^2 - 13x + 6 \end{array}$$

2. Factor and solve for  $x$

$$\begin{array}{ll} (i) 3x^2 - 4x = 0 & (ii) 3x^2 - 7x = 0 \\ (iii) 4x^2 - 1 = 0 & (iv) 4x^2 - 49 = 0 \end{array}$$

3. Solve for  $x$ :

$$(2x - 5)(x - 2) = 15.$$

4. Solve the equation:

$$(x + 3)^2 = (x + 1)(2x + 3).$$

5. Solve the equation

$$x^2 - 13x + 42 = 0.$$

Hence solve the equation

$$(2t - 3)^2 = 13(2t - 3) + 42 = 0.$$

6. Solve the equation

$$2x^2 + 9x - 5 = 0.$$

Hence solve the equation

$$2 \left( x - \frac{1}{2} \right)^2 + 9 \left( x - \frac{1}{2} \right) - 5 = 0.$$

7. Solve the equation

$$6x^2 - 11x - 10 = 0.$$

Hence solve the equation

$$6(t - 1)^2 - 11(t - 1) - 10 = 0.$$

8. Solve the equation

$$x^2 - 19x + 70 = 0.$$

Hence solve the equation

$$y^4 - 19y^2 + 70 = 0.$$

9. Solve each of the following equations

$$(i) \quad \frac{x+7}{3} + \frac{2}{x} = 4$$

$$(ii) \quad \frac{1}{x+1} + \frac{x}{5} = 1$$

$$(iii) \quad \frac{1}{x+1} - \frac{1}{x+2} = \frac{1}{2}$$

$$(iv) \quad \frac{2}{x-1} - \frac{2}{x} = \frac{1}{3}$$

$$(v) \quad \frac{3}{x-1} - \frac{2}{x+1} = 1$$

$$(vi) \quad \frac{5}{2x+3} - \frac{2}{4x-3} = \frac{1}{3}$$

10. Solve the equation  $x^2 + 5x - 14 = 0$ . Hence find the four values of  $t$  that satisfy the equation

$$\left(t - \frac{8}{t}\right)^2 + 5\left(t - \frac{8}{t}\right) - 14 = 0.$$

11. Solve the equation  $x^2 - 12x + 27 = 0$ . Hence find the four roots of the function

$$g(t) = \left(2t - \frac{5}{t}\right)^2 - 12\left(2t - \frac{5}{t}\right) + 27.$$

12. Find the roots of the following functions, correct to two decimal places:

$$(i) \ x^2 + 2x - 5 \quad (ii) \ 3x^2 - x - 1 \quad (iii) \ 5x^2 - 4x - 2$$

13. Solve for  $x$ : (i)  $3x^2 + 7x = 2$  (ii)  $3x^2 + 5x = 3$

14. Express the following equations in the form  $ax^2 + bx + c = 0$  and hence use the quadratic formula to solve for  $x$  correct to two decimal places:

$$(i) \quad \frac{7}{x} = 3 + 2x.$$

$$(ii) \quad \frac{1}{x+1} + \frac{2}{x-3} = 4.$$

15. Find the roots of:

$$3x^2 - 2x - 2.$$

Hence find the roots, correct to one decimal place, of the function

$$g(z) = 3(2z - 1)^2 - 2(2z - 1) - 2.$$

16. Given that  $x^2 + y^2 + z^2 = 29$ , calculate the two possible values of  $y$  when  $x = 2$  and  $z = -3$ .

17. If 2 is a root of  $f(x) = 3x^2 - 4x + c = 0$ , find the value of  $c$ .

18. Find the two values of  $y$  which satisfies the equation

$$y^3 + \frac{27}{y^3} = 28.$$



19. A real number is such that the sum of the number and its square is 10. Find the two real numbers which have this property.
20. Two consecutive numbers have product 42. Write down an equation and hence find two solutions (e.g. 5 and 6 AND 11 and 12).
21. Find the values of  $x$  for which:
  - (i)  $\sqrt{x+3} = x-3$    (ii)  $x + \sqrt{x} = 2$ .
22. \* Show that for all values of  $k \in \mathbb{R}$ ,  $f(x) = kx^2 - 3kx + k$  has real roots.
23. \* Show that the equation  $(x-1)(x+7) = k(x+2)$  always has two distinct solutions, where  $k \in \mathbb{R}$ .
24. \* By plugging in

$$x_+ = \frac{-b + \sqrt{b^2 - 4ac}}{2a},$$

into  $f(x) = ax^2 + bx + c$ , show that  $x_+$  is a root of  $f(x)$ .

25. \* By multiplying out, prove that:

$$f(x) = a \left( x + \frac{b}{2a} \right)^2 + \left( c - \frac{b^2}{4a} \right). \quad (3.3)$$

where  $f(x) = ax^2 + bx + c$ .

Now solve for  $x$ , thus deriving the formula for the roots of a quadratic function.

### 3.3 Transposition of Formulae

#### What is an Equation/ Formula?

#### Famous Examples

$$\pi = \frac{c}{d} \quad (3.4)$$

$$a^2 = b^2 + c^2 \quad (3.5)$$

$$E = mc^2 \quad (3.6)$$

$$F = \frac{Gm_1m_2}{r^2} \quad (3.7)$$

$$A = \pi r^2 \quad (3.8)$$

$$1 + 1 = 2 \quad (3.9)$$

$$y = mx + c \quad (3.10)$$

$$x_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (3.11)$$

$$\cos^2 \theta + \sin^2 \theta = 1 \quad (3.12)$$

$$e^{i\pi} + 1 = 0 \quad (3.13)$$



Figure 3.5: An equation or formula is like a weighing scales — when balanced.

Once we know the basic algebra facts we can solve almost all algebra problems using these in conjunction with combinations of these three *moves*. Given an equation  $\text{LHS} = \text{RHS}$ , any of the following are allowed:

(M1)

(M2)

(M3)

Think of the scale pan...

To solve an equation in  $x$ ; e.g.

$$x^2 + x = 0,$$

we implicitly suppose that  $x$  solves the equation. What we do then is apply a series of moves until we get  $x = \text{'something'}$ .

### Examples

1.  $x - 3 = -10$

2.  $x/5 = 1$

3. Suppose  $x > 0$ . Solve  $\sqrt{x} = 2$ .

See how we reacted to each equation: we saw something we didn't like, something that was getting in the way of us isolating  $x$  on the left-hand side and we chose something from the three moves to get rid of it. What are the obstacles to solving the following equations:

1.  $-3x = 12$

2.  $2x + 3 = 11$

3.  $6x - 7 = 2x + 13$

4.

$$\frac{2x}{3} + \frac{x}{4} = \frac{11}{6}$$

5.  $x^2 = 35$

6.  $4(x - 3) = 4$

7.  $3 - \sqrt{x} = -6$

8.

$$4 = \frac{5}{x}$$

9.

$$\frac{48}{\sqrt{x}} = 3$$

10.

$$\frac{1}{x-3} = 17$$

11. Let  $x > 0$ ;  $\sqrt{x+4} = 10$

12.

$$\sqrt{\frac{x}{4}} = 2$$

13.

$$\frac{1}{x-3} + \frac{5}{4} = 1$$

14. Let  $x > 0$ ;

$$4 = \sqrt{\frac{x^2}{81}}$$

Again, once we see what is in our way we must apply basic algebra and the four moves to solve the problem. First moves?

1.

2.

3.

4.

5.

6.

7.

8.

9.

10.

11.

12.

13.

14.

Now lets finish each on the board.

**Some Longer Examples**

1.  $3(2x + 1) - 3(x + 4) = 0$

2.  $5 - 4(x - 3) = x - 2(x - 1)$

3.  $2 + 5(3x - 1) = 4(2x - 3) + 2(x - 3)$

**Equations with More Than One Variable**

These are solved exactly the same as above. Remember letters just stand for real numbers — manipulate them as such.

**Example**

*Given  $a = b + (c^2 - 1)d$  write  $c$  in terms of  $a$ ,  $b$  and  $d$ .*

**A Slight Complication**

Sometimes the variable which you are trying to isolate appears twice:

**Example**

*If*

$$x = \frac{t - 3}{2t - 1},$$

*write  $t$  in terms of  $x$ .*

**A Harder Example**

*Solve for  $d$ :*

$$a = \frac{c}{2s} \left( \frac{h^2}{d - h} \right)$$

**An Example including Powers**

*Solve for  $x$ :*

$$5e^{3x} = 90$$

**Exercises**

1. Solve each of the following equations:

$$(i) 2x = 6$$

$$(ii) 6x = 30$$

$$(iii) -2x = -20$$

$$(iv) 5x = -15$$

$$(v) 5x = 0$$

$$(vi) 3x - 7 = 8$$

$$(vii) 5x - 2 = -12 \quad (viii) 5x = 12 + 8x \quad (ix) 7x + 40 = 2x - 10$$

$$(x) 2x - 5 = 1 - x$$

**Selected answers:** (iii)  $x = 10$  (vi)  $x = 5$  (ix)  $x = -10$ 

2. Solve each of the following equations:

$$(i) 3(x + 4) = 2(x + 8)$$

$$(ii) 4(x + 4) = 2(x + 3)$$

$$(iii) 3(x - 1) + 5(x + 1) = 18$$

$$(iv) 4(x + 5) - 2(x + 3) = 12$$

$$(v) 10(x + 4) - 3(2x + 5) - 1 = 0$$

$$(vi) 3(2x + 1) - 3(x + 4) = 0$$

$$(vii) 7(x - 6) + 2(x - 7) = 5(x - 4)$$

$$(viii) 5 - 4(x - 3) = x - 2(x - 1)$$

$$(ix) 11 + 4(3x - 1) = 5(2x + 1) + 2(2x - 5) \quad (x) 2 + 5(3x - 1) = 4(2x - 3) + 2(x - 3)$$

**Selected answers:** (iii)  $x = 2$  (vi)  $x = 3$  (ix)  $x = 6$ 

3. Solve each of the following equations:

$$(i) \frac{x}{3} + \frac{x}{4} = \frac{7}{12}$$

$$(ii) \frac{x}{3} - \frac{x}{5} = \frac{27}{20}$$

$$(iii) \frac{2x}{5} = \frac{3}{2} + \frac{x}{4}$$

$$(iv) \frac{x+4}{3} = \frac{x+1}{4} + \frac{1}{6}$$

$$(v) \frac{x+4}{3} - \frac{x+2}{4} = \frac{7}{6}$$

$$(vi) \frac{x-5}{3} + \frac{1}{15} = \frac{x-2}{5}$$

$$(vii) \frac{3x-1}{2} = \frac{x+8}{3} + \frac{x-1}{6}$$

$$(viii) \frac{x-1}{4} - \frac{1}{20} = \frac{2x-3}{5}$$

$$(ix) \frac{x-2}{2} = 5 - \frac{x+10}{9}$$

$$(x) \frac{x-1}{3} + \frac{x-3}{4} = x - 4$$

**Selected Answers:** (iii)  $x = 10$  (vi)  $x = 9$  (ix)  $x = 8$

**In the following we have “Lhs = Rhs; variable”. Solve for “variable”.**

4.  $E = P + k; k$ .
5.  $F = ma; m$ .
6.  $y = mx + c; m$ .
7.  $E = mc^2; c$ .
8.  $E = V/R; R$ .
9.  $v^2 = u^2 + 2as; u$ .
10.  $t = (3 + v)k + c; v$ .
11.  $s = ut + at^2/2; a$ .
12.  $z = p/(2s + q); s$ .
13.  $s = a/(1 - r); r$ .
14.  $A = 4\pi r^2; r$ .
15.  $x = \sqrt{y + z}; y$ .
16.  $E = mgh + mv^2/2; m$ .
17.  $v = \pi h(R^2 + Rr + r^2)/3; h$ .
18.  $s = n(2a + (n - 1)d)/2; d$ .
19.  $A = P(1 + r)^3; r$ .

**Selected answers:**

6.  $m = \frac{y - c}{x}$
9.  $u = \sqrt{v^2 - 2as}$
12.  $\frac{p - q^2}{2z}$
15.  $x^2 - z$
18.  $\frac{2(s - na)}{n(n - 1)}$



# Chapter 4

## Graphs

*11:15 Restate my assumptions:*

- 1. Mathematics is the language of nature.*
- 2. Everything around us can be represented and understood through numbers.*
- 3. If you graph these numbers, patterns emerge. Therefore: There are patterns everywhere in nature*

Max Cohen

### 4.1 Introduction

### 4.2 Direct & Indirect Proportion\*

[RATIO APPROACH]

### 4.3 Lines

#### Coordinate Geometry

Just as points on a numberline can be identified with real numbers, points in a plane can be identified with pairs of numbers. We start by drawing two perpendicular coordinate *axes* that intersect at the origin  $O$  on each line (where the coordinate is  $(0, 0)$ ). Usually one line is horizontal with positive direction to the right and is called the  $x$ -axis; the other line is vertical with positive direction upward and is called the  $y$ -axis:

This coordinate system is called the *Cartesian coordinate system* in honour of the French mathematician René Descartes.

#### Lines

We want to find the *equation* of a given line  $L$ ; such an equation is satisfied by the coordinates of the points on  $L$  and by no other point. To find the equation of  $L$  we use its *slope*, which is a measure of the steepness of the line:

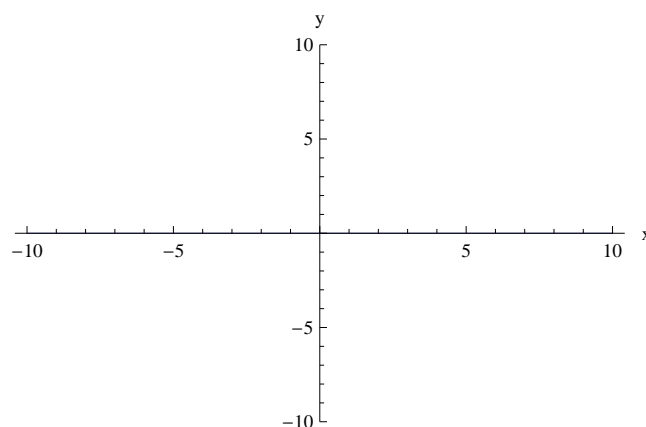


Figure 4.1: When graphing a function, usually the  $x$ -axis describes the input variable and the  $y$ -axis describes the output variable. To each point  $P$  we can associate a pair  $(a, b)$  and to each pair  $(c, d)$  we can associate a point  $Q$ .



Figure 4.2: The *slope* of a line is the ratio of how much you go up, as you across:  $\text{slope} = \frac{\text{rise}}{\text{run}}$

### Formula

The *slope* of a line that passes through the points  $P(x_1, y_1)$  and  $Q(x_2, y_2)$  is:

The slope of a line is constant — we can take the slope between *any* two points on the line and arrive at the same answer.

Now let's find the equation of the line that passes through a given point, say  $P(2, 3)$  with slope 4. Are we happy that this defines a line... We want the equation of the line to be like a membership card:

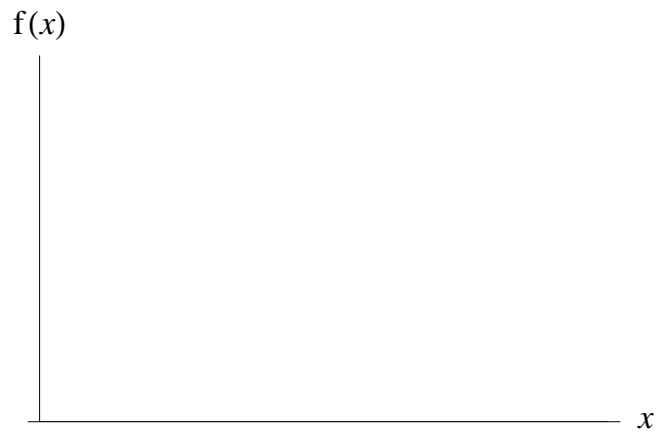


Figure 4.3: Notice that lines with positive slope slant upwards to the right, whereas lines with negative slope slant downwards to the right. Notice also that the steepest lines are those whose slope is big — be it in the positive or negative sense.

Now a point  $Q(x, y) \neq P(2, 3)$  is on this line if and only if:

That is,

Now, once again there is nothing special about  $(2, 3)$  nor 4 and we could abstract this process to a line defined by a point  $P(x_1, y_1)$  and slope  $m$ .

### Formula

The *equation of the line* passing through the point  $P(x_1, y_1)$  and having slope  $m$  is

### Examples

*Find the equation of the line through  $(1, -7)$  with slope  $-1/2$ .*

*Find the equation of the line through the points  $(-1, 2)$  and  $(3, -4)$ .*

Consider again the line defined by the point  $(2, 3)$  and slope 4:

$$y - 3 = 4(x - 2).$$

This shows that any equation of the form:

for  $m, c$  constants is another way of writing the equation of the line.  
What about  $c$ ?

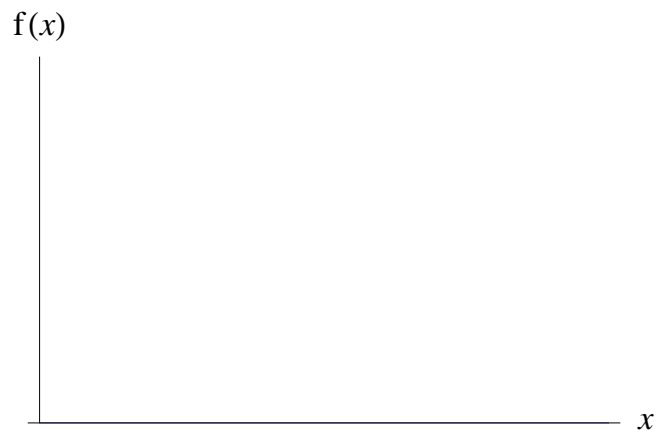


Figure 4.4: When the  $x$ -coordinate is zero, the graph of the line cuts the  $y$ -axis.

So  $c$  is where the line cuts the  $y$ -axis.

### Formula

The equation of the line with slope  $m$  and  $y$ -intercept  $c$  is given by:

Observe that every equation of the form:

$$AX + BY + C = 0 \tag{4.1}$$

is that of a line:

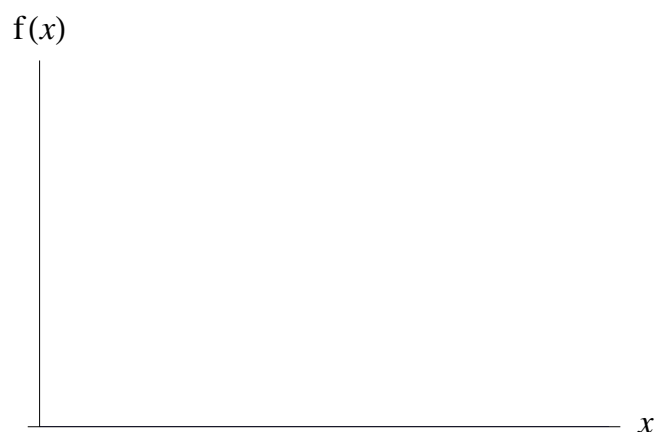
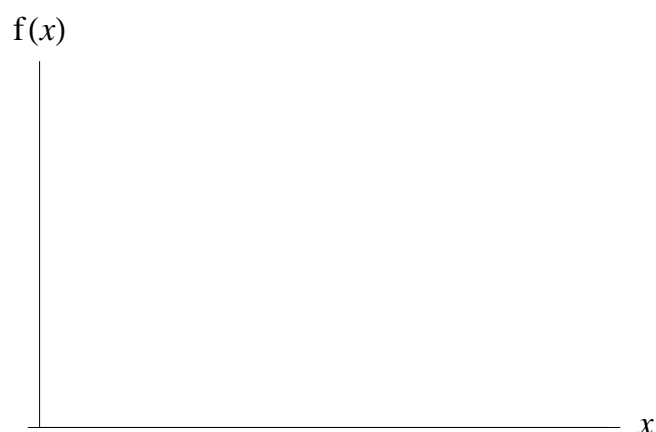


Figure 4.5: If a line is horizontal, then its slope is 0, so its equation is  $y = c$ , where  $c$  is the  $y$ -intercept. A vertical line does not have a slope, but we can write its equation as  $x = a$ , where  $a$  is the  $x$ -intercept, because the  $x$ -intercept of every point on the line is  $a$ .

We call an equation of the form (4.1) a *linear equation*.

### Example

*Sketch the graph of the line  $3x - 5y = 15$ .*



## Parallel and Perpendicular Lines

Slopes can be used to show that lines are parallel or perpendicular:

### Theorem

1. *Two lines are parallel if and only if they have the same slope.*
2. *Two lines with slopes  $m_1$  and  $m_2$  are perpendicular if and only if*

That is, if

### Example

*Find the equation of the line through the point  $(5, 2)$  that is parallel to the line  $4x + 6y + 5 = 0$ .*

### Example

*Show that the lines  $2x + 3y = 1$  and  $6x - 4y - 1 = 0$  are perpendicular.*

## Linear Models

When we say that an output  $y$  is a *linear function* of an input  $x$ , we mean that the graph of the function is a line, so we can use the slope-intercept form of the equation of a line to write a formula for the function as:

where  $m$  is the slope of the line and  $c$  is the  $y$ -intercept.

A characteristic feature of linear functions is that they grow at a constant rate. For instance

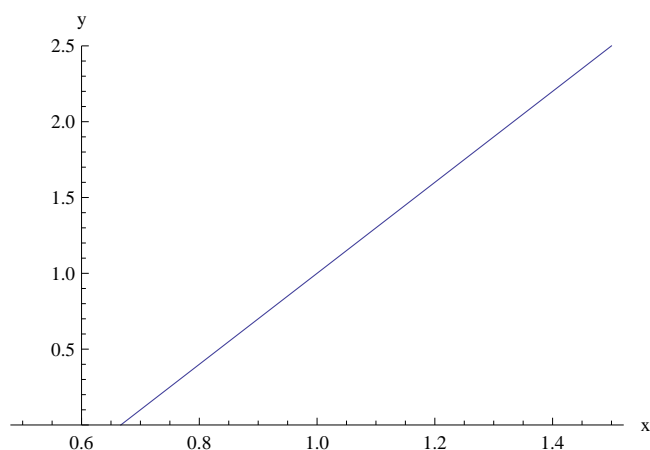


Figure 4.6: A graph of the linear function  $y = 3x - 2$ . Note that whenever  $x$  increases by 0.1, the value of  $y$  increase by 0.3. So  $y$  increases 3 times as fast as  $x$ . Thus, the slope of graph of  $y = 3x - 2$ , namely 3, can be interpreted as the rate of change of  $y$  with respect to  $x$ .

### Example

*As dry air moves upwards, it expands and cools. If the ground temperature is  $20^{\circ}\text{C}$  and the temperature at a height of 1 km is  $10^{\circ}\text{C}$ , express the temperature  $T$  (in  $^{\circ}\text{C}$ ) as a function of the height  $h$  (in kilometres), assuming that a linear model is appropriate. Draw the graph of the function. What does the slope represent. What is the temperature at a height of 2.5 km.*

If there is no physical law or principle to help us formulate a model, we construct an *empirical model*, which is based entirely on collected data. We seek a curve that “fits” the data in the

sense that it captures the basic trend of the data points.

### Example

The below table lists the average carbon dioxide level in the atmosphere, measured in parts per million at Mauna Loa Observatory from 1980 to 2000. Use the data to find a model for the carbon dioxide level. Use the linear model to estimate the average  $\text{CO}_2$  level for 1987 and to predict the level for 2010. According to this model, when will the  $\text{CO}_2$  level exceed 400 parts per million?

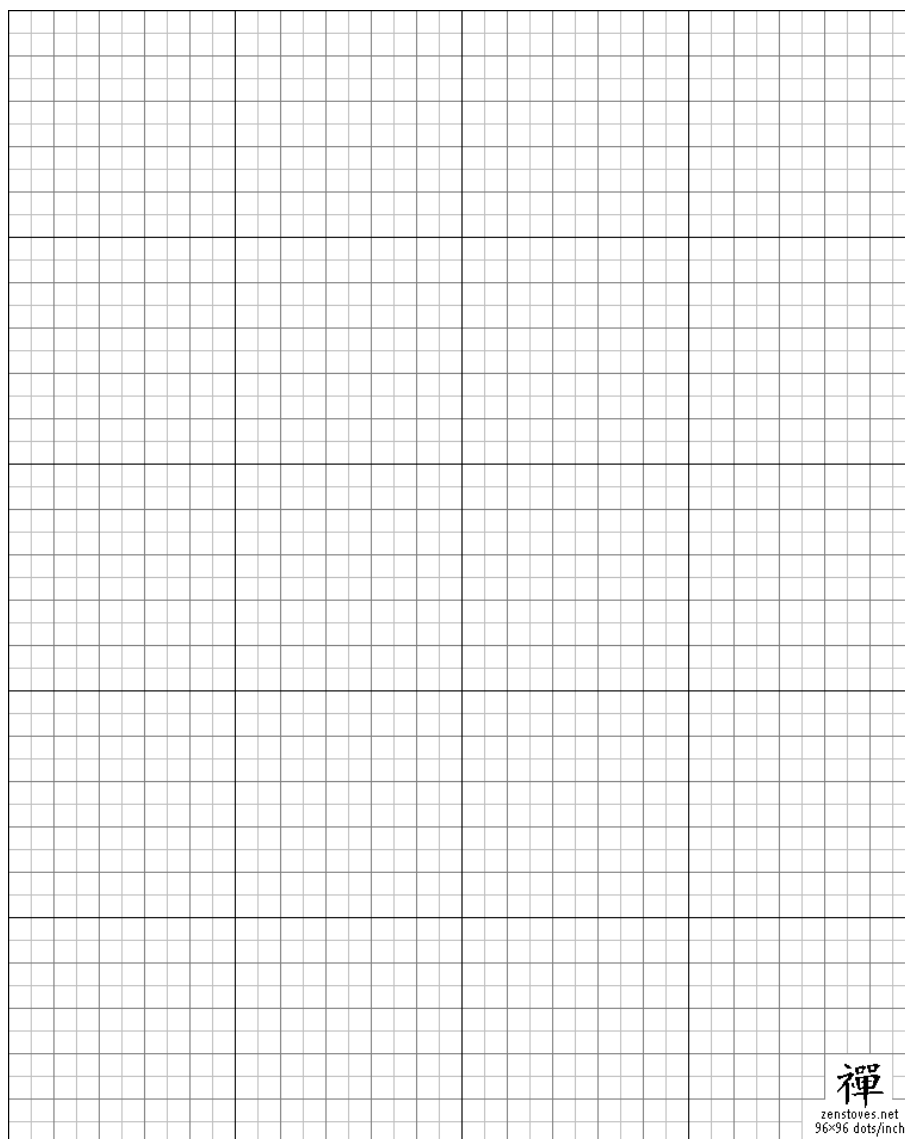
Year	$\text{CO}_2$ level (in ppm)
1980	338.7
1982	341.1
1984	344.4
1986	347.2
1988	351.5
1990	354.2
1992	356.4
1994	358.9
1996	362.6
1998	366.6
2000	369.4

### Exercises

- Find the slope of the line through  $P(-3, 3)$  and  $Q(-1, -6)$ .
- Use slopes to show that the triangle formed by the points  $A(6, -7)$ ,  $B(11, -3)$ ,  $C(2, -2)$  is a right-angled triangle.
- Use slopes to show that the points  $A(-1, 3)$ ,  $B(3, 11)$  and  $C(5, 15)$  lie on the same line.
- Sketch the graph of
  - $x = 3$
  - $y = -2$ .
- Find the graph of the equation that satisfies the given conditions:
  - Through  $(2, -3)$ , slope 6.
  - Through  $(-3, -5)$ , slope  $-7/2$ .
  - Slope 3,  $y$ -intercept  $-2$ .

**Selected Answers:** (iii)  $y = 3x - 2$ .





## 4.4 Reduction of a Non-linear Relationship to a Linear Form\*

### Review of Logs

We defined the *logarithm to base  $a$*  as the inverse function of  $f(x) = a^x$ . Another way of looking at logarithms is to consider them as a way of converting multiplication to addition; division to subtraction; and powers to multiplication — by way of the *laws of logs*:

**Examples**

1.  $\log(5x)$
2.  $\log((3x - 4)/y)$ :
3.  $\log \sqrt{10x^2 - 11}$ .

**Non-Linear Relationships**

We saw earlier that many mathematical relationships take the form of a linear relationship: In particular, if we know that some process is a linear one, but we don't know the *parameters*  $m$  and  $c$ , then we can make measurements of  $Y$  and  $X$ , and plot a straight-line graph of  $Y$  vs  $X$ :

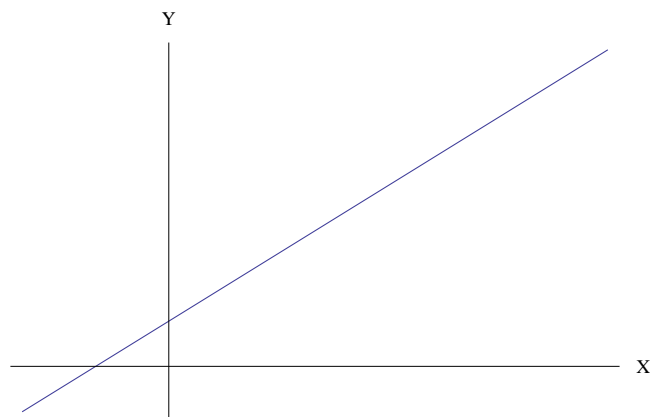


Figure 4.7: If  $Y$  and  $X$  are related via  $Y = mX + c$ , then the graph is a straight-line and we know that  $m$  is the slope and  $c$  is the  $y$ -intercept.

**Example: Radioactive Decay**

However not all mathematical relationships take such a tractable form. For example, it can be shown that the decay of a radioactive material is modeled by:

In this case  $N(t)$  is the number of particles at a time  $t$ ,  $N_0$  is the initial number of particles and  $k$  is a constant. If we were to take measurements of  $N(t_i)$  at various times  $t_i$ , and plotted them, we would get something like:

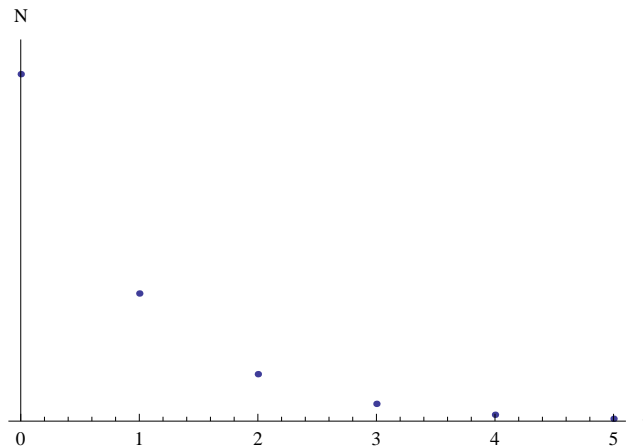


Figure 4.8: In Radioactive Decay, the number of particles reduced exponentially. It is very difficult to extract the parameters  $N_0$  and  $k$  from such a plot.

However — and this will be our tool in this section — we can convert the multiplication and powers in the decay equation by taking logs:

Now what we can do is, as before, measure  $N(t)$  and  $t$  at various  $t$ , but this time we will plot  $\log(N(t))$  vs  $t$  — and the graph will be a straight line. Furthermore, the parameters  $-k$  and  $\log N_0$  — which we want to calculate — will be the given by:

We will review plotting on Friday, but for today we just want to focus on writing non-linear relationships in linear form.

### Examples

In all of the following  $a$  and  $b$  are *constants*,  $e$  is the natural base, and the other two letters are *variables*. If the variables are  $x$  and  $y$  the aim is to write the following in the form:

for some functions  $f$  and  $g$  and constants  $m$  and  $c$  dependent on  $a$  and  $b$  alone. Please identify  $m$  and  $c$ .

1. Write in linear form:

$$F = ab^T.$$

2. Write in linear form:

$$v = ah^b.$$

3. Write in linear form:

$$v = ae^{bt}.$$

## What if it's not all Multiplication and Powers?

The distance traveled by an object in  $t$  seconds, with initial speed  $u$  and constant acceleration  $a$  is given by:

This is not in linear form; can we put it in linear form. The answer is yes, but we have to utilise a tricky method known as *completing the square*. We must outline the general method for completing the square.

### Completing the Square

Let  $q(x)$  be a quadratic (we'll write  $q$  for short):

What I am aiming to do is find real numbers  $h$  and  $k$  such that I could write  $q$  in the form:

Multiplying out<sup>1</sup>:

$$q = a(x^2 + 2hx + h^2) + k.$$

Now what does this compare with what we have:

$$q = a \left( x^2 + \frac{b}{a}x \right) + c$$

Now we will use a trick — we will add and subtract  $ah^2$  as follows:

$$a \left( x^2 + \frac{b}{a}x \underbrace{+ h^2}_{*} \right) + c \underbrace{- ah^2}_{**}.$$

Now to finish this off we need to compare the two:

$$q = a(x^2 + 2hx + h^2) + k = a \left( x^2 + \frac{b}{a}x + h^2 \right) + c - ah^2$$

We therefore need:

This we *can* do. As ever, examples shows off the theory.

### Examples

*Complete the Square:*

---


$$^1(x+y)^2 = (x+y)(x+y) = x^2 + xy + xy + y^2 = x^2 + 2xy + y^2$$

1.  $x^2 - 4x + 29$ .

2.  $2x^2 + 6x + 3$ .

Hence, looking at

$$ut + \frac{1}{2}at^2$$

## Exercises

For the following,  $a$  and  $b$  are the constants. Please write in linear form  $Y = mX + c$ ; identifying  $Y, m, X, c$ :

$$\begin{aligned} P &= ae^{bt}. \\ R &= aV^b. \\ y &= ae^{bx}. \\ y &= ax^b. \\ v &= ae^{tb}. \\ T &= aL^b. \\ p &= at^b. \\ G &= ax^2 + bx \\ F &= \frac{1}{3}at^2 + bt \end{aligned}$$

## 4.5 Manipulation of Data and Plotting of Graphs\*

### Introduction

Consider the spread of a disease *Armpitus shmellyus* in a population. Suppose a medical team recorded the number of people infected over a series of days and compiled the data:

Day, $t$	8	12	18	23	28
Infectants, $I$	36	59	96	129	164

Suppose the relationship is thought to be of the form:

$$I(t) = at^b, \tag{4.2}$$

for some constants  $a, b \in \mathbb{R}$ . This would be an example of *polynomial growth*. Today we will look at a method of testing whether this data satisfies this law.

### The Method

1. Write the equation in linear form. This means that starting with  $y = f(x)$  we end up with:

where  $Y = g_1(x, y)$  and  $X = g_2(x, y)$  (more usually each depend on  $y$  or  $x$ ); and  $m$  and  $c$  are independent of  $x$  and  $y$ .

2. Take your data for  $x$  and  $y$  and manipulate the data to come up with a table:

$x$	$y$	$X$	$Y$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

3. Plot the points  $X$  vs  $Y$  on graph paper.
4. If there is a reasonable straight-line fit we say that the law is verified. Draw the line if this is the case.

### Example

Show that Armpitus shmellyus data is a good fit to  $I = at^b$ .

1. Taking the log of both sides (which base??):

That is, we have  $Y = \log I$  and  $X = \log t$ .

2. We now manipulate the data to fit the linear model:

$t$	$I$	$X = \log t$	$Y = \log I$
8	36		
12	59		
18	96		
23	129		
28	164		

3. Now we plot  $X$  vs  $Y$  — morryah  $\log t$  vs  $\log I$ :
4. Hence the law is verified.

### Another Example: Summer 2007

Pressure  $P$  and velocity  $v$  are believed to be connected by the law  $v = ah^b$ , where  $a$  and  $b$  are constants.

$h$	10.6	13.4	17.2	24.6	29.3
$v$	9.77	11.0	12.44	14.88	16.24

Write the equation in linear form. Plot a suitable graph to verify that the law is true.

1. Taking the log of both sides:

That is, we have  $Y = \log v$  and  $X = \log h$ .

2. We now manipulate the data to fit the linear model:



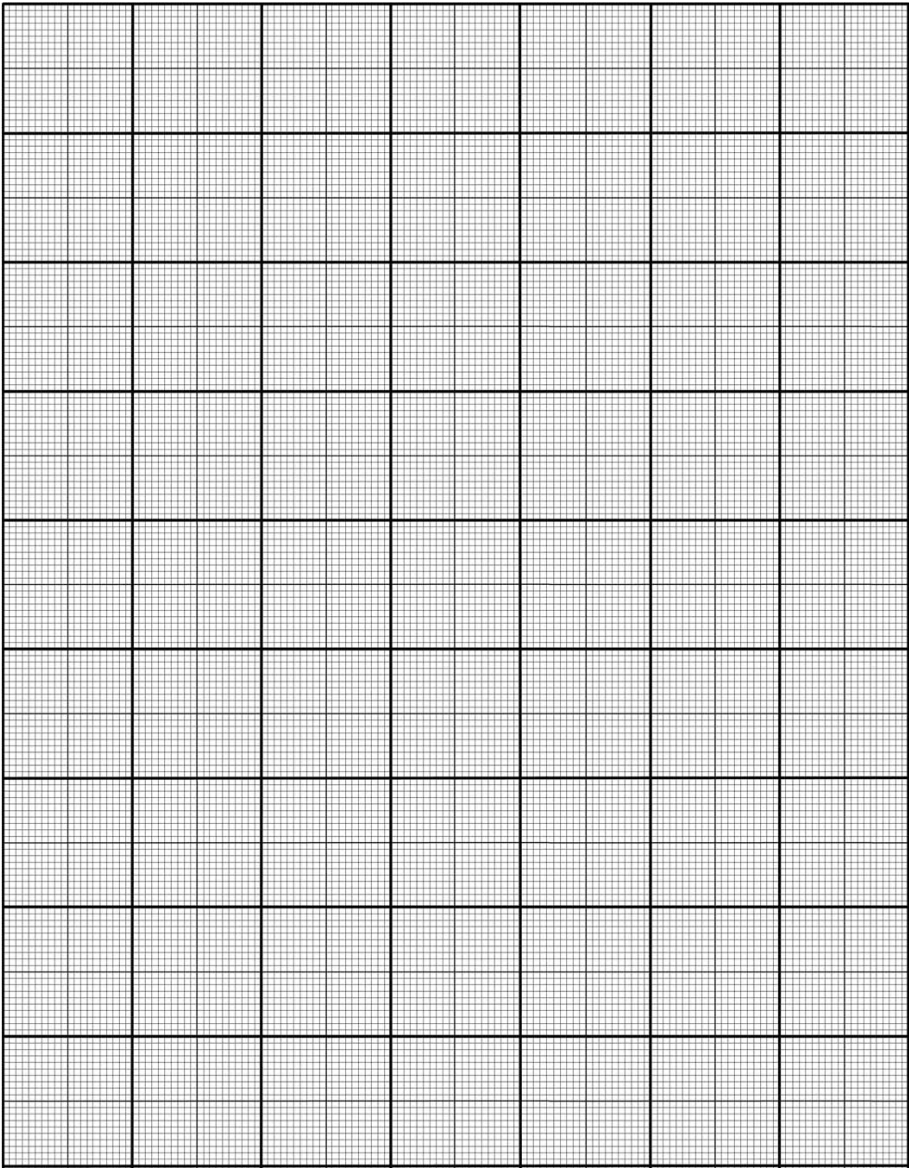


Figure 4.9: If  $Y$  and  $X$  are related via  $Y = mX + c$ , then the graph is a straight-line.

$h$	$v$	$X = \log h$	$Y = \log v$
10.6	9.77		
13.4	11.0		
17.2	12.44		
24.6	14.88		
29.3	16.24		

- 3. Now we plot  $X$  vs  $Y$  — morryah  $\log h$  vs  $\log v$ :
- 4. Hence the law is verified.

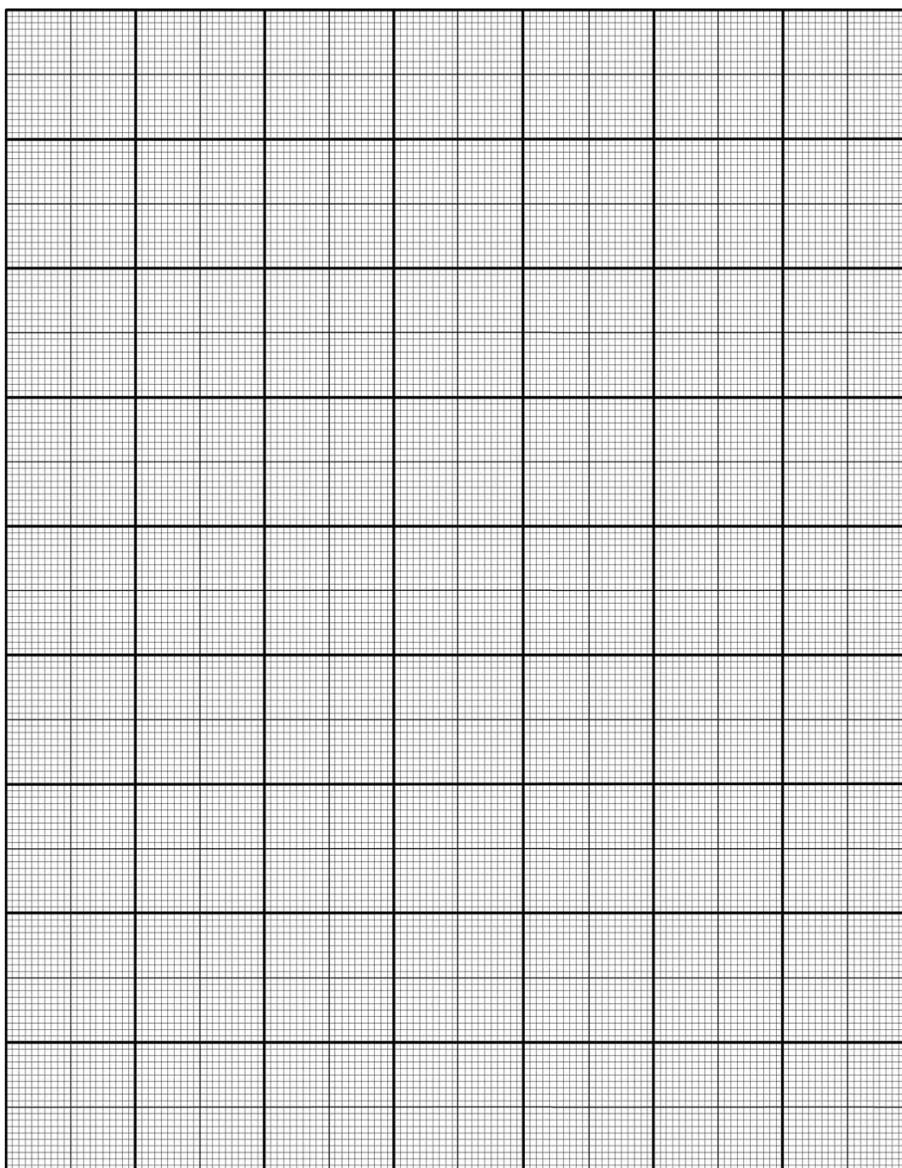


Figure 4.10: If  $Y$  and  $X$  are related via  $Y = mX + c$ , then the graph is a straight-line.

## Exercises

1.  $v$  is believed to be related to  $t$  according to the law  $v = ae^{bt}$ .

$t$	27.3	37.5	42.5	47.8	56.3
$v$	700	190	100	50	17

Show by plotting a graph of  $\log v$  against  $t$  that the law relating these quantities is as stated.

[HINT: The linear form is  $\log v = bt + \log a$ .]

2. In an experiment on moments, a bar was loaded with a load  $W$ , at a distance  $x$  from the fulcrum. The results of the experiment were:

$x$	10	20	30	40	50	60
$W$	55	27.5	18.33	13.75	11	9.17

Verify that a law of the form  $W = ax^b$  is true where  $a$  and  $b$  are constants by plotting  $\log W$  against  $\log x$ .

3. It is believed that  $x$  and  $y$  are related by a law of the form  $y = ar^{kx}$  where  $a$  and  $k$  are constants. Values of  $x$  and  $y$  were measured and the results are as shown:

$x$	0.25	0.9	2.1	2.8	3.7	4.8
$y$	6.0	10.0	25.0	42.5	85.0	198.0

Show by plotting  $\log y$  against  $x$  that the law is true.