MATH6015 — Technological Maths 2

J.P. McCarthy

October 10, 2012

Contents

0.1 Introduction

Lecturer

J.P. McCarthy

Office

Meetings before class by appointment via email only.

Email & Web:

jpmccarthymaths@gmail.com and *http://jpmccarthymaths.wordpress.com*

This page will comprise the webpage for this module and as such shall be the venue for course announcements including definitive dates for the tests. This page shall also house such resources as links (such as to exam papers), as well supplementary material. Please note that not all items here are relevant to MATH6015; only those in the category 'MATH6015'. Feel free to use the comment function therein as a point of contact.

Module Objective

This module introduces differential and integral calculus and treats applications pertinent to the student discipline.

Module Content

Differentiation

Introduction to limits. Definition and graphical interpretation of a derivative. Differentiation of common functions using the product, quotient and chain rules. Applications of differentiation.

Integration

Integration as anti-differentiation. Evaluation of standard integrals using table look-up and the method of substitution. Applications of the definite integral. Solutions of first-order differential equations.

Assessment

Total Marks 100: End of Year Written Examination 80 marks; Continuous Assessment 20 marks.

Continuous Assessment

The Continuous Assessment will be comprised of a one-hour written test worth 20%, in week 6.
Absence from a test will not be considered accept in truly extraordinary cases. Plenty of notice will be given of the test date. For example, routine medical and dental appointments will not be considered an adequate excuse for missing the test.

Lectures

It will be vital to attend all lectures as many of the examples, proofs, etc. will be completed by us in class.

Tutorials

The aim of the tutorials will be to help you achieve your best performance in the tests and exam.

Exercises

There are many ways to learn maths. Two methods which arent going to work are

1. reading your notes and hoping it will all sink in
2. learning off a few key examples, solutions, etc.

By far and away the best way to learn maths is by doing exercises, and there are two main reasons for this. The best way to learn a mathematical fact/ theorem/ etc. is by using it in an exercise. Also the doing of maths is a skill as much as anything and requires practise.
There are exercises in the notes for your consumption. The webpage may contain a link to a set of additional exercises. Past exam papers are fair game. Also during lectures there will be some things that will be *left as an exercise*. How much time you can or should devote to doing exercises is a matter of personal taste but be certain that effort is rewarded in maths.

Reading

Your primary study material shall be the material presented in the lectures; i.e. the lecture notes. Exercises done in tutorials may comprise further worked examples. While the lectures will present everything you need to know about MATH6015, they will not detail all there is to know. Further references are to be found in the library. Good references include:

- P. Tebbutt 1998, *Basic Mathematics*, John Wiley & Sons
- J.O.Bird 2005, *Basic Engineering Mathematics*, 4th Ed., Newnes

The webpage may contain supplementary material, and contains links and pieces about topics that are at or beyond the scope of the course. Finally the internet provides yet another resource. Even Wikipedia isn't too bad for this area of mathematics! You are encouraged to exploit these resources; they will also be useful for further maths modules.

Exam

The exam format will roughly follow last year's. Acceding to the maxim that learning off a few key examples, solutions, etc. is bad and doing exercises is good, solutions to past papers shall not be made available (by me at least). Only by trying to do the exam papers yourself can you guarantee proficiency. If you are still stuck at this stage feel free to ask the question come tutorial time.

0.2 Motivation: Rates of Change, Tangents to Curves & The Problem of Measure

Although this may seem a paradox, all exact science is dominated by the idea of approximation.

Bertrand Russell

Just to give a structure to the module, we will talk about what we want to achieve in this module. We will talk in the loosest terms possible — and won't get bogged down in too much detail.

0.2.1 Revision: Functions

We should have all have a passing acquaintance with the idea of a *function*. Suppose A is one collection of objects, and B is another collection. A function is like a map between A and B:

We can think of A as the collection of inputs, and B the collection of outputs. When we write $f : A \to B$ we mean that f is a function from A to B. In this module the collections we are interested in are collections of numbers. For example the real numbers (i.e. the numbers on the numberline) — and we are interested in functions — or maps — that send real numbers to real numbers. As an example, the function $f(x) = x^2$ takes a real number input — and the output is that real number squared. This is a purely algebraic picture, but we can also consider it in the geometric picture[1].

[1]and one of the major themes of modern mathematics is thus. See more: http://irishjip.wordpress.com/2011/03/14/my-understanding-of-non-commutative-geometry/

We can look at the *graph* of a function:

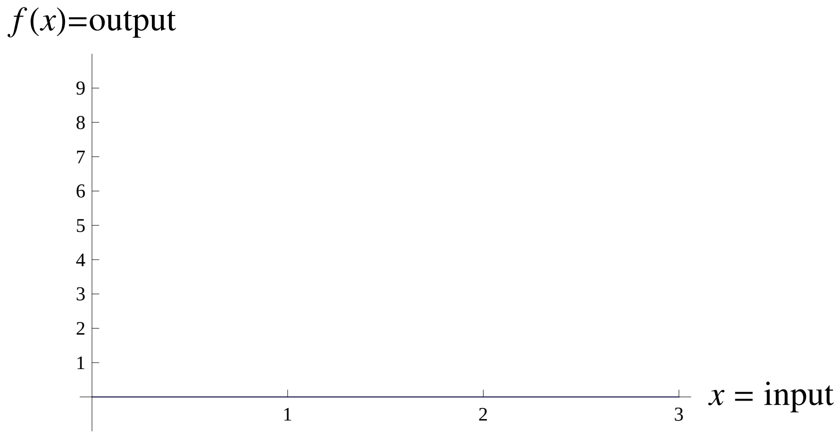


Figure 1: To plot the graph of a function — the collection of pairs $(x, f(x))$ as x runs over all the real numbers — you examine the outputs for various inputs.

Examples

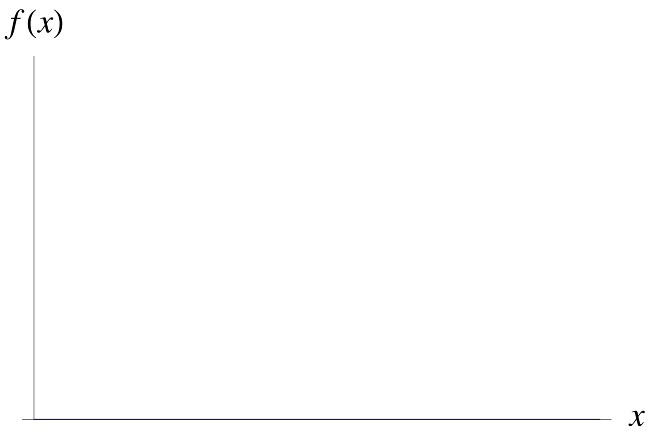


Figure 2: A constant function, e.g. $f(x) = 1$.

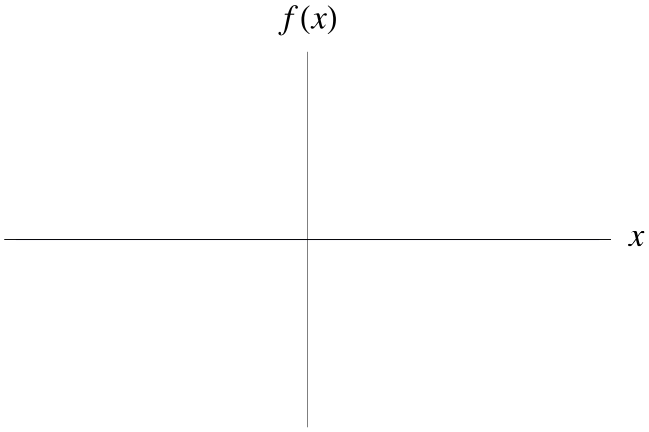


Figure 3: The line $f(x) = x$ is strictly increasing.

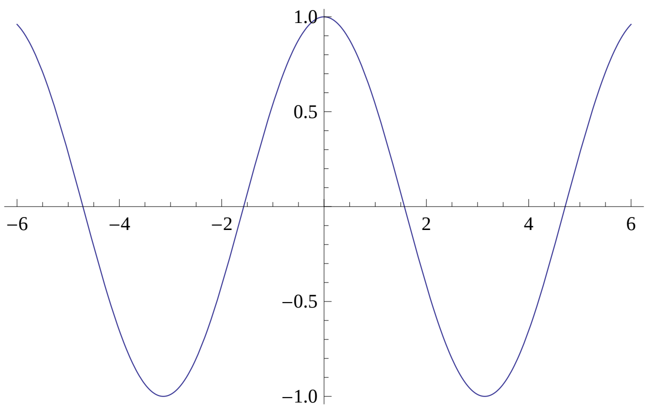


Figure 4: The cos function is strictly increasing on $[\pi, 2\pi]$ and symmetric about the y-axis.

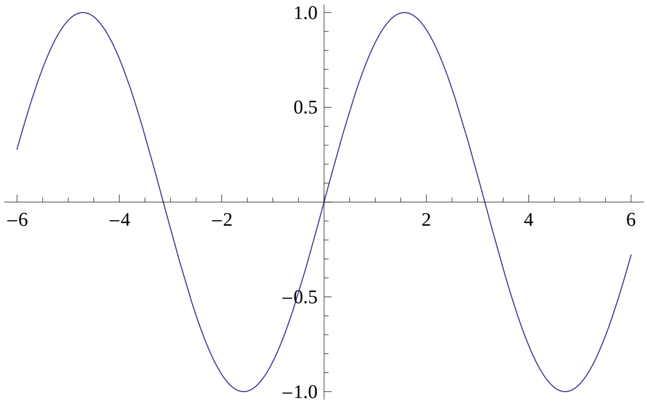


Figure 5: The sin function is strictly increasing on $[-\pi/2, \pi/2]$ and antisymmetric about the y-axis.

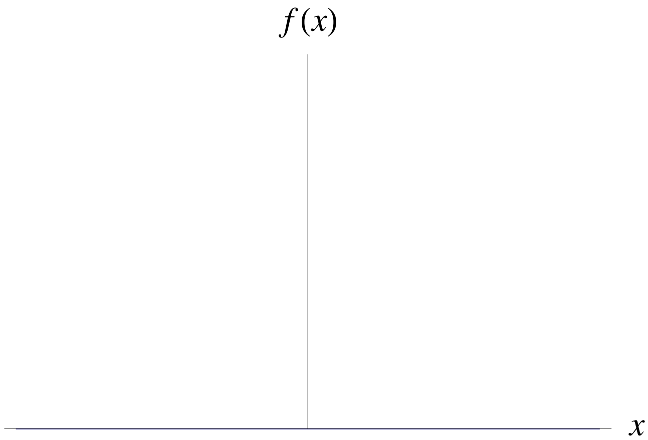


Figure 6: The function $f(x) = x^2$ is strictly increasing on $[0, \infty)$ and symmetric about the y-axis.

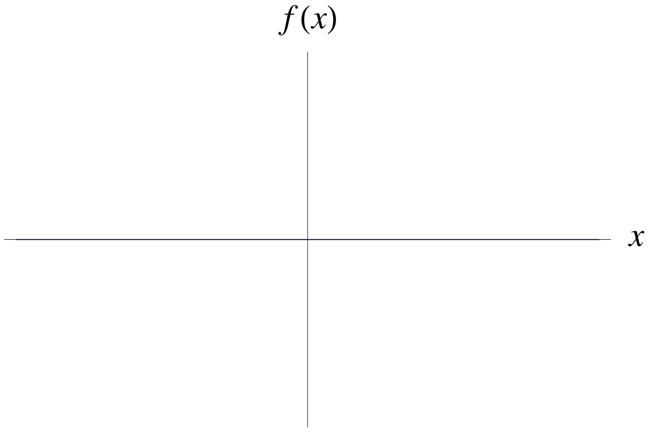


Figure 7: The function $f(x) = x^3$ is increasing *everywhere* and antisymmetric about the y-axis.

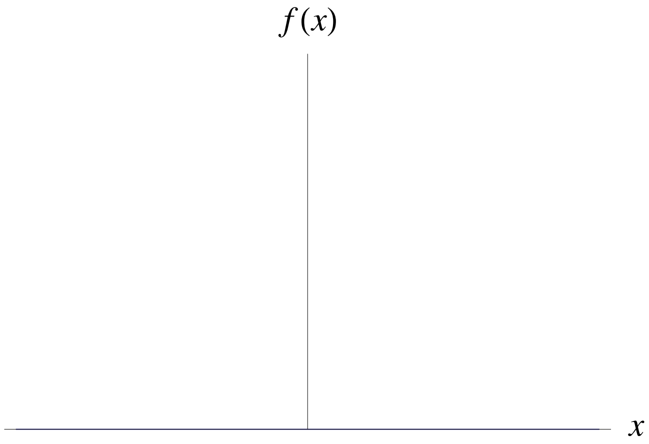


Figure 8: The function $f(x) = x^4$ is similar to the quadratic function in that it is symmetric about the y-axis and increasing on $[0, \infty)$

Remark

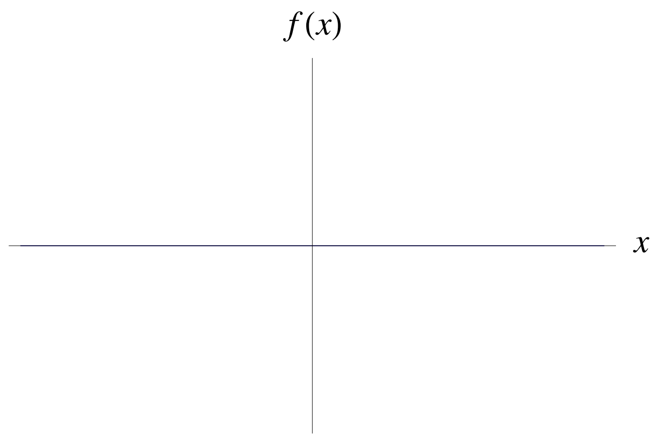
Suppose the graph of $y = f(x)$ is known. Then the graph of $y = f(x) + a$ is simply found by shifting the curve a units upwards (or indeed downwards if a is negative).

0.2.2 The Line

We need to have a very good handle on the line. A line of slope $m \in \mathbb{R}$ and y-intercept has equation:

What this means is that every point on the graph of $l(x)$ has coordinates $(x, mx + c)$ for some $x \in \mathbb{R}$. Conversely[2], *every* function of the form $f(x) = ax + b$ is a line of slope a and y-intercept b.

Examples



0.2.3 The Quadratic Function: Definition

For $a, b, c \in \mathbb{R}$, $a \neq 0$, any function $f : \mathbb{R} \to \mathbb{R}$ of the form:

$$f(x) = ax^2 + bx + c \tag{1}$$

is a *quadratic* function.

0.2.4 Definition

The *roots* of a function are:

$$\{k \in \mathbb{R} : f(k) = 0\} \tag{2}$$

i.e. the numbers when imputed into f produce 0, or where the graph of f cuts the x-axis.

[2]Conversely means *on the other hand*. [Ex]: The justification of these facts to yourself are left as an exercise.

0.2.5 Proposition

The roots of (1) are given by:

$$x_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{3}$$

Remark

1. It can be shown that every quadratic is simply a translation of x^2 and so *must* have shape $\bigcup$ or $\bigcap$.

2. Note that a quadratic is symmetric in the line $x = b/2a$ and indeed attains its minimum at this point. Suppose x gets 'big'. When x is 'big' $ax^2 \gg bx \gg c$ so that the function looks more and more like ax^2. If $a > 0$ the quadratic has shape like x^2 ($\bigcup$). If $a < 0$, $f(x)$ looks like an upside down x^2 ($\bigcap$).

3. Examining (3), note that if $b^2 - 4ac < 0$ then there is no (real) number equal to $\sqrt{b^2 - 4ac}$ as a real number squared is always positive. The roots are *complex*. In this case the graph of f does not cut the x-axis at any point:

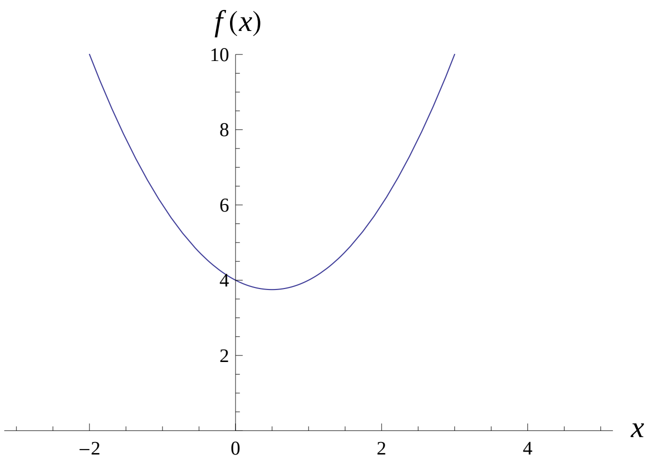


Figure 9: The function $f(x) = x^2 - x + 4$ has no real roots.

If $b^2 - 4ac = 0$ then the roots are real and equal,

$$x = \frac{-b \pm 0}{2a} = \frac{b}{2a}$$

In this case the graph has as a tangent the x-axis:

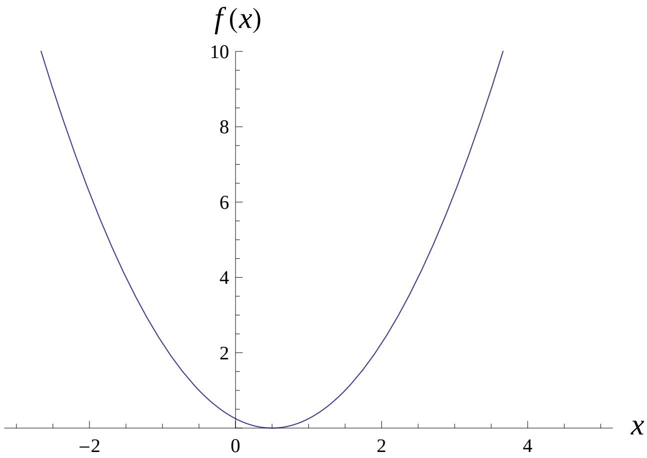


Figure 10: The function $f(x) = x^2 - x + 1/4$ has two equal, real roots.

Finally if $b^2 - 4ac > 0$ then the roots are real and distinct. In this case the function cuts the x-axis at two points:

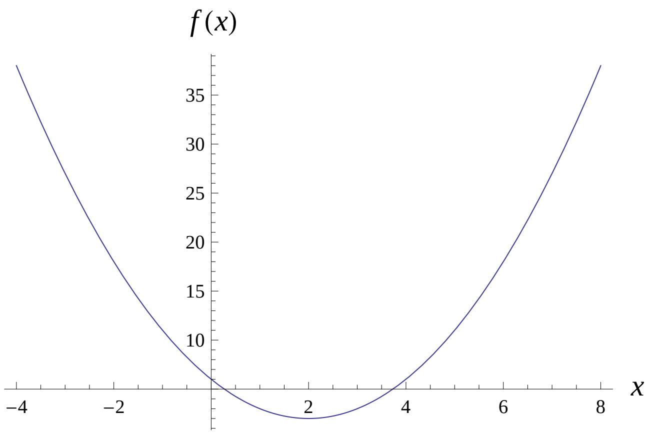


Figure 11: The function $f(x) = x^2 - 4x + 1$ has two distinct, real roots.

Exercises

1. True-False Quiz: Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why. If it is false, explain why or give an example that disproves the statement.
   (a) If f is a function, then $f(s+t) = f(s) + f(t)$.
   (b) If $f(s) = f(t)$ then $s = t$.
   (c) If f is a function, then $f(3x) = 3f(x)$.
2. An airplane flies from an airport and lands an hour later at another airport, 400 miles away. If t represents the time in minutes since the plane has left the terminal building, let $x(t)$ be the horizontal distance travelled and $y(t)$ be the altitude of the plane.
   (a) Sketch a possible graph of $x(t)$.
   (b) Sketch a possible graph of $y(t)$.
   (c) Sketch a possible graph of the ground speed.
   (d) Sketch a possible graph of the vertical speed.
3. If $f(x) = 3x^2 - x + 2$, find $f(2)$, $f(-2)$, $f(a)$, $f(-a)$, $f(a+1)$, $2f(a)$, $f(2a)$, $f(a^2)$, $[f(a)]^2$ and $f(a+h)$.
4. A spherical balloon with radius r inches has volumes $V(r) = \frac{4}{3}\pi r^3$. Find a function that represents the amount of air required to inflate a balloon from a radius of r inches to a radius of $r+1$ inches.
5. A rectangle has perimeter 20 m. Express the area of the rectangle as a function of the length of one of its sides.
6. A taxi company charges two euro for the first mile (or part of a mile) and 20 cent for each succeeding tenth of a mile (or part). Sketch the cost function C (in euros) of a ride as a function of the distance x travelled (in miles) for $0 < x < 2$.
7. (a) Find an equation for the family of lines with slope 2 and sketch several members of the family.
   (b) Find an equation for the family of lines such that $f(2) = 1$ and sketch several members of the family.
   (c) Which line belongs to both families.
8. The relationship between the Fahrenheit (F) and Celsius (C) temperature scales is given by the line $F = \frac{9}{5}C + 32$.
   (a) Sketch a graph of this function.
   (b) What is the slope of the graph and what does it represent.

0.2.6 Smooth Functions

A good question — although a difficult one — is which functions have a *smooth* graph?

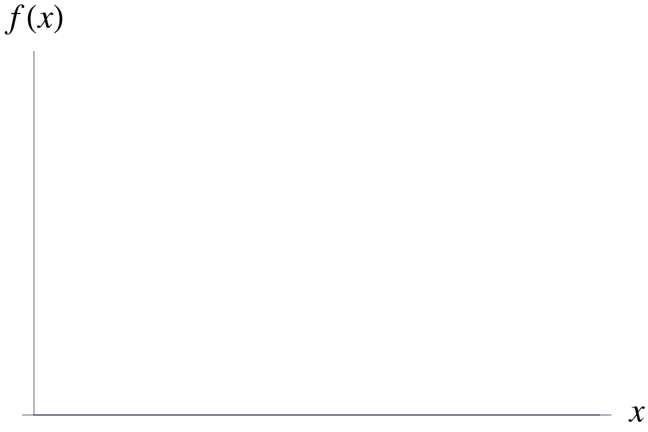


Figure 12: A graph of a smooth function and a non-smooth function. When I say smooth I mean that the graph of the function has no discontinuities or jagged edges.

What we can see is that a function has the property of being smooth if it has a well-defined tangent at each point. We will become very interested in the *slope of the tangent.*

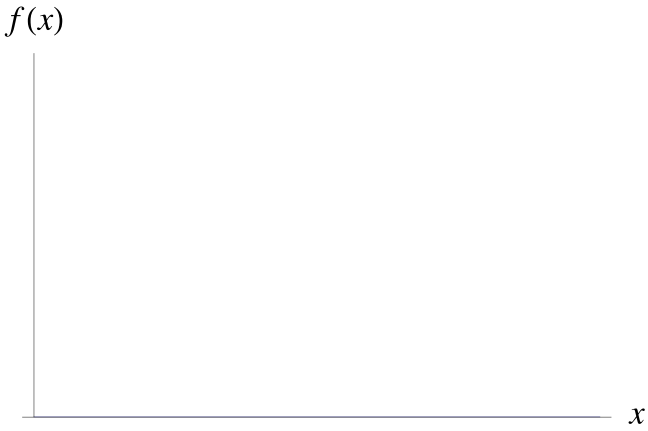


Figure 13: Only at point a does the graph of the function have a well defined tangent. At point b there are many tangents — and at c we can't even draw a tangent.

So smooth functions are basically those that are continuous with a nicely varying slope. Given a function $f(x)$, is it possible to estimate the slope of the tangent at a point a say? Now slope is nothing but the ratio of $\uparrow$ to $\rightarrow$ so we can say that the slope of the tangent is estimated by:

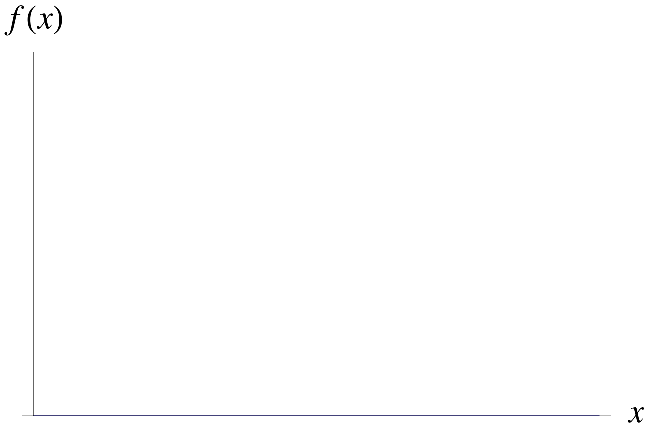


Figure 14: We can estimate the slope of the tangent at a by looking at the slope of the *secant*.

Now what we can do, is take h to be smaller and smaller — in fact look at the limit as h tends to zero. As h gets smaller and smaller, our estimate is getting better and better. We have then that the slope of the tangent is given by:

But be careful — h *cannot* equal zero. This is going to be a feature of limits — we will be interested in what happens when a number *approaches* another number — we don't care about what happens *at* that number. What happens here at $h = 0$? This expression here is the familiar *derivative*, and we write:

DON'T FORGET THIS!

0.2.7 Rates of Change

If you watch the speedo of a car, it is clear that the speed of the car is not constant. The speedo seems to tell is that the car has a definite speed at each moment: but how is the 'instantaneous' speed defined?

Suppose we take a car out on a test track and record it's distance along a straight as a function of time:

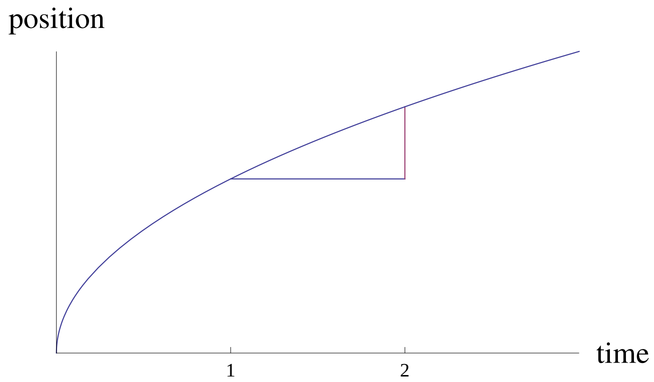


Figure 15: What is the speed of the car after 1 s?

Consider the triangle in the graph of position, $s(t)$; vs time, t. We can use this to find an average speed. The perpendicular height is $\Delta s = s(2) - s(1)$. Clearly the length of the base $\Delta t = 1$. Now how do we define average speed?

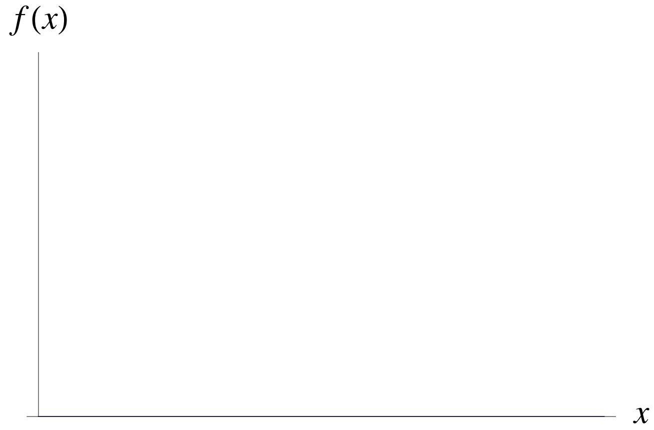
Therefore from $t = 1$ to $t = 2$ the car's average speed is given by

The difficulty in finding the speed after 1 s is that we are dealing with a single instant of time ($t = 1$), so no time interval is involved[3].

[3] see the Arrow Paradox

However we can approximate the desired quantity by computing the average velocity over a brief time interval of say a tenth of a second:

If we examine this geometrically it appears that the average speed over smaller and smaller intervals is becoming closer to a certain value... the slope of the tangent at $t = 1$.



What does the quantity

$$\lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt} \tag{4}$$

represent?

So speed is nothing but the derivative of position. By noting that speed is the rate of change of distance we can by analogy pretty much see the following.

Important Fact

Suppose that $F(x)$ is some variable then the rate of change of $F(x)$ with respect to x is given by

$$F'(x) = \frac{dF}{dx}$$

.

The speed of a particle is the rate of change of distance with respect to time. Physicists are interested in other rates of change as well — for instance, the rate of change of work with respect to time (which is called *power*). Chemists who study a chemical reaction are interested in the rate of change in the concentration of a reactant with respect to time (called the *rate of reaction*). A steel manufacturer is interested in the rate of change of the cost of producing x tonnes of steel per day with respect to x (called the *marginal cost*). A biologist is interested in the rate of change of a the population of a colony of bacteria with respect to time. In fact, the computation of rates of change is important in all of the natural sciences, in engineering, and even in the social sciences. All these rates of change can be interpreted as slopes of tangents. This gives added significance to the solution of the tangent problem.

0.2.8 The Problem of Area

The theme of the second chapter is arguably how to assign a size to certain sets — usually shapes. Let's have a closer look at areas:

An even more flexible shape than the rectangle is the triangle:

Triangles are actually more basic than rectangles since we can represent every rectangle, and actually and odd-shaped quadrangle, as the 'sum' of two non-overlapping triangles:

In doing so we have *tacitly* assumed a few things. For the triangles we have chosen a *particular* base line and the corresponding height arbitrarily. But the concept of *area* should not depend on such a choice and the calculation this choice entails. Independence of the area from the way we calculate it is called *well-definedness*. Plainly,

Notice that this allows us the most convenient method to work out in the area. In calculating the area of a quadrangle we actually used two assumptions:

- the area of non-overlapping (disjoint[4]) sets can be added, i.e.

- congruent triangles have the same area[5].

The above rules allow us to measure arbitrarily odd-looking *polygons*[6] using the following recipe: dissect the polygon into non-overlapping triangles and add their areas. But what about *curved* or even more complicated shapes, say,

[4]empty intersection

[5][Ex:] argue using the idea of congruent triangles why the area *should* be half the base times the perpendicular height — this argument here takes the area of a triangle as fundamental

[6]a figure formed by three or more points in the plane joined by line segments

Here is *one* possibility for the circle: inscribe a regular j-sided polygonal[7] into the circle, subdivide it into congruent triangles, find the area of each of these slices and then add all j pieces:

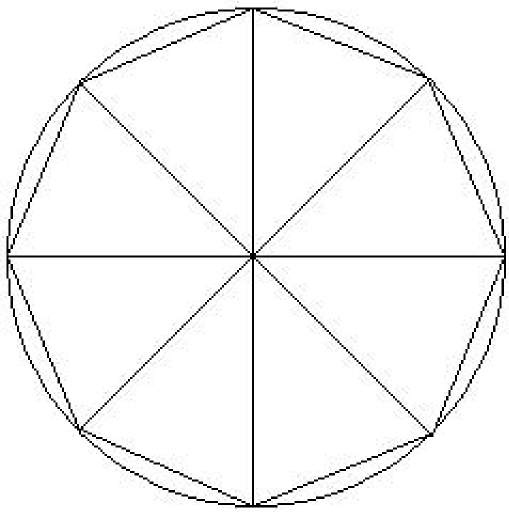


Figure 16: The area of the circle is approximated by triangles. Note the similarity with differentiation — we approximate the slope of the tangent and then take a limit. Here we repeat the trick.

In the next step increase $j \to j+1$ by increasing by one the number of points on the circumference and repeat the above procedure. Eventually[8],

Again, there are a few problems: does the limit exist? Is it submissible to subdivide a set into arbitrarily many subsets — each of vanishingly small area? Is the procedure independent of the particular subdivision? In fact, nothing should have stopped us from paving the circle with ever smaller squares! For a reasonable notion of area measure the answer to these questions must be assumed to be *yes*. It can be show that an area measure satisfying all these conditions is powerful enough to cater for all our everyday needs and for much more.

[7] made from points spaced at an equal distance around the circle

[8] [Ex]: find this limit by approximating the bases by the arc-length between each point — and the perpendicular height by the radius, r.

This good notion of area measure allows us to introduce integrals, basically starting with the naïve (but valid) idea that the integral of a positive function should be the same as the area of the set between the graph of the function and the x-axis:

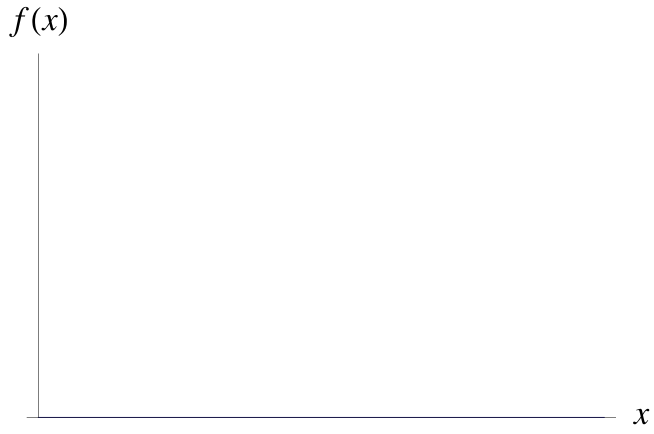


Figure 17: We shall *define* the object $\int_a^b f(x)\,dx$ as the area A.

Exercises

1. Estimate the area under the graph $f(x) = 1/x$ from $x = 1$ to $x = 5$ using four approximating rectangles. Sketch the graph and the rectangles. Is your answer an underestimate or overestimate.

2. Estimate the area under the graph $f(x) = 25 - x^2$ from $x = 0$ to $x = 5$ using five approximating rectangles. Sketch the graph and the rectangles. Is your answer an underestimate or overestimate.

Chapter 1

Differentiation

In the fall of 1972 President Nixon announced that the rate of increase of inflation was decreasing. This was the first time a sitting president used the third derivative to advance his case for reelection.

Hugo Rossi

1.1 Limits

Suppose we have a function $f : \mathbb{R} \to \mathbb{R}$ with a *hole discontinuity* at $a \in \mathbb{R}$ as shown:

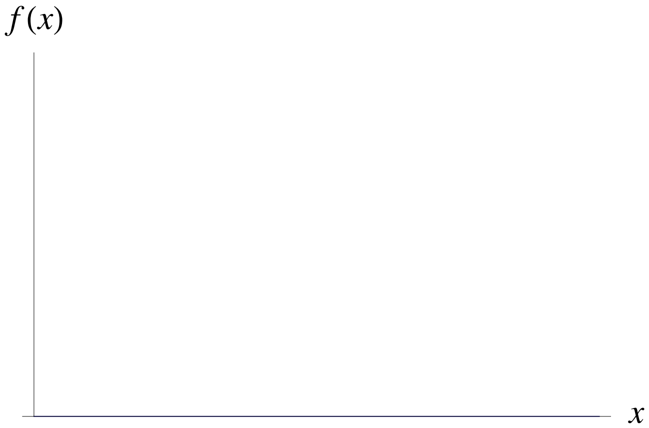


Figure 1.1: Although it appears that $f(a) = L$, this in fact cannot be the case as f is undefined at a: for the input a an output does not exist.

We can see in some way that as the inputs get closer and closer to a, that the outputs get closer and closer to L and informally at this stage we would write:

Remarks

1. If L is *the limit of f as x tends to a* then we write

   or $f(x) \to L$ as $x \to a$.

2. We do *not* require that $f(a) = L$: indeed we do not even require that $f(a)$ be *defined*.

3. If the limit exists then it is unique - but it *need not exist*.

1.1.1 Proposition (Calculus of Limits)

Suppose that f and g are two functions $\mathbb{R} \to \mathbb{R}$, and that for some point $a \in \mathbb{R}$ we have

$$\lim_{x\to a} f(x) = p \ , \quad \textbf{\textit{and}} \ \lim_{x\to a} g(x) = q.$$

for some $p, q \in \mathbb{R}$. *Then*

(i) $\lim_{x\to a}(f(x) + g(x)) = p + q$.

(ii) If $k \in \mathbb{R}$, $\lim_{x\to a} kf(x) = kp$.

(iii) $\lim_{x\to a}(f(x)g(x)) = pq$.

(iv) If $q \neq 0$, $\lim_{x\to a}(f(x)/g(x)) = p/q$.

(v) If $n \in \mathbb{N}$, *and* $p > 0$ *then* $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{p}$.

In everyday English:

- the limit of a sum is the sum of the limits.
- the limit of a constant times a function is the constant times the limit of the function.
- the limit of a product is the product of the limits.
- the limit of a quotient is the quotient of the limits.
- the limit of a root is the root of the limit.

1.1.2 Proposition

Suppose f and g are functions for which $f(x) = g(x)$ for all $x \neq a$. If $\lim_{x\to a} f(x)$ **exists** *then so does* $\lim_{x\to a} g(x)$, *and moreover* $\lim_{x\to a} f(x) = \lim_{x\to} g(x)$.

Remark

Along with the calculus of limits above this fact is all we need to evaluate any limits we will encounter in MATH6015.

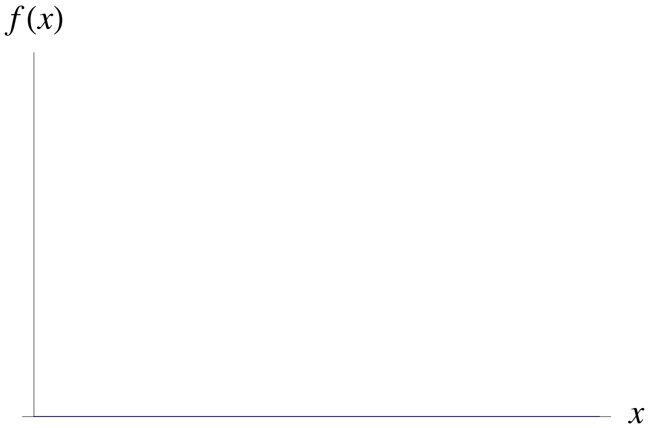


Figure 1.2: This is another geometrically plausible fact that we can prove. It's importance can be seen below.

Examples

1. Evaluate $\lim_{x\to 5}(2x^2 - 3x + 4)$.

   Solution: Here we can just plug in $x = 5$ as

2. Find the value of $\lim_{x\to 1}\dfrac{x-1}{x^2-1}$.

   Solution: If we put in $x = 1$ we get $0/0$ which is undefined. So we are going to have to look at $x \neq 1$. Let us simplfy in this region:

   Now take the limit

3. Evaluate $\lim_{h\to 0}\dfrac{(3+h)^2 - 9}{h}$.

   Solution: Plugging in $h = 0$ is not allowed here as this would be division by zero... Hence we look at '$f(h)$' for $h \neq 0$:

Now we can just plug in $h = 0$:

So we can see our technique is going to be:

1. plug in the point. If this doesn't work:
2. simplify the function and plug in the point.

Exercises:

1. Evaluate the limit
   (a) $\lim_{x \to -2} (3x^4 + 2x^2 - x + 1)$
   (b) $\lim_{t \to -1} (t^2 + 1)^3 (t + 3)^5$
   (c) $\lim_{x \to 4^-} \sqrt{16 - x^2}$

   Selected Answer: (iii) 0

2. Evaluate the limit
   (a) $\lim_{x \to -4} \dfrac{x^2 + 5x + 4}{x^2 + 3x - 4}$
   (b) $\lim_{t \to -3} \dfrac{t^2 - 9}{2t^2 + 7t + 3}$
   (c) $\lim_{x \to 1} \dfrac{x^3 - 1}{x^2 - 1}$
   (d) $\lim_{x \to 2} \dfrac{x^4 - 16}{x - 2}$

   Selected Answer: (iii) $\frac{3}{2}$

1.2 The Derivative

In a rough sense, a function is *smooth* if it has a well-defined tangent at each point:

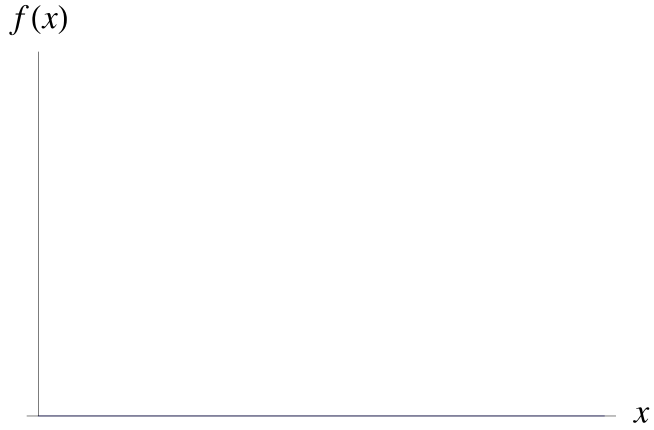


Figure 1.3: We want to develop an algebraic picture of what a smooth function looks like.

We have actually discussed this issue in the Motivation, so we will move right into the definition.

1.2.1 Definition

A function $f:\mathbb{R}\to\mathbb{R}$ is *smooth* at $a\in\mathbb{R}$ if

In general we deal with smooth-everywhere functions and instead use $x\in\mathbb{R}$:

$f'(x)$ is the *derivative of* $f(x)$. If $f'(x)$ exists for all $x\in\mathbb{R}$, then $f(x)$ is said to be *smooth*. In this case, $f':\mathbb{R}\to\mathbb{R}$ is a new function.

Remark

The derivative of $f(x)$ is denoted $f'(x)$. Other names for the derivative of $f(x)$ include:

- the differentiation of $f(x)$
- the derived function for $f(x)$
- the slope of the tangent at $(x, f(x))$
- the gradient
- $\dfrac{df}{dx}$

There is an alternate and superior notation to that of 'f'(x)'. This is the *Leibniz notation*[1]. If we zoom in on a curve:

we see that the slope of the secant is:

This Δ is the capital of the Greek letter *delta* — and it usually signifies 'a change in'. When we make h (morryah Δx) small, Δf will also get small and we end up using

where the capital 'D's and turned into little 'd's. Given a function $y = f(x)$, there is no difference between $f'(x)$ and dy/dx.
To reiterate if $y = f(x)$; then dy/dx is the same thing as:

- the derivative of $f(x)$
- the differentiation of $f(x)$
- the derived function for $f(x)$
- the slope of the tangent at $(x, f(x))$
- the gradient
- $f'(x)$

1.3 Differentiation of Common Functions

The equation of a line of slope m, and y-intercept c is $l(x) = mx + c$. What is the tangent to a line?

[1]The derivative was independently developed by Isaac Newton and G.W. Leibniz. The 'f'(x)' notation is Newton's.

1.3.1 Proposition

If $l : \mathbb{R} \to \mathbb{R}$ **is a straight line of slope** m**, given by** $l(x) = mx + c$, *then* $l'(x) = m$.

Proof.

As we expect. What is the slope of a constant function?

Example

Define $f : \mathbb{R} \to \mathbb{R}$ *by* $f(x) = x^2 + 5x + 2$. *Find* $f'(x)$.

Solution:

Winter 2010: Question 2(a)

Differentiate $y = 4x^2 - 3x + 3$ from first principles.

Solution: Here we just have to use the formula that we derived in the motivation:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \tag{1.1}$$

A good idea is to take your time. From the discussion on notation we should know that $y = y(x) = f(x)$. First we calculate $f(x+h)$:

Now we take away $f(x) = 4x^2 - 3x + 3$ to find $f(x+h) - f(x)$:

Now divide by h to get $\dfrac{f(x+h) - f(x)}{h}$:

Finally we take the limit as $h \to 0$ to find to find the derivative

Winter 2011: Question 2(a)

Differentiate $y = 3x - 3x^2$ from first principles.

Solution: Let $f(x) = 3 - x - 3x^2$. We calculate

Now divide by h to get $\dfrac{f(x+h) - f(x)}{h}$:

Finally we take the limit as $h \to 0$ to find to find the derivative

Exercises: Differentiate each of the following with respect to x from first principles:

$$(i)\, x^2 - 2x + 5 \quad (ii)\, x^2 + 5x \quad (iii)\, 3x + 2$$

$$(iv)\, 2x^2 - 5x \quad (v)\, 2x - x^2$$

Selected Answer: (iii) 3

1.3.2 Proposition (Power Rule)

For each $n \in \mathbb{Q}$, $f_n(x) = x^n$ is differentiable on $\mathbb{R}$ with derivative $f_n'(x) = nx^{n-1}$.

Proof. The proof is split into a number of cases. For $n \in \mathbb{N}$ we can use either induction or the binomial theorem. For negative $n \in \mathbb{Z}$ we can use the Quotient Rule, which we will see in Section 1.4.2. Finally to prove it for genuine fractions $n \notin \mathbb{Z}$ we use implicit differentiation, which we won't be covering in MATH6015 •

Remark

So for all integer powers, we can simply say, *bring down the power, and lower the power by* 1.

1.3.3 Linearity of Differentiation

Let $f, g : \mathbb{R} \to \mathbb{R}$ **be functions.** *Then*

(i) $f + g$ *has derivative* $(f + g)'(x) = f'(x) + g'(x)$ *[Sum Rule].*

(ii) ***For*** $k \in \mathbb{R}$**,** kf *has derivative derivative* $(kf)'(x) = kf'(x)$.

Remark

In the Leibniz notation,

Examples

1. Find $f'(x)$ where

$$f(x) = x^{17} - 3x^8 + 4x^5.$$

Solution:

2. Find $g'(x)$ where
$$g(x) = x^7 + \frac{1}{x^2} - \frac{5}{x^{10}}.$$

   Solution: To differentiate, write g slightly differently — all in terms of powers of x:

3. Find $h'(x)$ where
$$s(x) = \sqrt{x}.$$

   Solution: Can we write $\sqrt{x}$ as a power...

4. Differentiate $\sqrt{x^3}$ with respect to x.

   Solution: Again write the function a power of x if possible:

Winter 2010: Question 1 (a)

Differentiate by rule $y = x^5 - 2\sqrt{x} + \dfrac{4}{x}$.

Solution: First we write everything as powers of x:

Now we use linearity to differentiate term-by-term:

Winter 2010: Question 2(b)(i)

Differentiate by rule $y = 3x^4 + \dfrac{2}{x^4} + 5\sqrt{x} + 6$.

Solution: Same again rewrite the function:

and differentiate

Exercises:

1. Differentiate the function

$$(i)\, f(x) = 5x - 1 \quad (ii)\, g(x) = 5x^8 - 2x^5 + 6 \quad (iii)\, V(r) = \tfrac{4}{3}\pi r^3$$

$$(iv)\, R(x) = \tfrac{\sqrt{10}}{x^7} \quad (v)\, y = x^{-2/5} \quad (vi)\, g(u) = u\sqrt{2} + \sqrt{3u}$$

   Selected Answers: (iii) $4\pi r^2$ **(vi)** $\sqrt{2} + \frac{\sqrt{3}}{2\sqrt{u}}$.

2. Differentiate $2x^5$.
3. Differentiate $9 + 3x - 5x^2$ with respect to x.
4. Find the derivative of $4(3 - x)^2$.
5. Differentiate $(1 + 3x)^2$.
6. Differentiate $x^3 + 2\sqrt{x}$.
7. Differentiate $\sqrt{x}(x + 2)$.
8. Differentiate $(1 + 7x)^3$ with respect to x.
9. Differentiate $Y(u) = (u^{-2} + u^{-3})(u^5 - 2u^2)$.
10. Find the equation of the tangent to $y = \dfrac{1}{x}$ at the point $\left(2, \dfrac{1}{2}\right)$.

1.3.4 Proposition

(i) **If** $f(x) = \sin x$**,** *then* $f'(x) = \cos x$ *for all* $x \in \mathbb{R}$*.*

(ii) **If** $f(x) = \cos x$**,** *then* $f'(x) = -\sin x$ *for all* $x \in \mathbb{R}$*.*

(iii) **If** $f(x) = \tan x$**,** *then* $f'(x) = \sec^2 x = (\sec x)^2$ *for all* $x \in \mathbb{R}$ *such that* $\cos x \neq 0$*. (*$\sec = 1/\cos$*)*

(iv) **If** $f(x) = e^x = \sum_{i=0}^{\infty} x^i/i!$**,** *then* $f'(x) = e^x$ *for all* $x \in \mathbb{R}$*. Note that* $e^x > 0$ *for all* $x \in \mathbb{R}$*.*

(v) **If** $f(x) = \log x$**,** *then* $f'(x) = 1/x$ *for all* $x > 0$*.*

Proof. (i) Using the identity[2]:

$$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right) \tag{1.2}$$

We can write $\sin(x+h) - \sin x$ as a product of a cosine and sine. Then we use the limit

$$\lim_{x\to} \frac{\sin x}{x} = 1 \tag{1.3}$$

to complete the proof.

(ii) Similarly to $\sin x$ except we use

$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right) \tag{1.4}$$

(iii) Now $\tan x = \sin x/\cos x$ and then we use the Quotient Rule (Section 1.4.2).

(iv) We can actually *define* e^x as the function $f(x)$ which is equal to its own derivative with $f(0) = 1$.

(v) We can also *define* $\log x$ as the function $f(x)$ which has derivative $1/x$ for $x > 0$ with $f(1) = 0$.

Remark

We might see the following notations:

These are all the same function.

Examples

1. Differentiate $f(x) = x - 3\cos x$.

---

[2]in the tables

2. Find the equation of the tangent line to the curve $y = \tan x$ at the point $(\pi/4, 1)$.

   Solution: To write down the equation of a line we need either two points, or a point and the slope. We have a point. How do we get the slope?

   Now what is $\cos(\pi/4)$?

   Now we use the equation of the line formula:

$$y - y_1 = m(x - x_1) \tag{1.5}$$

Exercises:

1. Differentiate $y = \sin x + 10 \tan x$.
2. Find the equation of the tangent line to the curve $y = x + \cos x$ at the point $(0, 1)$.
3. For what values of x does the graph of $f(x) = 2x^3 + 3x^2 - 36x + 11$ have a horizontal tangent?

1.4 Differentiating Products, Quotients and Compositions

Adding functions together or multiplying a function by a constant are not the only ways of generating new functions from old. For example we have products of functions:

and quotients of functions

How do we differentiate functions like these? By analogy with the Sum Rule, one might be tempted to guess that the derivative of a product is the product of the derivatives, We can see that this is wrong by looking at a particular example.

Example

Let $f(x) = x$ and $g(x) = x^2$. Calculate $f'(x)g'(x)$ and $(fg)'(x)$.

Solution: Firstly we have $f'(x) = 1$ and $g'(x) = x^2$ so that $f'(x)g'(x) = x^2$. However

so we can say that, in general,

1.4.1 Proposition: Product Rule

Let $f, g : \mathbb{R} \to \mathbb{R}$ **be smooth functions**. *Then* fg *is smooth with*

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x) \tag{1.6}$$

Or, in the Leibniz Notation

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} \tag{1.7}$$

Proof. We can prove this from first principles[3], but there is also a way to see *why* the theorem is true.

[3] see almost any reference with Calculus in the title

We want to see how the product $P(x) = f(x)g(x)$ changes when x changes by a small amount. Hence we can look at the quantity

$$\lim_{\Delta x \to 0} \frac{\Delta P}{\Delta x} = P'(x) \tag{1.8}$$

the derivative of the product. Here ΔP signifies a small change in P caused by a small change in x, Δx. Now when x changes by Δx we have that $f(x)$ and $g(x)$ also change:

Now we examine what happens to $f(x)g(x)$ when x is changed by Δx:

That implies that the small change in $P(x) = f(x)g(x)$ caused by the change in x, Δx is given by

$$\Delta P = f\Delta g + \Delta f g + \Delta f \Delta g \tag{1.9}$$

Now we want to find $\dfrac{\Delta P}{\Delta x}$:

Now take the limit as $\Delta x \to 0$. If $\Delta x \to 0$ then both Δf and Δg go to zero also but the ratios $\dfrac{\Delta f}{\Delta x}$ and $\dfrac{\Delta g}{\Delta x}$ go to $f'(x)$ and $g'(x)$ respectively and we are left with

Remark

In words, the Product Rule says that the derivative of a product of two functions is the first function times the derivative of the second function plus the second function times the derivative of the first function.

Examples

1. Differentiate $y = x^2 \sin x$.

   Solution: Using the Product Rule:

2. Differentiate $f(x) = x^2 e^x$

   Solution: Using the Product Rule

3. Differentiate $g(u) = \sin u \log(u)$.

   Solution: Using the Product Rule

Exercises:

1. Differentiate $f(x) = x \sin x$.
2. Differentiate $y = \dfrac{\sin x}{x^2}$. [HINT: Write y in the form $x^n \sin x$]
3. If $f(x) = \sqrt{x} \sin x$, find $f'(x)$.
4. Differentiate $y = e^x(\cos x + 3x)$.
5. Differentiate $f(x) = \sqrt{x} \log x$.
6. Differentiate $y = \dfrac{\ln x}{x^2}$.
7. Find the equation of the tangent line to the curve $y = e^x/x$ at the point $(1, e)$.
8. Find the equation of the tangent line to the curve $y = (x^2 - 1)\log(x)$ at the point $(1, 0)$.

1.4.2 Quotient Rule

What about quotients (fractions) of functions? We probably suspect, correctly, that the derivative of a quotient is *not* given by the quotient of the derivatives.

1.4.3 Proposition: Quotient Rule

Let $f, g : \mathbb{R} \to \mathbb{R}$ **be smooth functions**. *Then* $\dfrac{f}{g}$ *is smooth for* $g(x) \neq 0$ *with*

$$\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2} \tag{1.10}$$

Or, in the Leibniz Notation

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \tag{1.11}$$

Proof. Once again, we can prove this from first principles[4], but as in the product rule there is also a way to see *why* the theorem is true. We want to see how the product $Q(x) = \dfrac{f(x)}{g(x)}$ changes when x changes by a small amount. Hence we can look at the quantity

$$\lim_{\Delta x \to 0} \frac{\Delta Q}{\Delta x} = Q'(x) \tag{1.12}$$

[4]see almost any reference with Calculus in the title

the derivative of the quotient. Here ΔQ signifies a small change in Q caused by a small change in x, Δx. Now when x changes by Δx we have that $f(x)$ and $g(x)$ also change:

Now we examine what happens to $\dfrac{f(x)}{g(x)}$ when x is changed by Δx:

Now we look at the small change in $Q(x) = \dfrac{f(x)}{g(x)}$ caused by the change in x, Δx is given by

$$\Delta Q = \frac{f + \Delta f}{g + \Delta g} - \frac{f}{g} \tag{1.13}$$

Now we want to find $\dfrac{\Delta Q}{\Delta x}$:

Now take the limit as $\Delta x \to 0$ and we are left with

Examples

1. Differentiate $y = \frac{x^2+x-2}{x^3+6}$.

   Solution: First we find the derivatives of the top and bottom:

Now we use the Quotient Rule:

Inasmuch as possible we should simplify our answers as much as possible:

2. Differentiate $y = \dfrac{x}{\cos x}$

   Solution: First we find the derivatives of the top and bottom

   Now we apply the Quotient Rule and simplify:

3. Differentiate $y = \dfrac{e^x}{1+x}$

   Solution: First we find the derivatives of the top and bottom

Now we apply the Quotient Rule and simplify:

4. Differentiate $f(t) = \dfrac{1 + \ln t}{1 - \ln t}$

   Solution: First we find the derivatives of the top and bottom

   Now we apply the Quotient Rule and simplify:

Exercises:

1. Differentiate
$$g(x) = \frac{3x - 1}{2x + 1}.$$
2. Differentiate
$$f(t) = \frac{t^3 + t}{t^4 - 2}.$$
3. Differentiate
$$y = \frac{1}{x^4 + x^2 + 1}.$$
4. Differentiate
$$y(t) = \frac{t^2}{3t^2 - 2t + 1}.$$
5. Differentiate
$$y = \frac{\sqrt{x} - 1}{\sqrt{x} + 1}.$$
6. Where m is a constant, differentiate
$$f(x) = \frac{mx}{1 + mx}.$$
7. Differentiate
$$y = \frac{x}{x^2 - 1}.$$
8. Differentiate
$$y = \frac{1 + \sin x}{x + \cos x}.$$

9. Differentiate
$$y = \frac{e^x + e^{-x}}{e^x - e^{-x}}.$$

10. Differentiate
$$f(u) = \frac{\log u}{1 + \log(2u)}.$$
[HINT: Use the identity $\log(ab) = \log a + \log b$ to rewrite the bottom/denominator]

11. Find y' where
$$y = \frac{\ln x}{x^2}.$$

12. If
$$f(x) = \frac{x}{\ln x},$$
find $f'(e)$.

13. Find the equation of tangent line to the curve
$$y = \frac{2x}{x+1}$$
at the point $(1, 1)$.

14. Find the equation of the tangent line to the curve
$$y = \frac{1}{\sin x + \cos x}$$
at the point $(0, 1)$.

15. Find the derivative of the function
$$F(x) = \frac{x - 3x\sqrt{x}}{\sqrt{x}}$$
by (i) simplifying first by writing in terms of powers of x and (ii) by using the Quotient Rule.

1.4.4 Chain Rule

There are more ways to combine functions than adding, making products and quotients. We also have *compositions* of functions. Recall that a function which assigns to each input a unique output. A lot of the time, for example in this module, the inputs are real numbers are real numbers and the outputs are also real numbers. That means that we could treat the outputs of one function as the inputs of another. As an example consider the functions $f(x) = \tan x$ and $g(x) = e^x$:

Now the object that takes as input x and outputs $e^{\tan x}$ is a perfectly well-defined function. We might call it $h(x) = e^{\tan x}$. It is called the *composition* of f and g and we write:

It turns out the composition of two smooth functions is also smooth, and there is also a formula for the derivative of a composition in terms of the constituent functions.

1.4.5 Proposition: Chain Rule

Let $g, f : \mathbb{R} \to \mathbb{R}$ **be smooth functions, and let** F **denote the composition** $F = f \circ g$ **(that is** $F(x) = f(g(x))$**).** *Then* F *is smooth with*

$$F'(x) = f'(g(x))g'(x) \tag{1.14}$$

Proof. Once again the proof from first principles is a bit beyond us but we can argue *why* this formula makes sense.

Recall that we can interpret derivatives as rates of change. Regard $\dfrac{dg}{dx}$ as the rate of change of g with respect to x. That is if x changes by one unit, $g(x)$ changes by

Now that $\dfrac{df}{dg}$ is the rate of change of $f(x)$ with respect to $g(x)$, if $g(x)$ changes by one unit, $f(x)$ changes by

Suppose that $g(x)$ changes twice as fast as x so that and $f(x)$ changes, say, three times as fast as $g(x)$, then it is reasonable that $F(x) = f(g(x))$ changes six times as fast as x as this schematic shows:

Therefore we expect

$$\frac{dF}{dx} = \frac{df}{dg}\frac{dg}{dx}, \text{ or}$$
$$F'(x) = f'(g(x)) \times g'(x).$$

Remark

In theory this sounds pretty straightforward. We have a function of the form

and the derivative is

> *Differentiate the 'outside' function at the 'inside' function, and multiply by the derivative of the 'inside' function*

It may not be that easy to identify which is the 'inside' function and which is the 'outside' function. Usually the 'inside' function will really look inside — except when we are dealing with something of the form $e^{g(x)}$. Some authors use $e^x = \exp(x)$ to get around this problem as it is easy to see the 'inside' function when you write $y = \exp(\sin x)$, but somehow harder when we write $y = e^{\sin x}$.

Examples

1. Differentiate $y = (x^2 + 1)^8$.

   Solution: We *could* multiply out $(x^2 + 1)^8$ but frankly this would take too long[5]. What we do instead is recognise it as a composition:

[5] it is *not* equal to $(x^2)^8 + 1^8$

We have $x^2 + 1$ at the 'inside' and x^7 on the outside. So

2. Differentiate $y = \sin(e^x)$.

   Solution: Once again we have to recognise this is a composition:

   We have e^x at the 'inside' and $\sin x$ on the outside. So

3. Differentiate $f(t) = \log(t^3 - \sqrt{t})$.

   Solution: O.K. we know that $t^3 - \sqrt{t}$ is the inside and $\log t$ the outside:

4. Differentiate $e^{\tan x}$.

   Solution: This is the one that we have been warned about. What we have here is

   So in fact $\tan x$ is the inside and e^x is the outside:

5. Differentiate $f(x) = \cos^2(\log x)$.

   Solution: First a remark. When we write $\sin x^2$ do we mean

   This ambiguity is cleared up by saying

   So this function here really is

   So for the moment at least, the inside function is $\cos(\log x)$ and the outside function is x^2. Let us apply the Chain Rule:

   Now when we differentiate $\cos(\log x)$ we need a second Chain Rule. For this function the inside function is $\log x$ and the outside function is $\cos x$. So, using the Chain Rule, the derivative of $\cos(\log x)$ should be given by

   Now we can put the whole thing together to find $f'(x)$:

The split here of products, quotients and compositions seems to suggest that we are dealing with either a product rule, a quotient rule or a chain rule. Nothing could be further from the truth. For example, to differentiate the function

$$f(x) = \frac{(\sin x + 3)^5 e^x}{\tan x + \ln x}$$

requires a quotient rule as it is a quotient. We will need to differentiate the top thus and we will need a product rule for this. In turn we will need a chain rule for $(\sin x + 3)^5$ when differentiating that for the product rule. One of the big problems we will have is correctly interpreting, for example, that $\log(2x + 1)$ is "log of $2x + 1$", not "log by $(2x + 1)$" which makes no sense whatsoever (log is a function, it needs an input).

Winter '10: Question 1 (b)

Differentiate by rule

$$y(x) = \frac{x}{\sqrt{1-x^2}}$$

Solution: Firstly we recognise this as a quotient and it might make sense to differentiate the top, $u = x$, and the bottom, $v = \sqrt{1-x^2}$, separately.

Before we differentiate $v = \sqrt{1-x^2}$ we use $\sqrt{a} = a^{1/2}$ to rewrite v:

Now as this is a composition, with $1 - x^2$ inside, and $x^{1/2}$ outside we must use the Chain Rule to differentiate this:

Now we use the formula

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$$

In terms of an exam we have a lot of the marks here but we should really simplify if possible. What is annoying me is that $\sqrt{1-x^2}$ on the bottom on the numerator. To get rid of it I could multiply above and below by $\sqrt{1-x^2}$...

Winter '10: Question 2 (b) (i)

Differentiate the following by rule

$$y = \frac{2\sin x}{(\cos x + 1)^2}$$

Solution: Again we have a quotient function so we will need to use the Quotient Rule. Differentiating the top and bottom separately is as ever a good idea. The top, u, is not troublesome:

However the bottom, $v = (\cos x + 1)^2$, is troublesome. It is the composition of the inside function $\cos x + 1$ and the outside function x^2 so we apply the Chain Rule:

Now we put these into the Quotient Rule formula:

In terms of an exam we have a lot of the marks but we should simplify if possible. We can divide above and below by $(\cos x + 1)$ but we should be careful that[6] $\cos x + 1 \neq 0$:

Can we go further? The above would have gotten full marks but we could go a little further... in fact a lot further in this example.

[6]when does this happen? $\cos x + 1 = 0 \Leftrightarrow \cos x = -1 \Leftrightarrow x \neq \pi + 2k\pi = (2k+1)\pi$ for $k \in \mathbb{Z}$

Winter '10: Question 2 (b) (ii)

Differentiate by Rule

$$y = 4\sin(3x^2 + 5)\ln(2x).$$

Solution: In this example we have a product so we will need a product rule. Because of linearity we can pull the 4 to the front and just differentiate the $\sin(3x^2 + 5)\ln(2x)$. Now there are two functions multiplied together here:

When we differentiate we will use the Product Rule formula:

$$\frac{d(uv)}{dx} = uv' + vu' \tag{1.15}$$

They are both compositions so both need Chain Rules to differentiate them: i.e. to find u' and v'. Usually with products we can just use the Product Rule formula but when the functions are compositions it makes more sense to differentiate them separately:

Now we plug everything back into the Product Rule formula, but not forgetting the four at the front:

Again we simplify as much as possible:

Winter '11: Question 1 (b)

Given that $f(t) = 3t^2\sqrt{1 + t^2}$, find $f'(1)$.

Solution: The question wants us to find $f'(1)$, the slope of the tangent at $t = 1$. So we find $f'(t)$, the slope of the tangent at t, and plug in $t = 1$. We *cannot* plug in $t = 1$ first as this will just give us a number or constant $f(1)$ whose derivative will be zero... First thing we do is rewrite the function so that the square root is now a power of $1 + t^2$:

This is the product of two functions[7], $u = 3t^2$ and $v = (1 + t^2)^{1/2}$. To do the Product Rule we need the derivative of u, u'; and the derivative of v, v'.

[7] we could 'fix' the 3 here as below but it's not to troublesome with the t^2

The differentiation of u is straightforward but the differentiation of v requires a Chain Rule:

Now implement the Product Rule formula:

Normally we would simplify this but since we only want $f'(1)$ we could plug in $t = 1$ at this point rather than later:

As far as I am concerned this is the answer but if you want an idea of the size of number we have you can plug it into the calculator and get $f'(1) \approx 10.61$. This means that the function $f(t)$ is increasing quickly around $t = 1$.

Alternate Solution: There is often more than one way to skin a cat. We saw in MATH6014 $\sqrt{a}\sqrt{b} = \sqrt{ab}$. We can actually write any positive number as a square root. For example, $e^x > 0$, so it can be written as a root:

This means we can rewrite $f(t)$ as

Now we can differentiate using the Chain Rule:

Now again instead of simplifying again we can just input $t = 1$:

Winter '11 Question 2 (b)

Differentiate by rule

(i) $y = \sqrt{x^2 + x + 1}$.

(ii) $f(t) = 10e^{-0.2t}\sin(4\pi t - \pi)$

(iii) $y = \dfrac{4 - x^2}{4 + x^2}$.

Solution

(i) After a quick rewriting this is a straightforward application of the Chain Rule:

(ii) This looks quite messy so we break it up a little.

Now we can forget about the constant until the end. After that it is a product rule where both $u = e^{-0.2t}$ and $v = \sin(4\pi t - \pi)$ are compositions so will need Chain Rules to differentiate them. Therefore we are better off differentiating separately. Personally, 0.2 looks all kinds of wrong to me. Also, in this case e^x is the outside function, and rewriting as $\exp(x)$ makes the Chain Rule easier to see:

Now we can differentiate using the Chain Rule recalling that $\frac{1}{5}$ and 4π are nothing but constants

Now implement the formula $(uv)' = uv' + vu'$, not forgetting the 10:

Plenty of marks here but we should do our best to simplify:

(iii) A straightforward application of the Quotient Rule formula:

$$\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2} \tag{1.16}$$

where $u = 4 - x^2$ and $v = 4 + x^2$. We have

and hence derivative, remembering to simplify:

Exercises: To check answers that are not here use WolframAlpha.com; an amazing online computational engine. To differentiate, say $e^{\sin x}$ please input

$$D[E\hat{}(\text{Sin}[x]), x] \tag{1.17}$$

Other functions should be fairly intuitive and Wolfram Alpha is very forgiving if you do get the code wrong. For example Wolfram Alpha will also interpret

$$\text{Differentiate } e\hat{}(\sin\ x) \tag{1.18}$$

exactly the same way. Be aware that your answer might be correct but not as simplified as the ones given below.

1. Find the derivative of $f(x) = \sin(4x)$.
2. Where a is a constant, find the derivative of $f(x) = \cos(ax)$.
3. Find the derivative of $f(t) = e^{-4t}$ **Ans:** $-4e^{-4t}$.
4. Find the derivative of $y = \log(5x)$.
5. Find the derivative of $F(x) = (x^3 + 4x)^7$.
6. Differentiate $y = (x^3 - 1)^{100}$ **Ans:** $-300x^2(x^3 - 1)^{99}$.

7. Find $F'(x)$ if $F(x) = \sqrt{x^2+1}$.
8. Find the derivative of $\sqrt[4]{1+2x+x^3}$.
9. Find the derivative of $g(x) = \dfrac{1}{(t^4+1)^3}$ **Ans:** $\dfrac{-12t^3}{(1+t^4)^4}$.
10. Where a is a constant, differentiate the function $y = \cos(a^3 + x^3)$.
11. Where a is a constant, differentiate $y = a^3 + \cos^3 x$.
12. Differentiate (a) $y = \sin(x^2)$ and (b) $y = \sin^2 x$ **Ans:** **(a)** $2x\cos x$ **(b)** $2\sin x \cos x = \sin 2x$.
13. Differentiate the function $y = e^{\tan x}$.
14. Find $f'(x)$ if
$$f(x) = \frac{1}{\sqrt[3]{x^2+x+1}}.$$
15. Find the derivative of the function
$$g(t) = \left(\frac{t-2}{2t+1}\right)^9.$$
Ans: $\dfrac{45(t-2)^8}{(1+2t)^{10}}$.
16. Find the derivative of $g(x) = (1+4x)^5(3+x-x^2)^8$.
17. Differentiate $y = (2x+1)^5(x^3-x+1)$.
18. Find the derivative of $y = x\cos(3x)$ **Ans:** $\cos(3x) - 3x\sin x$.
19. Find the derivative of $y = x\sin\sqrt{x}$.
20. Find the derivative of $y = \sin(x\cos x)$.
21. Find the derivative of
$$f(x) = \frac{x}{\sqrt{7-3x}}.$$
Ans: $\dfrac{14-3x}{2(7-3x)\sqrt{7-3x}} = \dfrac{14-3x}{2(7-3x)^{3/2}}$.
22. Differentiate
$$F(z) = \sqrt{\frac{z-1}{z+1}}.$$
23. Find the derivative of
$$y = \frac{\sin^2 x}{\cos x}.$$
24. Find the derivative of $y = \sin\sqrt{1+x^2}$ **Ans:** $\dfrac{x\cos\sqrt{1+x^2}}{\sqrt{1+x^2}}$.

25. Find the derivative of $r(\theta) = \tan^2(3\theta)$.

26. Differentiate $y = xe^{-x}$.

27. Differentiate $f(x) = x^2e^x$ **Ans:** $xe^x(2 + x)$.

28. Differentiate $y = e^{\frac{1}{x}}$.

29. Find y' if $y = e^{-4x}\sin(5x)$.

30. Differentiate $y(u) = e^u(\cos u + 3)$ **Ans:** $e^u(3 + \cos u - \sin u)$.

31. Differentiate $F(t) = e^{t\sin t}$.

32. Differentiate $y = \cos(e^{\pi x})$.

33. Differentiate $y = \ln(x^3 + 1)$ **Ans:** $\dfrac{3x^2}{x^3 + 1}$.

34. Find $\dfrac{d}{dx}\log_e(\sin x)$.

35. Differentiate $y = \sqrt{\log x}$.

36. Differentiate the function $f(\theta) = \ln(\cos\theta)$ **Ans:** $-\tan x$.

37. Differentiate
$$f(u) = \frac{\ln u}{1 + \ln(2u)}.$$

38. Find the derivative of $y = (1 + \cos^2 x)^6$.

39. Find the derivative of $y = x\sin\dfrac{1}{x}$ **Ans:** $\sin\left(\dfrac{1}{x}\right) - \dfrac{\cos\left(\frac{1}{x}\right)}{x}$.

40. Find the derivative of $\sin(\sin(\sin(x)))$.

41. Find the derivative of $y = \sqrt{x + \sqrt{x}}$.

42. Find $\dfrac{d}{dx}\ln\dfrac{x + 1}{\sqrt{x - 2}}$ **Ans:** $\dfrac{5 - x}{4 + 2x - 2x^2}$.

43. Differentiate the function
$$f(x) = \log\left(\frac{x}{x - 1}\right).$$

44. Differentiate $h(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$.

45. Differentiate $y = \log(e^{-x} + xe^{-x})$ **Ans:** $\dfrac{-x}{1 + x}$.

46. Find the equation of the tangent line to the curve $y = (1 + 2x)^{10}$ at the point $(0, 1)$.

47. Find the equation of the tangent to the curve $y = \sqrt{5 + x^2}$ at the point $(2, 3)$.

48. Find the equation of the tangent line to the curve $y = \tan(\pi x^2/4)$ at the point $(1, 1)$
Ans: $y = -\frac{2}{e}x + \frac{3}{e}$.

49. Find the equation of the tangent to the curve $y = \frac{e^{-x}}{x}$ at the point $(1, 1/e)$.

50. Find the equation of the tangent line to the curve $y = \ln(x^3 - 7)$ at the point $(2, 0)$.

51. Find all points on the graph of the function

$$f(x) = 2\sin x + \sin^2 x$$

where the tangent line is horizontal.

1.5 Rates of Change: Applied Problems

We have already seen that if $F(x)$ is some quantity which depends on x, then the rate of change of the quantity with respect to x is given by $F'(x)$.

Examples

1. Speed, $v(t)$, is the rate of change of distance, $s(t)$, with respect to time, t:

2. *Velocity*, $v(t)$, is the rate of change of displacement, $s(t)$, with respect to time, t:

3. *Acceleration*, $a(t)$, is the rate of change of speed/velocity, $v(t)$, with respect to time, t:

   If a particle has a negative acceleration it is often said to have a *retardation*. In other words an acceleration of -10 m s^{-2} is the same as a retardation of +10 m s^{-2}.

4. The *angular velocity*, $\omega(t)$, of a particle in circular motion is the rate of change of angle, $\theta(t)$, with respect to time, t:

5. The *angular acceleration* of a particle in circular motion is the rate of change of angular velocity, $\omega(t)$, with respect to time, t.

6. The *linear density* of a rod, $\rho(x)$, is the rate of change of mass, $m(x)$, with respect to length, x:

7. *Current*, $I(t)$, is the rate of change (or flow) of *charge*, $Q(t)$, with respect to time, t:

8. The *growth rate of a population* is the rate of change of population, $N(t)$, with respect to time, t:

9. The *marginal cost of production* is the rate of change of the cost of producing x items, $C(x)$, with respect to x:

   The marginal cost of production is the rate at which costs are increasing with respect to x:

   Economies of Scale occur when the marginal cost $C'(x)$ decreases.

Remark

We have seen here that if we differentiate $s(t)$ twice we have:

We say here that acceleration, $a(t)$, is the *second derivative* of $s(t)$ (with respect to time, t), and we write

Winter 2011: Question 1 (b)

A force F is described by $F(t) = 30(1 - e^{-0.1t})$, where t is the time elapsed in seconds. Find the rate at which the force is changing with respect to time after 20 seconds.

Solution: We know that the rate of change is given by the derivative $F'(t)$:

We want to calculate this at $t = 20$:

Summer 2012: Question 1(b)

The displacement s in metres of an object after t seconds is given by

$$s(t) = 5t - 2\ln(1 - 2t).$$

Write down the velocity and acceleration at any time t.

Solution: We know that velocity is the rate of change of displacement with respect to time, t... in other words the derivative of displacement, $s(t)$:

We know that acceleration is the rate of change of velocity with respect to time, t... in other words the derivative of velocity, $v(t)$. To differentiate the second term we use a quotient rule:

Summer 2012: Question 2(b)

The angle θ radians that a rotating wheel has turned after a time t seconds is given as:

$$\theta(t) = 50t - \frac{t^2}{4}.$$

(i) Write down the angular velocity in radian s^{-1}. Find the angular velocity after 10 seconds.

(ii) Write down the angular acceleration in radian s^{-2} . Show that the wheel has constant retardation.

Solution:

(i) We know that angular velocity, $\omega(t)$, is the rate of change of angle, $\theta(t)$, with respect to time, t:

After $t = 10$ s:

(ii) We know that angular acceleration is the rate of change of angular velocity, $\omega(t)$, with respect to time, t:

That is the angular acceleration is negative and constant; that is the angular retardation is constant. Hence the wheel has constant retardation[8].

[8]The relationship between angular velocity and normal (linear) velocity is given by $v = \omega r$. In this example any point on the wheel is on a circle of constant radius so we have that $a = r\frac{d\omega}{dt}$ as r is a constant. Therefore for particles in circular motion, if the angular acceleration is constant then so it the normal, linear acceleration.

Exercises:

Wi '10 Q.1 (c) The temperature θ of a body in degrees Celsius of an object at time t seconds is given by

$$T(t) = 20 + 80(1 - e^{-0.1t}).$$

Find the rate at which the temperature is changing after 10 seconds.

Answer: $+\frac{8}{e}\,^\circ$ **C s**$^{-1} \approx 2.94^\circ$ **C s**$^{-1}$**.**

Au '10 Q.1 (b) The displacement, $s(t)$, of a mass from its starting point after t seconds is given by

$$s(t) = \frac{10t}{1+t}.$$

Find the velocity and the acceleration of the body at $t = 4$ seconds.

Answer: $\frac{2}{5}$ **m s**$^{-1}$**.**

Au '09 Q.1 (b) The displacement, $s(t)$, of a mass from its starting point after t seconds is given by

$$s(t) = 12t + 2t^2.$$

Find the velocity and the acceleration of the mass after 2 seconds.

Answer: 20 m s^{-1}**, 4 m s**$^{-2}$**.**

Wi '09 Q.1 (c) The temperature T in degrees Celsius of an object at time t seconds is given by

$$T(t) = 20 + 80e^{-0.05t}.$$

Find the rate of change of temperature initially and after 20s.

Answer: -4 $^\circ$ **C s**$^{-1}$ **and** $-\frac{4}{e}\,^\circ$ **C s**$^{-1} \approx -1.472^\circ$ **C s**$^{-1}$**.**

Wi '09 Q.2 (c) A body moves along a line and its displacement, $s(t)$, in metres at any instant t seconds is given by

$$s(t) = 2t^3 + 6t^2 - 8t.$$

(i) Find the velocity and the acceleration at any instant t. In particular find the velocity after 4 seconds.

(ii) Find the value of t where the velocity is zero.

Answer: $v(t) = 6t^2 + 12t - 8$**, 136 m s**$^{-1}$**,** $\frac{1}{3}(-3 + \sqrt{21})$ **m s**$^{-1} \approx 0.528$ **m s**$^{-1}$**.**

1.6 Duality of Algebra & Geometry

One of the most remarkable and powerful aspects of mathematics is that many questions about geometry may be answered algebraically.

Example 1

Consider the two circles as shown. Find the coordinates where they intersect.

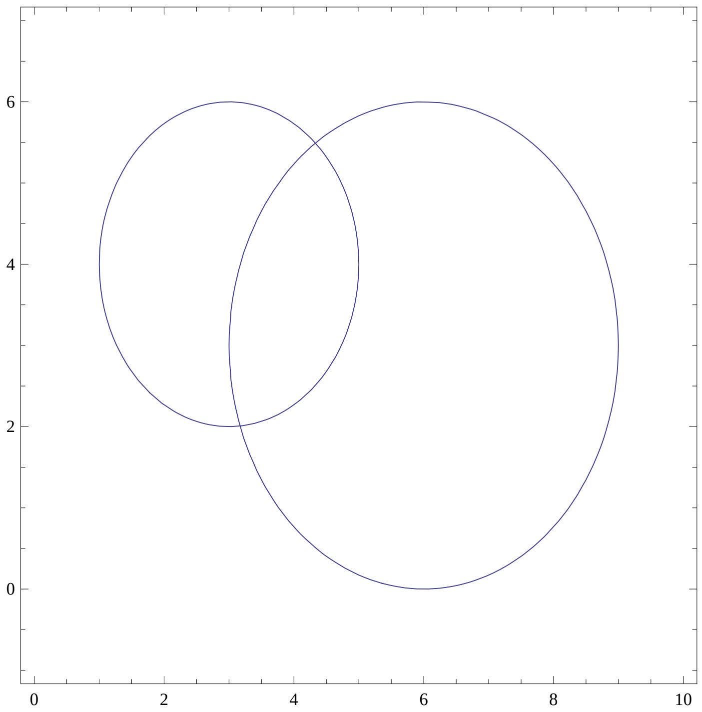


Figure 1.4: Find the coordinates of the intersections of the two circles. The left circle has centre $(3, 4)$ and radius 2. The right-circle has centre $(6, 3)$ and radius 3.

Solution: We show how to answer this algebraically. First of all what is a circle?

Now take a general point (x, y) on the plane. What does it take for the point (x, y) to be on the left circle? It can be either too far from the centre, too close to the centre or just right:

That is (x, y) is on the circle if and only if the distance from (x, y) to $(3, 4)$ is equal to 2. We can calculate the distance from a general point (x, y) to $(3, 4)$ using the distance formula from coordinate geometry[9]:

What we have here is a dictionary between the *geometric*

and the *algebraic*

If we take the general case of a circle with radius r and centre (h, k) we can see that all the points of such a circle satisfy the equation

$$(x - h)^2 + (y - k)^2 = r^2. \tag{1.19}$$

So we have a dictionary that sends the geometric object "circle with centre $(6, 3)$ and radius 3" to the algebraic object

Now what are intersections of curves?

So to find the intersections we find points (x, y) that satisfy

and

at the same time:

[9]this formula comes from Pythagoras Theorem

Conversely many questions about algebra can be answered geometrically!

Example 2

Consider the family of simultaneous system of equations

$$x^2+y^2+2ax+2by+c=0$$
$$x^2+y^2+dx+ey+f=0$$

where $a,b,c,d,e,f\in\mathbb{R}$ are constants. How many solutions can elements of this family have?

Solution: A careful rewriting shows that these equations may be rewritten as

$$(x-a)^2+(y-b)^2=(\sqrt{a^2+b^2-c})^2$$
$$(x-d)^2+(y-e)^2=(\sqrt{d^2+e^2-f})^2$$

So both of these equations represent circles so the question asks how many intersections can two circles have:

So the answer is none, one, two or an infinite number.

What we want to do in this section is set up a dictionary

that applies to functions, particularly smooth functions (smooth means 'has a derivative'). We will call this the AG & GA Dictionary. At times the below can get quite confusing: are we in the algebraic picture or the geometric picture??! This is exactly the point — they might look different, but algebra and geometry are pretty much the same thing — to work in algebra is to work in geometry and vice versa. This is called a duality:

1.6.1 Positive Regions, Negative Regions and Roots

If we have a real-valued function $f(x)$ then we can say that $f(x)$ is *positive* at k if

Similarly we say that $f(x)$ is *negative* at k if

We can say that a function is *zero* at a or *has a root* at a if

These are algebraic definitions. It would be very nice if we could see what this looks like geometrically:

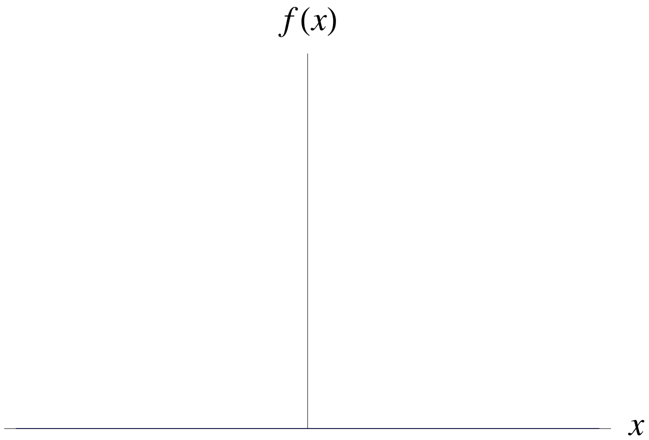


Figure 1.5: We see that a function is positive when the graph is *above* the x-axis, and negative when it is below the x-axis. Where the graph cuts the x-axis we have a root.

1.6.2 Increasing and Decreasing Regions

The algebraic definition of increasing isn't very natural to be honest:

Definition

Let $I \subset \mathbb{R}$ be an interval of the real line. Then we say that $f(x)$ is *increasing on I* if whenever $x_1 < x_2$ are elements of I we have

$$f(x_1) < f(x_2).$$

We define a function to be decreasing in a similar way.

We could dissect this algebraic definition and see what it means geometrically. However it still won't feel natural because of this:

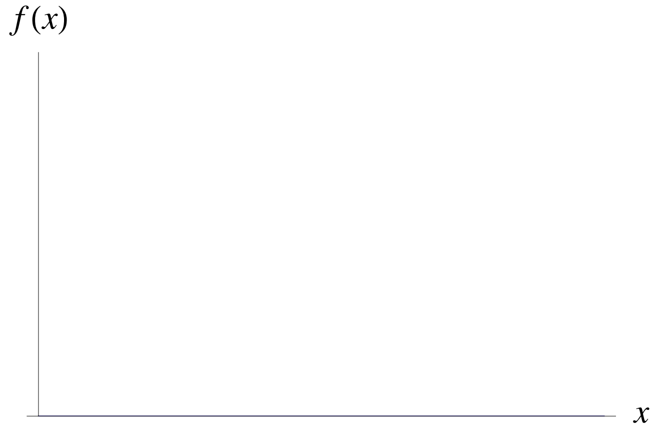


Figure 1.6: According to this definition, the function is increasing to left of a, and decreasing to the right of a. But what if we went right-to-left this would appear the wrong way around. This is confusing!

So we will start with a more natural example where we only think left-to-right: time!

Example

Suppose that a product is launched at time $t = 0$ and thereafter the daily sales of the product (in thousands of units say) is given by

where t is the number of days after the launch. Now the sales on the day of the launch are given by

Now rewrite the sales function slightly

Now as t gets large so does $e^t \approx 2.718^t$. Therefore $1/e^t \to 0$ and we have that as the number of days increases, the daily sales increase to 20,000 per day.

If we graph this function:

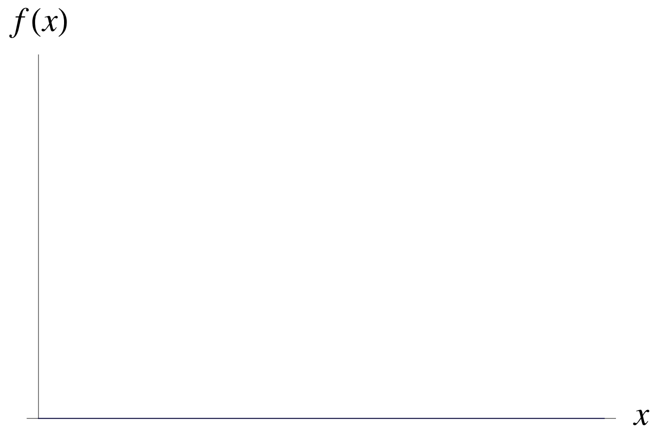


Figure 1.7: When we look at a function of time, $f(t)$, we see why we like to go from left-to-right. Our definition of increasing works like this — we always ask how is $f(x)$ changing when x is getting *bigger* — moving to the right.

Now we should be happy with what increasing and decreasing look like geometrically. However we can go a little further in two equivalent directions.

Increasing means Positive Derivative I

If a function is increasing, what can we say about its rate of change?

What have we shown is equal to a function's rate of change?

Therefore we have that if a function is increasing, then its derivative must be positive. Similarly if a function is decreasing, its rate of change is negative that is its derivative is negative.

What about a function that isn't changing at all, a function that is constant? What is its rate of change?

Increasing means Positive Derivative II

If a function is increasing:

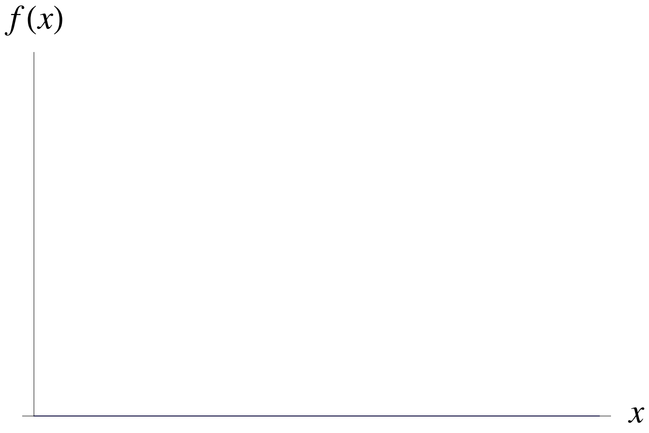


Figure 1.8: When a function is increasing, the slope (of the tangent) is up and so positive.

In other words,

Similarly if a function is decreasing the slope (of the tangent) is negative

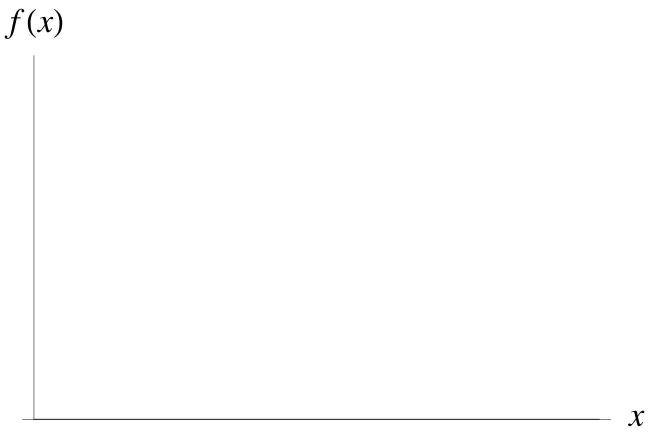


Figure 1.9: When a function $f(x)$ is decreasing, we have $f'(x) < 0$.

Further Remark

Remember a dictionary works both ways: if increasing means positive derivative then positive derivative better mean increasing. However, in general just because A implies B does not imply that B implies A. For example, "today is Wednesday" implies that "today is a weekday" but "today is a weekday" does not imply that "today is Wednesday".

To make these arguments precise we have to be quite careful. We can show that positive derivative implies increasing, but to go the other way, to show that increasing implies positive derivative, is a little trickier. A function can be increasing but non-smooth — its derivative does not exist. If we are careful and restrict to smooth functions we can make all of these arguments rigorous if we want. However this section is just to help us visualise these algebraic concepts geometrically and are to aid understanding so we don't have to get too caught up like this.

Our AG & GA Dictionary works both ways. We should be able to start with a geometric picture and translate to an algebraic picture. What about a function like this:

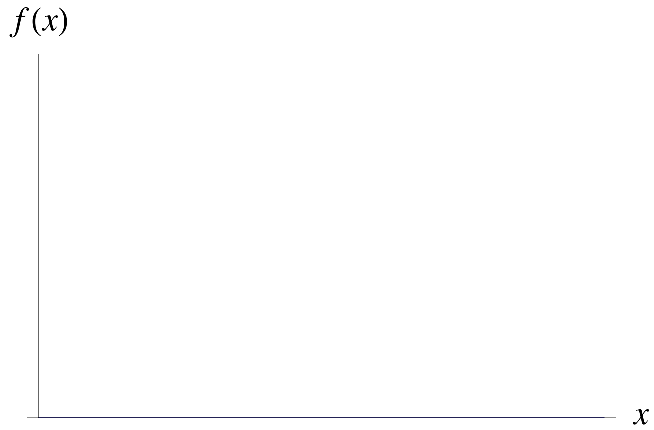


Figure 1.10: What's going on with this function around $x = a$?

Now its slope is zero around $x = a$. That is to say that its derivative $f'(x)$ is zero. What is another way of thinking about derivatives:

Now if the rate of change of a function is zero what about it?

1.6.3 Turning Points: Maxima & Minima

Now we start with a geometric picture and translate it into the algebraic picture. Consider the turning points of a graph

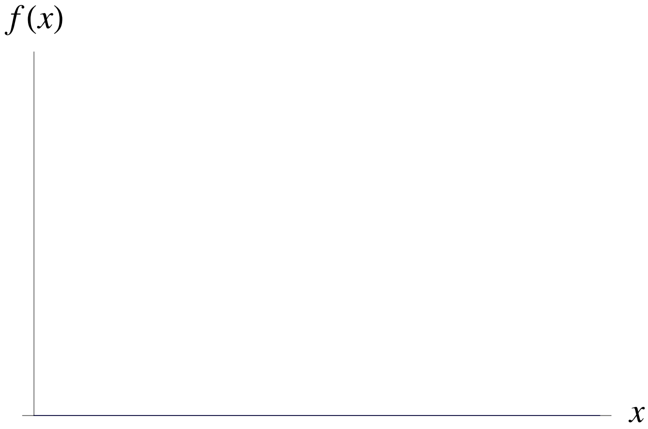


Figure 1.11: Can we say something about $f(a)$ and $f(b)$? Or maybe $f'(a)$ and $f'(b)$?

What we could say about $f(a)$ is that *near* the point $x = a$, that it is a maximum. We call this a *local maximum*. Similarly *near* the point $x = b$ we have that $f(b)$ is a local minimum. Now we argue that the derivatives at these points must be zero.

Suppose that the derivative at a maximum is positive:

However a positive derivative is equivalent to the function increasing. If the function is still increasing at $x = a$ then it has not yet reached its maximum. Therefore the the derivative cannot be positive. Suppose that the derivative is negative:

However a negative derivative is equivalent to the function decreasing. If the function is decreasing it must have passed its maximum already. Therefore the derivative cannot be negative either. Therefore at a local maximum we have

A similar argument shows that at a local minimum the derivative must be zero. So we have for turning points

Now does

Nearly but not quite! Constant functions and *shunt points* also have zero slope/derivative but are not turning points:

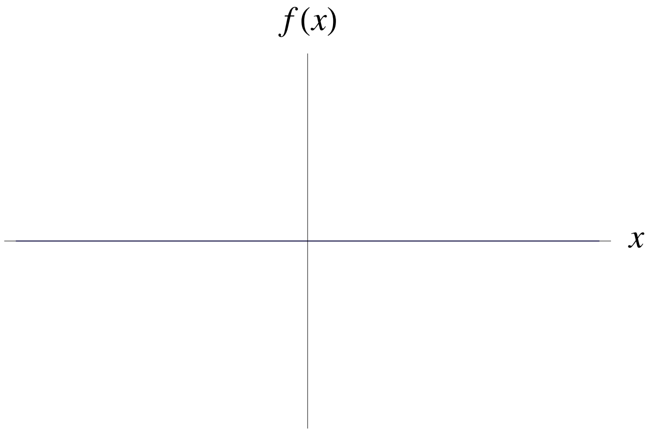


Figure 1.12: $f(x) = x^3$ has a shunt point at $x = 0$: $f'(0) = 0$. However it is not a turning point. $c(x) = 1$ has derivative $c'(x) = 0$ everywhere but no turning points.

A question might ask us to find the turning points of a function. Our argument here shows that any point such that

is certainly a candidate. However it could equally be a shunt point or maybe constant. We will see how to distinguish between these candidates next week using the next section.

1.6.4 Concavity: The Orientation of a Graph

Note the following geometric difference between local maxima and local minima:

Figure 1.13: Somehow at the maximum the graph is 'pointing down', and at the local minimum the graph is 'pointing up'. Can we say anything about $f(a)$, $f(b)$, $f'(a)$ or $f'(b)$.

We say that a graph is *concave down* at local maxima and *concave up* at local minima. The point which separates regions of concavity are known as *points of inflection*:

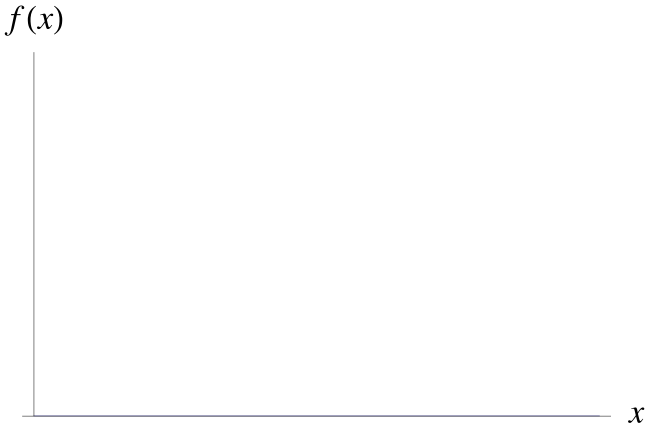


Figure 1.14: Concave down, concave up and a point of inflection... but still no algebraic equivalent.

Let us examine a concave down graph around a local maximum:

As we move over the 'hump', the slope moves from positive — to zero — to negative. What is happening to the slope if it is going from positive, to zero, to negative?

Now for a second let us think in terms of slope $= f'(x)$, it's algebraic equivalent. We have that $f'(x)$ is decreasing. Now $f'(x)$ is function that can be graphed:

Figure 1.15: The derivative, $f'(x)$, of some function, $f(x)$, is decreasing. Can we think of $f'(x)$ decreasing algebraically?

If $f'(x)$ is decreasing, that means its slope of its graph is negative. The slope (of the tangent to) of a function is nothing but the derivative:

This means that if $f'(x)$ is decreasing, then so is its derivative:

So we have that a function is concave down if its slope is decreasing that is if

Similarly we can show that the second derivative is positive for concave up. This is our algebraic equivalent for concave up and down:

Now we will have enough to find the local maxima and minima of functions next week. A turning point in a concave down region

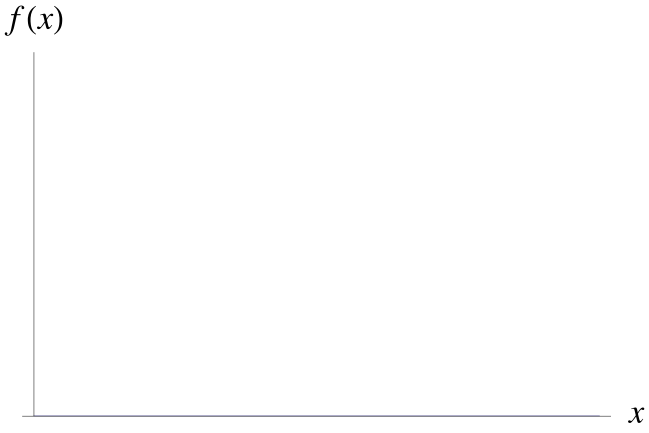
is a local maximum. While a turning point in a concave up region

is a local minimum. Now using the algebraic equivalents of 'turning point' and 'concave':

1.6.5 AG & GA Dictionary

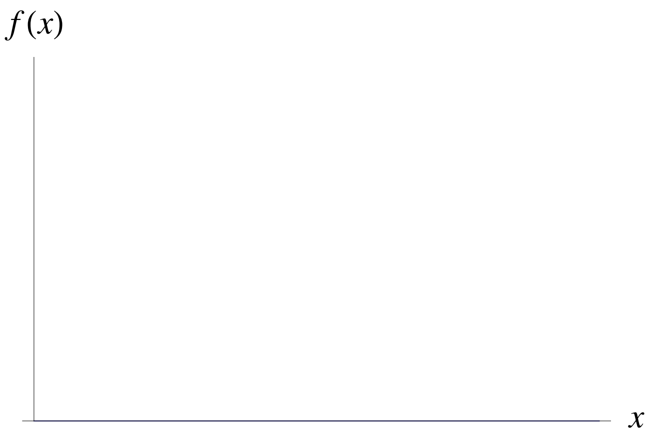
Positive Negative and Roots

A function is positive where its graph is above the x-axis, negative where its graph is below the x-axis, and has a root where the graph cuts the x-axis.



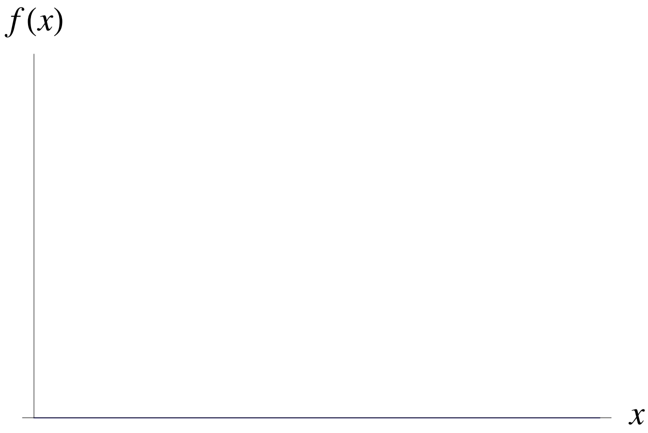
Increasing, Decreasing and Constant

A function is increasing when $f'(x) < 0$, decreasing where $f'(x > 0$ and constant where $f'(x) = 0$.



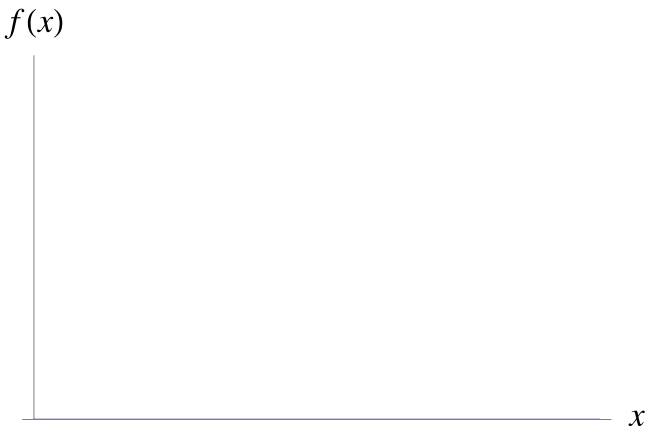
Turning Points

If a function $f(x)$ has a turning point, then the derivative is zero at that point.



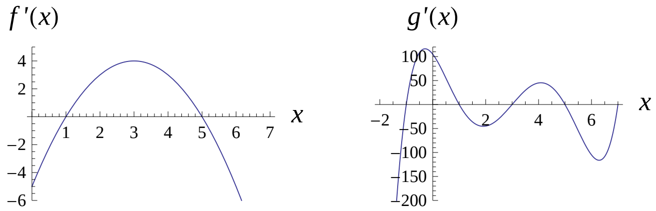
Concavity

A function $f(x)$ is concave down where $f''(x) < 0$ and concave up where $f''(x) > 0$.



Note that all we really want to take out of this section is an understanding that will help us in the next section. Don't get too worried if these questions seem too difficult — feel free to focus on the sample for test 1. *Exercises:*

1. The graphs of the derivatives of functions $f(x)$ and $g(x)$ are shown.



   For each function state

   (a) The intervals where the function is increasing/decreasing.

   (b) Where are the turning points

2. Sketch the graph of a function whose first and second derivatives are always negative.

3. Sketch the graph of a function that satisfies the conditions $f'(x) > 0$ except at $x = 1$, $f''(x) > 0$ for $x < 1$ & $x > 3$ and $f''(x) < 0$ for $1 < x < 3$.

4. Let $K(t)$ be a measure of the knowledge you gain by studying for a test for t hours. Which do you think is larger, $K(8) - K(7)$ or $K(3) - K(2)$? Is the graph of K concave up or concave down? Why?

5. Show that a cubic function $f(x) = ax^3 + bx^2 + cx + d$ always has an inflection point (a point where the concavity changes from up to down or down to up).

1.7 Maxima & Minima

1.7.1 Definition

Let $f : \mathbb{R} \to \mathbb{R}$ be a function and a, $b \in \mathbb{R}$. Then f has a *local maximum at a* if there is some open interval $I_1 \subset \mathbb{R}$ such that $a \in I_1$ and

Similarly, f has a *local minimum at b* if there exists some open interval $I_2 \subset \mathbb{R}$ such that $b \in I_2$ and

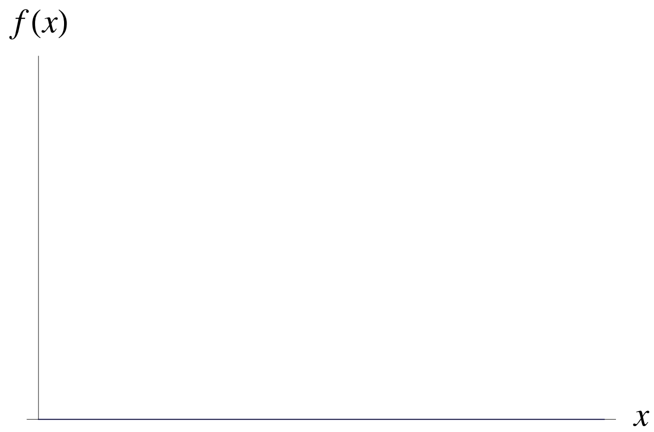


Figure 1.16: Note that we don't require a local max to be an absolute maximum.

1.7.2 Proposition

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. If f has a local maximum at some $x_1 \in \mathbb{R}$ and is differentiable at this point, *then* $f'(x_1) = 0$. **Similarly if f has a local minimum at some $x_2 \in [a, b]$ and is differentiable at this point,** *then* $f'(x_2) = 0$.

Proof. [Ex]: Left as an exercise. Mimic the proof of Rolle's Theorem •

Example

Find the location of the local maxima/ minima of

$$f(x) = ax^2 + bx + c$$

Solution: From our knowledge that f either has $\bigcup$ or $\bigcap$ geometry, we know that f has a single maximum or minimum. We also know that f is differentiable, so at this point we know that $f'(x) = 0$ — so we solve:

1.8 Second Derivative Test

1.8.1 Definition

Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function with differentiable derivative $f'(x) : \mathbb{R} \to \mathbb{R}$. That is for each $x \in \mathbb{R}$, the *second derivative*,

$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} \tag{1.20}$$

exists. Then f is *twice differentiable.* A function is defined to be n-times differentiable in the obvious way. If derivatives of all orders exists, then f is said to be *infinitely differentiable.*
[Ex]: Show that polynomials are infinitely differentiable.
What if we want to look for the local extrema of $f : \mathbb{R} \to \mathbb{R}$ on the entire real number line? Assuming that f is differentiable, we can certainly search for candidates by solving $f'(x) = 0$ — but how do we know if we've caught a max or a min? Occasionally we will be able to exploit the following situation:

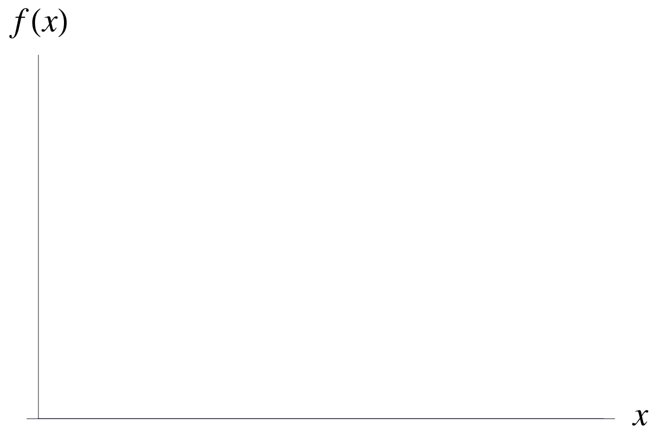


Figure 1.17: If a and b are stationary points of f; then $f''(a) < 0$ implies that a is a local maximum and $f''(b) > 0$ implies that b is a local minimum.

Case (c) below shows us that the second derivative test is not perfect — I include it as some of ye may have encountered it before. The First Derivative Test, which we develop below is far superior if a little harder to use. Also the First Derivative Test works when the function is not differentiable[10].

1.8.2 Proposition: Second Derivative Test

Suppose $f : \mathbb{R} \to \mathbb{R}$ **is twice differentiable and that** $f'(a) = 0$ **for some** $a \in \mathbb{R}$.

(a) **If** $f''(a) < 0$ *then* $x = a$ *is a local maximum.*

(b) **If** $f''(a) > 0$ *then* $x = a$ *is a local minimum.*

(c) **If** $f''(a) = 0$ *then we have no information.*

Proof. Omitted but once again uses the Mean Value Theorem •

[10] well, not differentiable at a finite number of points.

Examples

Find the location of the local maxima/ minima of the following functions — where they are differentiable.

1. $f(x) = \log(3x) - 3x$.
   Solution: First we find the candidates:

   Now we find $f''(x)$ and test $x = 1/3$:

   That is there is a local maximum at $x = 1/3$ •

2. $g(x) = 3x^4 - 2x^3 - 9x^2 + 8$.
   Solution:

   That is there are stationary points at -1, 0 and $3/2$:

3. $h(x) = x^4$, $i(x) = -x^4$, $j(x) = x^3$.
   Solution: Big problems with all of these!

   But

   So no information. As it happens, h has a min, i a max, and j a *saddle point* (why??).

Chapter 2

Integration

I'm very good at integral and differential calculus, I know the scientific names of beings animalculous; in short, in matters vegetable, animal, and mineral, I am the very model of the modern Major General.

W.S. Gilbert in the Pirates of Penzance.

2.1 Definite Integrals

Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function. If $f \geq 0$, then one can approximate the area under $y = f(x)$ on $[a, b]$ by drawing rectangles:

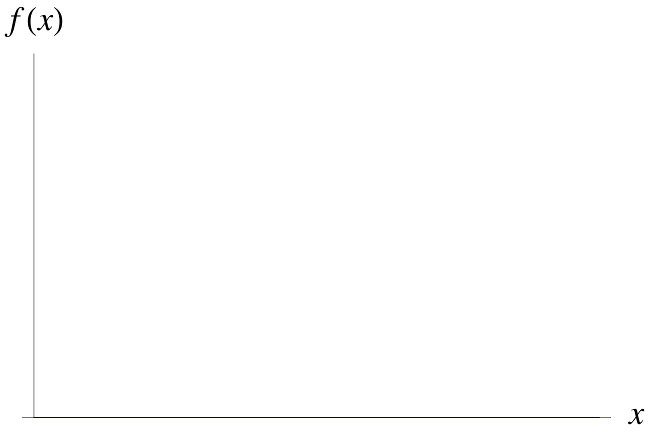


Figure 2.1: Follow the process below to approximate the area under the curve using rectangles.

(i) Divide the interval $[a, b]$ into $n \geq 2$ equal pieces.

(ii) Draw a rectangle on each subinterval with height equal to the value of $f(x)$ at the midpoint of each interval.

Suppose that the length of each subinterval is Δx. Then we have

In particular we have $x_1 = a+\Delta x$, $x_2 = a+2\Delta x$,... , $x_i = a+i\Delta x$. Let $\overline{x_i}$ be the midpoint of the ith subinterval. In this notation we have that the area under the curve is approximated by

Intuitively, one expects that if we choose a larger n (i.e., more subintervals, and consequently narrower rectangles) then the total area of the rectangles is a better approximation of the area under $y = f(x)$. We take the limit as $n \to \infty$ to therefore define this area:

2.1.1 Definition

Let $f : [a, b] \to \mathbb{R}$ be continuous. Then *the integral of f on* $[a, b]$, in the notation above, is given by:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(\overline{x_i})\Delta x.$$

Remarks

1. The sums on the right-hand side here are known as Riemann sums. That f is continuous is a sufficient condition for the convergence of such a sum.

2. What if $f(x) \ngeq 0$??

3. Here the function $f(x)$ is the *integrand*, the a is the *lower limit of integration* and b is the *upper limit of integration*. When a and b are constants, then the definite integral is a number and does not depend on x; in fact

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(s)\,ds, \quad \text{etc.}$$

In situations like this where a variable such as x, t, s appears but does not affect the value of the expression, the variables x, t, s are called *dummy variables*.

2.1.2 Proposition

Suppose that $f : [a,b] \to \mathbb{R}$, $g : [a,b] \to \mathbb{R}$ are continuous and $k \in \mathbb{R}$, with $a < b$. Then we have the following:

1. $$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx.$$

2. $$\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx.$$

3. **If** $f(x) \geq 0$ **for all** $x \in [a,b]$ *then*
$$\int_a^b f(x)\,dx \geq 0.$$

4. **If** $f(x) \geq g(x)$ **for all** $x \in [a,b]$**,** *then*
$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

5. $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

6. **Suppose that** f, g **are continuous on a closed interval containing** a, b **and** $c \in \mathbb{R}$**:**
$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

7. **Where** $m \in \mathbb{R}$ **and** $M \in \mathbb{R}$ **are the minimum and maximum of** f **on** $[a,b]$**:**
$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a). \tag{2.1}$$

8. $$\int_a^b f(x)\,dx = (b-a)f(c) \quad \text{for some } c \in [a,b].$$

The Mean Value Theorem for Integrals.

Examples

1. Given that $\int_4^9 f(x)\,dx = 38$, can you deduce the value of $\int_9^4 f(t)\,dt$? Justify your answer.
   Solution:

2. Use (2.1) to find largest and smallest possible values of

$$\int_1^5 (x-2)^2\,dx.$$

Solution: Using the Closed Interval Method, the maxima and minima of $f(x) = (x-2)^2$ on $[1,5]$ are found at endpoints, points where $f' = 0$ and points where f is not differentiable:

Now applying (6):

In the next two sections we shall examine how one computes the value of a definite integral — hopefully not from first principles!

Exercises

1. Find lower and upper bounds for:

(i) $\displaystyle\int_2^5 (3x+1)\,dx$ Ans: 21 and 48 (ii) $\displaystyle\int_{-1}^2 \frac{x}{x+2}\,dx$ Ans: -3 and $3/2$.

2.2 Anti-Derivatives

2.3 The Second Fundamental Theorem of Calculus

Now we move on to the *Second Fundamental Theorem of Calculus*, which is the key to evaluating almost all definite integrals. First we must talk about anti-derivatives.

Given $F(x)$, we know how to compute its derivative $F'(x)$. Now consider the converse problem: given the derivative of a function, find the function itself.

2.3.1 Definition

We say that $F(x)$ is an *antiderivative* of $f(x)$ on any interval if $F'(x) = f(x)$ on that interval.

Examples

1. Suppose that $f(x) = \cos x$. Find an antiderivative of $f(x)$.
   Solution:

   This example is typical. Once we have found one particular antiderivative $F(x)$ of a function $f(x)$, then all antiderivatives of $f(x)$ are given by the formula $F(x)+C$, where C is an arbitrary constant.

2. Suppose that $f(x) = x^5$.
   Solution:

   If we are given some extra numerical information, this will pin down the value of C.

3. A curve satisfies $\dfrac{dy}{dx} = 3x^2$ and passes through the point (2, 5). Find the equation of the curve.
   Solution: A particular antiderivative of $3x^2$ is x^3. Hence all functions of the form $x^3 + C$ are anti-derivatives.

2.3.2 Theorem

Let $F(x)$ **be an antiderivative of** $f(x)$ **for** $x \in [a, b]$**.** *Then for* $x \in [a, b]$ *every antiderivative of* f *has the form* $F(x) + C$ *for some constant* C*.*

Remark

This is a geometrically obvious fact:

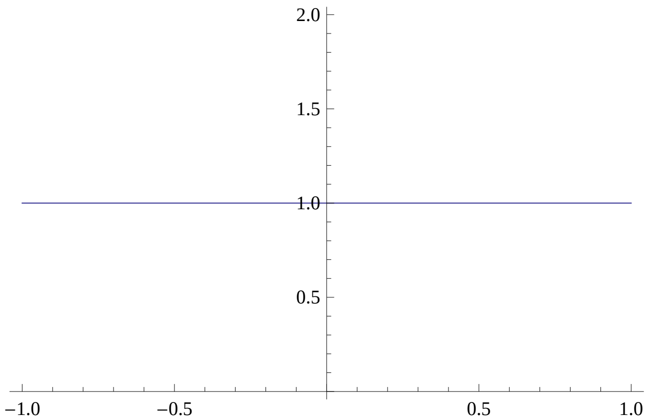


Figure 2.2: Shifting a graph up or down does not change its slope (here we have x^2, x^2+2, x^2-1). Shifting a graph up or down is equivalent to adding a (positive or negative) constant.

Proof. Let $G(x)$ be another antiderivative of f. Then $G'(x) = f(x)$ for $x \in [a,b]$. Set $H(x) = G(x) - F(x)$. Let x_1 and x_2 be any two points in the interval $[a,b]$. Apply the Mean Value Theorem for Derivatives to the function H on the interval $[x_1, x_2]$; there exists a point $c \in (x_1, x_2)$ such that:

Now by the Sum Rule for Differentiation $H'(x) = G'(x) - F'(x) = f(x) - f(x)$;

But c and d were any two points in $[a,b]$ so $H(x)$ must be constant for all $x \in [a,b]$, i.e., $H(x) = G(x) - F(x) = C \Rightarrow G(x) = F(x) + C$ for $x \in [a,b]$ •

Note that this proof was examinable last year — this won't be the case this year.
Notation: the *indefinite integral* $\int f(x)\,dx$ means the antiderivatives of $f(x)$.

2.3.3 The Fundamental Theorem of Calculus

2.3.4 Second Fundamental Theorem of Calculus

Suppose that $f : [a,b] \to \mathbb{R}$ is continuous. Then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f (i.e. $F'(x) = f(x)$).

Remark

This theorem wants to say that when you integrate the derivative of a function you get back the original function. In a very rough sense (*heuristically*) we show *why* this should be true. The theorem could also be cast as

Now we have already seen that integration is really summation — we are adding up all the $f'(\overline{x_i})\Delta x$. What are these[1] $f'(x_i)\Delta x$? As ever, a picture helps:

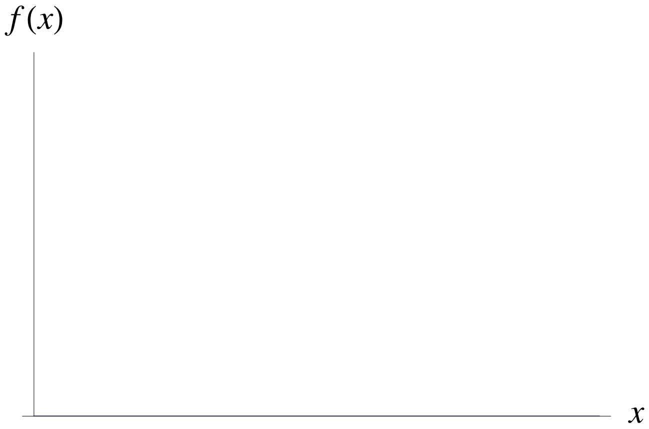


Figure 2.3: $dy = f'(x_i)\Delta x$ estimates Δy. As the partition $[x_i, x_{i+1}]$ becomes finer as $n \to \infty$, the curve looks flatter so that $f'(x_i)\Delta x$ becomes a better and better estimate of Δy.

That is, in the limit, $f'(x_i)\Delta x \approx \Delta y$:

[1] The first set of further remarks explains why we can replace $\overline{x_i}$ by x_i

Examples

1. Evaluate $\int_2^3 2x\,dx$.
   Solution: An antiderivative of $2x$ is x^2. By Theorem 2.3.4,

   Note that one always inserts the upper limit first, then subtracts the value attained at the lower limit, irrespective of whether the upper limit is bigger or smaller than the lower limit. Also, you do not need to use the "$+C$" that appears in indefinite integrals (why?).

2. Evaluate

$$\int_{1/2}^{4} (x^3 - 6x^2 + 9x + 1)\,dx$$

   Solution: Anti-derivatives are readily seen to be $x^4/4$, $-2x^3$, $9x^2/2$ and x:

   And a calculator will show that this is 679/74.

To evaluate a definite integral $\int_a^b f(x)\,dx$, one finds an antiderivative $F(x)$ (in other words, one evaluates the indefinite integral $\int f(x)\,dx$) then one applies the Second Fundamental Theorem of Calculus: $\int_a^b f(x)\,dx = F(b) - F(a)$. This last step is merely arithmetic so it's easy, but the first step—evaluating the indefinite integral—can be tricky.

Exercises Evaluate each of the following integrals.

1. $\displaystyle\int_2^7 (x^2 - 2x)\,dx$ Ans: $\dfrac{200}{3}$

2. $\displaystyle\int_4^0 (y^3 - y^2 + 1)\,dy$ Ans: $-\dfrac{140}{3}$

3. $\displaystyle\int_1^{64} \left(\sqrt{t} - \frac{1}{\sqrt{t}} + \sqrt[3]{t}\right) dt$ Ans: $\dfrac{6215}{12}$

4. $\displaystyle\int_0^{\pi} (1 + \sin x)\,dx$ Ans: $\pi + 2$

5. $\displaystyle\int_{\pi/2}^{0} \sin x\, dx$ Ans: -1

6. $\displaystyle\int_{0}^{\pi/4} \sec^2 t\, dt$ Ans: 1

2.4 Integration of Elementary Functions

2.4.1 Proposition

Suppose that f and g have anti-derivatives and that $k \in \mathbb{R}$. *Then we have*

1.
$$\int kf(x)\, dx = k \int f(x)\, dx.$$

2.
$$\int [f(x) \pm g(x)]\, dx = \int f(x)\, dx \pm \int g(x)\, dx.$$

Proof. Follows easily from the scalar and sum rules for differentiation •

2.4.2 Proposition

Suppose that $n \in \mathbb{Q}$, $n \neq -1$. *Then*

$$\int x^n\, dx = \frac{x^{n+1}}{n+1} + C.$$

Proof. Simply differentiate the right-hand side •

Examples

1. Evaluate
$$\int_0^1 (x^{4/3} + 4x^{1/3})\, dx.$$

   Solution:

2. Find

$$\int (x^2 - 4x + 2)\, dx.$$

Solution: Using Proposition 2.4.1 we write

Observe that we wrote down only one "$+C$" in the second line here; it is unnecessary to have a separate constant C for each integral since all three constants can be added to form a single constant. The same simplification—using a single $+C$ when integrating even though several integrals are present—will be used frequently.

3. Find

$$\int \left(3x^2 + \sqrt{x} - \frac{5}{x^3}\right) dx.$$

Solution:

2.4.3 Definitions

Define the following functions:

$$\sec x = \frac{1}{\cos x},$$
$$\csc x = \frac{1}{\sin x},$$
$$\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}.$$

2.4.4 Definition: Inverse Trigonometric Functions

Consider the graph of $\sin x$ in $[-\pi, \pi]$:
What angle has the sine of a half? Looking at the graph it can be seen there are two values; about $x = 0.5$ and 2.5 (in fact $x = \pi/6$ and $5\pi/6$). Also if the graph is inverted:
It is seen that in this range the function defined as the inverse of sine is not well-defined as one input permits two outputs. However if restricted to $[-\pi/2, \pi/2]$:
This inverse function is a well-defined function. Hence the inverse sine function is defined.

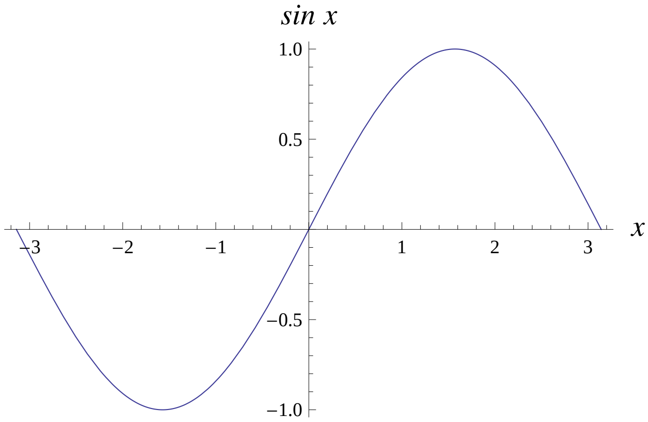
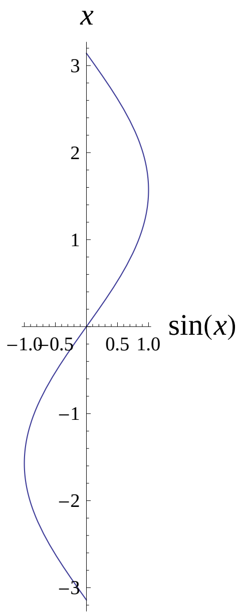


Figure 2.4: The graph of $y = \sin x$ in $[-\pi, \pi]$.



2.4.5 Definition

The *inverse sine* or *arcsin* function is:

$$\arcsin : [-1, 1] \to [-\pi/2, \pi/2] \; : \; x \mapsto y = \arcsin(x) \Leftrightarrow x = \sin y. \tag{2.2}$$

Similarly it can be seen that the inverse tan function makes sense in the same co-domain:

2.4.6 Definition

The *inverse tangent* or *arctan* function is:

$$\arctan : \mathbb{R} \to [-\pi/2, \pi/2] \; : \; x \mapsto y = \arctan(x) \Leftrightarrow x = \tan y. \tag{2.3}$$

Here are some standard indefinite integrals, each of which can be verified by differentiating the right-hand side. The ones of any use will be in mathematical tables (attached at the

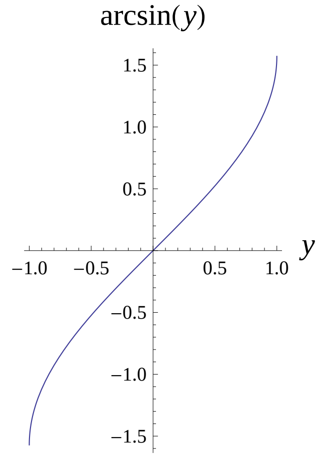


Figure 2.5: The graph of $x = \arcsin y$ in $[-1, 1]$.

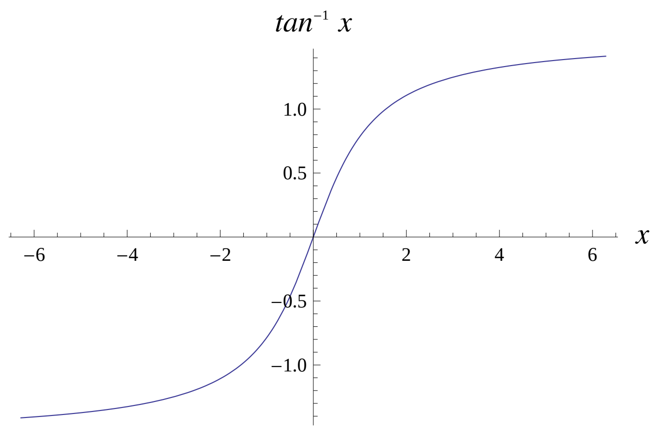


Figure 2.6: The graph of $x = \arctan y$ in $[-\pi/2, \pi/2]$.

end of this set of notes) and don't need to be learnt off (note that we don't include the exponential and logarithmic functions — they are defined properly in a later chapter):

2.4.7 Proposition

$$\int \sin x\, dx = -\cos x + C.$$
$$\int \cos x\, dx = \sin x + C.$$
$$\int \sec^2 x\, dx = \tan x + C.$$
$$\int \csc^2 x\, dx = -\cot x + C.$$
$$\int \sec x \tan x\, dx = \sec x + C.$$
$$\int \csc x \cot x\, dx = -\csc x + C.$$
$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin(x/a) + C.$$
$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\arctan(x/a) + C.$$

Exercises Evaluate the following indefinite integrals.

1. $\int \sqrt{3t}\, dt$ Ans: $\frac{2}{\sqrt{3}}\, t^{3/2} + C$
2. $\int (2 - \sqrt{x})^2\, dx$. (Hint: multiply out) Ans: $4x - \frac{8}{3}\, x^{3/2} + \frac{1}{2}\, x^2 + C$
3. $\int (2x^3 - 3x^2 + 4x)\, dx$ Ans: $\frac{1}{2}x^4 - x^3 + 2x^2 + C$
4. $\int \sqrt[3]{x^2}\, dx$ Ans: $\frac{3}{5}x^{5/3} + C$
5. $\int \left(\frac{1}{x^4} + \frac{1}{\sqrt[4]{x}}\right) dx$ Ans: $-\frac{1}{3x^3} + \frac{4}{3}x^{3/4} + C$
6. $\int (\sin\theta + \cos\theta)\, d\theta$ Ans: $-\cos\theta + \sin\theta + C$
7. $\int (s+1)^2\, ds$ Ans: $\frac{1}{3}s^3 + s^2 + s + C$
8. $\int \frac{x^2+1}{x^2}\, dx$ Ans: $x - \frac{1}{x} + C$
9. $\int \frac{t^2 - 1}{\sqrt{t}}\, dt$ Ans: $\frac{2}{5}t^{5/2} - 2\sqrt{t} + C$

2.5 The Substitution Method

A table of indefinite integrals is helpful but no table can cover all possible integrands $f(x)$. Nevertheless, the usefulness of such a table is greatly increased by the following technique, which can transform an unfamiliar integrand into a recognizable form.

The basic strategy for integration is as follows:

1. Direct — straight from the tables.
2. Manipulation — use trigonometric identities or rewrite the integrand.
3. Substitution — the method developed in this section.
4. Integration by Parts — a technique using the Product Rule for Differentiation.

The substitution method is a technique of integration that comes from the chain rule. Suppose that $f(x)$ is a function with anti-derivative $F(x)$; i.e. $F'(x) = f(x)$. Now consider, for some other function $F(g(x))$ and differentiate with the Chain Rule:

This seems to look like a particularly difficult pattern to spot. However, if we let $u = g(x)$ we can make the following (justified by the above comments) calculation, starting with:

$$\int f(g(x))g'(x)\,dx,$$

Now u is just a dummy variable so hopefully we can integrate away with respect to u. The key here is that, starting from the complicated integral $\int f\left(g(x)\right) g'(x)\,dx$, find the a function-(multiple of the)derivative pattern and *choose* the substitution $u =$ function. Then everything should hopefully work out. Note we have *not* evaluated the integral; we have replaced it by a simpler integral.

Oral Exercise

Spot the function-derivative pattern and state what the substitution should be:

$\displaystyle\int \sin 2x\,dx$	$\displaystyle\int \frac{\cos x}{1+\sin x}\,dx$	$\displaystyle\int \frac{\sin x}{1+\cos x}\,dx[1ex]$
$\displaystyle\int \sin x\sqrt{1+\cos x}\,dx$	$\displaystyle\int 3x^2\sin(x^3)\,dx$	$\displaystyle\int x\sqrt{x^2+9}\,dx$
$\displaystyle\int x(1+x^2)^3\,dx$	$\displaystyle\int \frac{2x+1}{x^2+x+1}\,dx$	$\displaystyle\int \sin x\cos^3 x\,dx[1ex]$
$\displaystyle\int \frac{2x}{\sqrt{1+x^2}}\,dx$	$\displaystyle\int \frac{2x-4}{x^2-4x+29}\,dx$	$\displaystyle\int \frac{x+4}{x^2+8x+1}\,dx[1ex]$
$\displaystyle\int \frac{x}{x^2+4}\,dx$	$\displaystyle\int (x+3)\sec^2(x(x+6))\,dx$	$\displaystyle\int \frac{x-2}{x^2-4x+5}\,dx$

Remarks

1. With indefinite integrals, always transform back to the original variable after the integration is over.

2. One usually cannot integrate a mixture of variables such as $\int x^2\,du$ or $\int \sin\theta\,dx$. Thus when using the substitution method, be careful to transform all variables from x to u — and do this using the equations

$$g(x) = u \text{ and } dx = \frac{du}{g'(x)}.$$

   Sometimes after a substitution both an x and u are present — can you do a *back-substitution*: write x in terms of u?

3. When the integrand contains different types of functions, the selection of the expression to be substituted by u is often clarified by invoking the LIATE rule-of-thumb: in order of preference, the type of function to set equal to u is

   L ogarithmic — more on this later.

   I nverse Trigonometric — arcsin/arctan.

   A lgebraic — polynomials.

   T rigonometric — sin/cos/tan.

   E xponenetial — more on this later.

   The reason this seems to work shall be clarified in a later section (Integration by Parts).

4. When choosing your substitution $u = g(x)$, the only fixed rule is that $kg'(x)$ (some constant multiple of the derivative) must be a factor of integrand. Usually $g(x)$ appears "inside another a function".

5. If a sum or difference of terms (e.g., $x^2 + x - 6$) appears in the integrand, never break them up when choosing u (i.e., don't try $u = x^2$ or $u = x^2 - 6$; but $u = x^2 + x - 6$ may work).

Examples

Evaluate each of the following:

1. $\displaystyle\int 3x^2\sqrt{x^3+9}\,dx.$
   Solution: The integrand is not in the tables and has no obvious manipulation. We try a substitution. Function-Derivative pattern:

2. $\displaystyle\int \sqrt{2x+1}\,dx.$
   Solution: The integrand is not in the tables and has no obvious manipulation. We try a substitution. Function-Derivative pattern:

3. $\displaystyle\int x^5\sqrt{1+x^2}\,dx.$
   Solution: The integrand is not in the table and has no obvious manipulation. We try a substitution. Function-Derivative pattern... In this case our strategy has failed. According LIATE we should choose $u = x^5$ as this has the higher degree... Try $u = 1 + x^2$.

It seems as if all is lost but in fact we can do a *back-substitution*:

4. $\displaystyle\int x\sin(x^2)\,dx.$

   Solution: The integrand is not in the table and has no obvious manipulation. We try a substitution. Function-Derivative pattern:

Exercises

Evaluate the following integrals. Note that you can differentiate your answer for each indefinite integral to check its correctness.

1. $\displaystyle\int 2x^2\sqrt{x^3+1}\,dx$ Ans: $\displaystyle\frac{4}{9}(x^3+1)^{3/2}+C$
2. $\displaystyle\int \sqrt{3x+4}\,dx$ Ans: $\displaystyle\frac{2}{9}(3x+4)^{3/2}+C$
3. $\displaystyle\int t(5+3t^2)^8\,dt$ Ans: $\displaystyle\frac{1}{54}(5+3t^2)^9+C$

4. $\displaystyle\int s^2\,\sqrt[5]{7-4s^3}\,ds$ Ans: $-\dfrac{5}{72}(7-4s^3)^{6/5}+C$

5. $\displaystyle\int x^2\sqrt{1+x}\,dx$ Ans: $\dfrac{2}{7}(1+x)^{7/2}-\dfrac{4}{5}(1+x)^{5/2}+\dfrac{2}{3}(1+x)^{3/2}+C$

6. $\displaystyle\int \frac{t}{\sqrt{t+3}}\,dt$ Ans: $2\left[\dfrac{1}{3}(t+3)^{3/2}-3(t+3)^{1/2}\right]+C$

7. $\displaystyle\int \frac{27r^2-1}{\sqrt[3]{r}}\,dr$

8. $\displaystyle\int \sqrt{1+\frac{1}{3x}}\,\frac{dx}{x^2}$ Ans: $-2\left(1+\dfrac{1}{3x}\right)^{3/2}+C$

9. $\displaystyle\int \cos 5x\,dx$ Ans: $\dfrac{1}{5}\sin 5x+C$

10. $\displaystyle\int (x^2+1)\sin(x^3+3x)\,dx$ Ans: $-\dfrac{1}{3}\cos(x^3+3x)+C$

11. $\displaystyle\int x^2\sec^2(x^3+1)\,dx$ Ans: $\dfrac{1}{3}\tan(x^3+1)+C$

12. $\displaystyle\int \sin^2 x\cos x\,dx$ Ans: $\dfrac{1}{3}\sin^3 x+C$

2.5.1 The Substitution Method in Definite Integrals

When evaluating a definite integral by means of a substitution, you must transform the limits of integration — alternatively you can suppress the limits and when you have integrated with respect to u, and transformed back into x, use the original limits. Either method is correct. Personally I much prefer the latter but I'll do the first example by transforming the limits.

Examples

Evaluate each of the following:

1. $\displaystyle\int_4^9 \frac{\sqrt{x}}{(30-x^{3/2})^2}\,dx.$

   Solution: No direct integration or manipulation:

   We must also transform the limits:

2. $\displaystyle\int_{-1}^{3} \frac{dy}{(y+2)^3}$.
Solution: No direct integration or manipulation:

3. $\displaystyle\int_{0}^{\pi^2/4} \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$.
Solution: No direct integration or manipulation. Lookout for $\sqrt{x}$ in the denominator. Recall

$$y = \sqrt{x} = x^{1/2}$$
$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}.$$

A $\sqrt{x}$ in the denominator is a (multiple of a) derivative of $\sqrt{x}$. Hence let $u = \sqrt{x}$:

4. $\displaystyle\int_{0}^{3} x\sqrt{1+x}\,dx$.
Solution: No direct integration or manipulation — and seemingly no substitution. Chance our arm with the more complicated $u = 1 + x$ and maybe hope for a back-substitution:

Now transform back to x and plug in the limits:

Exercises Evaluate

1. $\displaystyle\int_0^1 x(1-x^2)^5\,dx$ Ans: $\dfrac{1}{12}$.

2. $\displaystyle\int_0^{\sqrt{\pi/2}} x\cos(x^2)\,dx$ Ans: $\dfrac{1}{2}$.

3. $\displaystyle\int_1^3 \frac{t^2+9}{t^2}dt$ Ans: 8.

2.6 Trigonometric Integrals

For $\displaystyle\int \sin^m x\cos^n x\,dx$, if m or n is an odd positive integer, make the substitution

$$u = \textit{other trigonometric function}.$$

and possibly use the well-known trigonometric identity $\sin^2 x+\cos^2 x = 1$. This should work because say we have $\sin^{2n}(x)\cos^{2m+1}(x)$ as the integrand, this can also be written as

$$\underbrace{(\sin x)^{2n}}_{\text{function}}\;\underbrace{\cos x}_{\text{derivative}}\;\underbrace{\cos^{2m}(x)}_{\text{back-substitution with }\sin^2+\cos^2=1}.$$

Examples

Find the following integrals:

1. $\displaystyle\int \sin x\cos^2 x\,dx$.

   Solution: Here the power of $\sin x$ is odd so we let $u = \cos x$:

2. $\displaystyle\int \frac{\sin^3 x}{\cos^4 x}\,dx$.

   Solution: First we have
   $$\int \sin^3 x\cos^{-4} x\,dx,$$
   so as sin is the odd power we let $u = \cos x$:

Autumn 2011 Question 1(b)(ii)

Evaluate

$$\int \sin^3(x)\,dx.$$

Solution: There is no direct integration but a number of possible manipulations. The method here suggests we use $u = \cos x$ as the power of sin is odd:

Note that we have a back-substitution via the relationship between $\sin^2 x$ and $\cos^2 x$:

Exercises

1. Evaluate the following integrals:

   (a) $\int \sin^2 x \cos x\,dx$ Ans: $\frac{1}{3}\sin^3 x + C$

   (b) $\int \sin^3 x \cos x\,dx$ Ans: $\frac{1}{4}\sin^4 x + C$

   (c) $\int \sin^3 x \cos^4 x\,dx$ Ans: $\frac{1}{7}\cos^7 x - \frac{1}{5}\cos^5 x + C$

   (d) $\int \sin^3 x\,dx$ Ans: $\frac{1}{3}\cos^3 x - \cos x + C$

   (e) $\int \sin^5 x\,dx$ Ans: $-\frac{1}{5}\cos^5 x + \frac{2}{3}\cos^3 x - \cos x + C$

   (f) $\int \frac{\cos^3 x}{\sqrt{\sin x}}\,dx$ Ans: $2\sin^{1/2} x - \frac{2}{5}\sin^{5/2} x + C$

2. (a) Consider $\int \sin x \cos x\,dx$. Here both $\sin x$ and $\cos x$ have odd positive integer powers, so one has a choice of substitutions that work: $u = \cos x$ or $v = \sin x$. Evaluate this integral using both methods. The answers that you get will look different from each other at first sight, but show that in fact they are the same. Now use an identity from the tables to integrate.

(b) Similarly to the previous exercise, evaluate the integral $\int \sin x \cos^3 x\, dx$ in two different ways then show that the answers you get are the same.

2.7 The Natural Logarithm Function

From Section 2.4 we recall that

$$\int x^n\, dx = \frac{1}{n+1}x^{n+1} + C \quad \text{provided that } n \neq -1.$$

What happens when $n = -1$? We are about to answer this question.

Unlike the situation when $n \neq -1$, the integral $\int x^{-1}\, dx$ is not equal to some power of x. Instead, to evaluate this integral we must introduce a brand-new type of function — but it will turn out to be same as the $\log_e$ function that one meets in secondary school.

2.7.1 Definition

The *natural logarithm function* is

$$\ln x = \int_1^x \frac{1}{t}\, dt \quad \text{for all } x > 0.$$

By the first fundamental theorem of calculus (Theorem **??**),

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \text{ for } x > 0.$$

Note however that $\ln x$ is not precisely the anti-derivative of $1/x$ when $x < 0$ (Note that $\ln 0$ is not defined.):

$$\ln|x| = \begin{cases} \ln x & \text{if } x > 0, \\ \ln(-x) & \text{if } x < 0. \end{cases}$$

From this we can see by differentiating that

$$\int \frac{1}{x}\, dx = \ln|x| + C \quad \text{for all } x \neq 0.$$

Examples

1. Find, where $\cos x > 0$

$$\frac{d}{dx}(\ln \cos x).$$

Solution:

2. Evaluate $\displaystyle\int \frac{dx}{x\ln x}$, where $x > 0$.
   Solution: This does not yield to a direct integration nor a manipulation. Is there a function-derivative pattern...

3. Evaluate
$$\int \frac{3}{2x-1}\,dx.$$
   Solution: No direct integration or manipulation. From function-derivative or LIATE:

   The same type of substitution is used in the next example but note that $1/f(x)$ as an integrand does *not* necessarily mean that the integral is a log.

4. Evaluate
$$\int \frac{1}{(2x-1)^2}\,dx.$$
   Solution: No direct integration or manipulation. From function-derivative or LIATE:

Integrals similar to these arise frequently in the *method of partial fractions* (more on these later).

2.7.2 Properties of the natural logarithm

As you would expect, this definition of the natural logarithm (ln) has the same properties as those discovered at school. To show this we first prove a lemma (a theorem that is proved only for the purpose of proving another).

2.7.3 Lemma

If a **and** b **are positive,** *then*

$$\int_a^{ab} \frac{1}{t}\,dt = \int_1^b \frac{1}{t}\,dt = \ln(b).$$

Proof. Make the substitution $u = t/a$ in the first integral and this time transform the limits:

2.7.4 Theorem

For all positive a **and** b**, and any rational number** r**,** *we have*

$$\begin{array}{ll} (i) & \ln 1 = 0, \\ (ii) & \ln(ab) = \ln a + \ln b, \\ (iii) & \ln\left(\frac{a}{b}\right) = \ln a - \ln b, \\ (iv) & \ln a^r = r \ln a. \end{array}$$

Proof. (i) This is trivial.

(ii)

(iii) Firstly

Now note

$$\ln\left(\frac{1}{b}\right) = \int_1^{1/b} \frac{1}{t}\,dt = -\int_{1/b}^{1} \frac{1}{t}\,dt$$
$$= -\int_{1/b}^{b\times 1/b} \frac{1}{t}\,dt = -\ln(b).$$

(iv) First we prove the case with $r = n$ is a natural number:

Now suppose $r = p/q$ is a fraction (less than 1). In fact it suffices to prove the fact in the case where $f = 1/q$ as

$$\ln(a^r) = \ln(a^{p/q}) = \ln((a^{1/q})^p) = p\ln(a^{1/q}).$$

Now

Use the substitution $u = t^q$:

This completes the proof as with $r = p/q$

$$\ln(a^{p/q}) = \ln((a^{1/q})^p) = p\ln(a^{1/q}) = p\left(\frac{1}{q}\ln(a)\right) = r\ln(a) \bullet$$

Remark

Both the lemma and the theorem were examinable over the last few years and have been on both papers in 2009 and 2011. They are also examinable this year and as those years are the years this course is derived from there is a *very high chance* that they will be on your summer paper.

2.7.5 Graph of $y = \ln x$

The function $y = \ln x$ is defined for $x > 0$ and we know already that its derivative is $1/x$. This is positive, so $\ln x$ is an increasing function. Differentiating again,

This is negative, so the graph is concave down. To determine what happens to $\ln x$ as $x \to \infty$ and $x \to 0$, we first use a picture to show that

$$\frac{1}{2} < \ln 2 < 1 :$$

Figure 2.7: Using the lower and upper bound techniques of Section 2.4 allows us to estimate $\ln 2$.

Consequently,

and thus

$$\lim_{x\to\infty} \ln x = \infty.$$

Similarly,

so that

$$\text{for } x > 0, \ \lim_{x\to 0} \ln x = -\infty.$$

Putting all these facts together, the graph is
Exercises

Figure 2.8: A rough sketch of $y = \ln(x)$.

1. Find the derivative of each of the following functions.

   (a) $\ln(3x^2 - 6x + 8)$ Ans: $\dfrac{6x - 6}{3x^2 - 6x + 8}$

   (b) $\ln[(4x^2 + 3)(2x - 1)]$ Ans: $\dfrac{24x^2 - 8x + 6}{(4x^2 + 3)(2x - 1)}$

   (c) $\ln\left(\dfrac{x}{x+1}\right)$ Ans: $\dfrac{1}{x(x+1)}$

   (d) $\ln(2x - 1)^3$ Ans: $\dfrac{6}{2x - 1}$

   (e) $\ln^3(2x - 1)$ Ans: $\dfrac{6\ln^2(2x - 1)}{2x - 1}$

2. Evaluate the following integrals.

   (a) $\displaystyle\int \frac{x^2\,dx}{x^3 + 1}$ Ans: $\dfrac{1}{3}\ln|x^3 + 1| + C.$

   (b) $\displaystyle\int \frac{x^2 + 2}{x + 1}\,dx$ (Hint: first divide the bottom into the top)
   Ans: $\dfrac{1}{2}x^2 - x + 3\ln|x + 1| + C.$

   (c) $\displaystyle\int \frac{\ln x}{x}\,dx$ (Hint: recall LIATE) Ans: $\dfrac{1}{2}\ln^2 x + C.$

   (d) $\displaystyle\int \frac{dx}{3 - 2x}$ Ans: $-\dfrac{1}{2}\ln|3 - 2x| + C.$

   (e) $\displaystyle\int_3^5 \frac{2x}{x^2 - 5}\,dx$ Ans: $\ln 5.$

   (f) $\displaystyle\int \frac{dx}{\sqrt{x}(1 + \sqrt{x})}$ Check your answer by differentiation

   (g) $\displaystyle\int \frac{\cos(\ln x)}{x}\,dx$ Ans: $\sin(\ln x) + C.$

(h) $\displaystyle\int \frac{\sin x}{1-\cos x}\,dx$ Ans: $\ln|1-\cos x| + C$.

3. Use logarithmic differentiation to compute the derivatives of the following functions.

   (a) $\dfrac{x^3+2x}{\sqrt[5]{x^7+1}}$ Ans: $\dfrac{8x^9-4x^7+15x^2+10}{5(x^7+1)^{6/5}}$

   (b) $\dfrac{3x}{\sqrt{(x+1)(x+2)}}$ Ans: $\dfrac{3}{2}(3x+4)[(x+1)(x+2)]^{-3/2}$

   (c) $\dfrac{\sqrt[3]{x+1}}{(x+2)\sqrt{x+3}}$ Ans: $\dfrac{-7x^2-23x-12}{6(x+1)^{2/3}(x+2)^2(x+3)^{3/2}}$

4. In each of the following equations, find dy/dx by implicit differentiation.

   (a) $\ln xy + x + y = 2$ Ans: $-\dfrac{xy+y}{xy+x}$

   (b) $\ln\dfrac{y}{x} + xy = 1$ Ans: $\dfrac{y(1-xy)}{x(1+xy)}$
   [Autumn 2011 Question 3 (b)]

2.8 The Natural Exponential Function

From the graph of §2.7.5 we see that $y = \ln x$ has domain $(0,\infty)$, range $(-\infty,\infty)$ and is one-to-one. Consequently it has an inverse function, which we call the *natural exponential function* and write as $x = \exp y$. That is, $y = \ln x$ means exactly the same as $x = \exp y$. It follows that

These are called the *cancellation equations*.

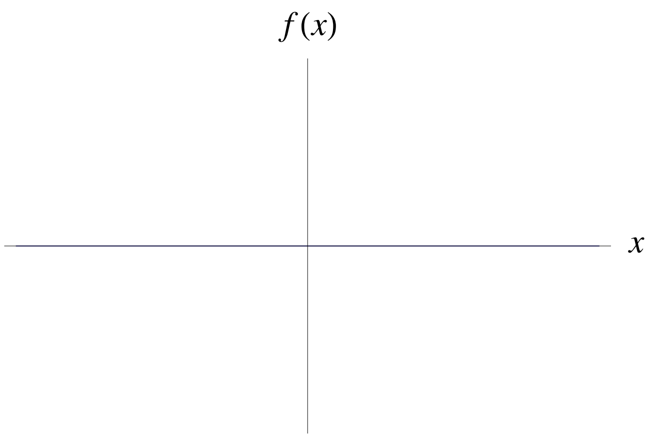


Figure 2.9: We can figure this graph out from the graph of $y = \ln x$.

It turns out that in fact $\boxed{\exp x = e^x}$, where the number $e \approx 2.71828$. That is, $y = \ln x$ means exactly the same as $x = e^y$. From the standard definition of logarithms to any base, this means that $\boxed{\ln x = \log_e x}$ for all $x > 0$.

In particular $\ln 1 = 0$ implies that $e^0 = 1$. From the graph, we have $e^x > 0$ for all x, $\lim_{x\to-\infty} e^x = 0$, $\lim_{x\to\infty} e^x = \infty$.

To find the derivative of e^x we differentiate the $\ln(e^y) = y$ with respect to y, using the chain rule and $d(\ln y)/dy = 1/y$:

We are more used to dealing with functions of x, so we write this as

$$\frac{d(e^x)}{dx} = e^x.$$

Of course this implies that

$$\int e^x\,dx = e^x + C.$$

Asymptotically as $x \to \infty$, $e^x \gg x$; that is e^x grows much, much larger than x so we have, for example

Examples

1. Find the absolute minimum value of the function $g(x) = e^x/x,\ x > 0$.
   Solution: For this we note first that g has a vertical asymptote at $x = 0$ and also that g tends to infinity as x does. Also note that g is differentiable for $x \neq 0$ (why?) and so we can use the second derivative test to hopefully locate the minimum (as, at the very least, g has '$\bigcup$' geometry). So, by the quotient rule, differentiate:

2. Find the derivative of $e^{2\sqrt{x}}$.
   Solution: Using the Chain Rule:

3. Find $\displaystyle\int \frac{e^{1/x}}{x^2}\,dx$.
   Solution: Again, no direct integration nor obvious manipulation but note

$$\int \frac{e^{1/x}}{x^2}\,dx = \int e^{1/x}\frac{1}{x^2}\,dx,$$

   and the derivative of $1/x$ is a constant multiple of $1/x^2$:

   [Ex]: Check your answer by differentiation.

4. Evaluate $\displaystyle\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}}$.
   Solution: Again it's hard to see a function-derivative pattern but note:

$$\int \frac{dx}{x\sqrt{\ln x}} = \int \frac{1}{\sqrt{\ln x}} \times \frac{1}{x}\,dx,$$

Exercises

1. Find the derivatives of the following functions.

   (a) e^{1/x^2} Ans: $-\dfrac{2e^{1/x^2}}{x^3}$

   (b) $e^{2x+\ln x}$ Ans: $e^{2x} + 2xe^{2x}$

   (c) e^{-3x^2} Ans: $-6xe^{-3x^2}$

   (d) $\dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ Ans: $\dfrac{4}{(e^x + e^{-x})^2}$

   (e) $e^{\cos 2x}$ Ans: $-2e^{\cos 2x}\sin 2x$

(f) $e^{x\ln x}$ Ans: $x^x(\ln x + 1)$

2. Sketch the graph of $y = e^{-x}$

3. In each of the following equations, find dy/dx by implicit differentiation.

   (a) $e^x + e^y = e^{x+y}$ Ans: $\dfrac{e^x(1-e^y)}{e^y(e^x-1)}$
   [Summer 2011 Question 3(b)]

   (b) $y^2e^{2x} + xy^3 = 1$ Ans: $-\dfrac{y^2+2ye^{2x}}{2e^{2x}+3xy}$

4. Evaluate the following integrals.

   (a) $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$ Ans: $2e^{\sqrt{x}} + C$

   (b) $\displaystyle\int e^{2-5x}\,dx$ Ans: $-\dfrac{1}{5}e^{2-5x} + C$

   (c) $\displaystyle\int \frac{1+e^{2x}}{e^x}\,dx$ (Hint: divide bottom into top) Ans: $e^x - e^{-x} + C$

   (d) $\displaystyle\int \frac{e^{3x}}{(1-2e^{3x})^2}\,dx$ Ans: $\dfrac{1}{6(1-2e^{3x})} + C$

   (e) $\displaystyle\int \frac{e^{2x}}{e^x+3}\,dx$ Ans: $e^x - 3\ln(e^x+3) + C$

   (f) $\displaystyle\int_0^1 e^2\,dx$ Ans: e^2

   (g) $\displaystyle\int_0^2 xe^{4-x^2}\,dx$ Ans: $\dfrac{1}{2}(e^4-1)$

5. Let $f(x) = xe^{-x}$. Find the local extrema of f and its points of inflection.

2.9 Partial Fractions

This technique concerns the integration of rational functions[2]. *When integrating a rational function, the degree of the numerator must be less than the degree of the denominator.* If this is not the case, then use polynomial long division to divide the denominator into the numerator. For example, $x^2/(x+1)$

[2]recall f is a rational function if $f(x) = p(x)/q(x)$ where p and q are polynomials

and $x-1$ is easily integrated, so

$$\int \frac{x^2}{x+1}\,dx = \frac{x^2}{2} - x + \int \frac{1}{x+1}\,dx,$$

where in the remaining integral the degree of the numerator is 0 while the degree of the denominator is 1, as desired.

A *proper* rational function is a quotient $p(x)/q(x)$ of two polynomials $p(x)$ and $q(x)$ with degree $p <$ degree q. This section deals only with integrands that are proper rational functions.

Many methods of integration (e.g., substitution, integration by parts) are general in nature: they can be applied to many types of integrand, but one may not know in advance whether they will work or fail. In contrast, the *partial fractions method* that we now describe is suitable only for the *integration of proper rational functions* but it is *guaranteed to work!*

The method is mostly algebraic in nature. Note that we can add together fractions; e.g.

Also we can decompose fractions into simpler ones; e.g.
The key idea is that $p(x)/q(x)$ can be written as a sum of simpler terms, each term having a denominator corresponding to a factor of $q(x)$. Consequently the original integral can be replaced by a sum of simpler integrals. Furthermore, it can be shown, that this *partial fraction expansion* is unique.

2.9.1 General Method for Partial Fractions

Let $f(x) = p(x)/q(x)$ be a rational function.

1. Ensure that $f(x)$ is a proper rational function[3].
2. Factor $q(x)$ as far as possible (every polynomial can be factored as a product of linear factors $ax+b$ and quadratic factors ax^2+bx+c).
3. To each factor of $q(x)$ we associate a term in the partial fraction decomposition via the following rule:

   I: To each *non-repeated* linear factor of the form $(ax+b)$ (i.e. no other factor of $q(x)$ is a constant multiple of $(ax+b)$) there corresponds a partial fraction term of the form:

   Example: *Suppose $f(x) = p(x)/q(x)$, with deg(q) < deg(p), and $q(x) = (x-1)(2x-1)(-x+2)$. What is the partial fraction expansion of $f(x)$?*

[3]this will always be the case in MS 2002

To integrate each of the right-hand side terms, let u =linear factor in each of the denominators.

II: To each linear factor of the form $(ax+b)^n$ (i.e. a repeated linear factor of $q(x)$) there corresponds a sum of n partial fraction terms of the form:

Example: *Suppose* $f(x) = p(x)/q(x)$, *with deg(q)* $<$ *deg(p), and* $q(x) = (x-1)^3(2x-1)(-x+2)^2$. *What is the partial fraction expansion of* $f(x)$*?*

Again, to integrate each of the right-hand side terms, let u =linear factor in each of the denominators.

III: To each non-repeated *quadratic* factor of $q(x)$ of the form (ax^2+bx+c) (i.e. no other factor of $q(x)$ is a constant multiple of (ax^2+bx+c)) there corresponds a partial fraction term of the form:

Example: *Suppose* $f(x) = p(x)/q(x)$, *with deg(q)* $<$ *deg(p), and* $q(x) = (x-1)^2(x^2+x+1)(2x^2+3)$. *What is the partial fraction expansion of* $f(x)$*?*

To integrate the terms that come from quadratic factors, complete the square and work from there.

IV: To each quadratic factor of the form $(ax^2+bx+c)^n$ (i.e. a repeated linear factor of $q(x)$) there corresponds a sum of n partial fraction terms of the form:

Example: *Suppose* $f(x) = p(x)/q(x)$, *with deg(q)* $<$ *deg(p), and* $q(x) = (x-1)^2(2x-1)(2x^2+3)^2$. *What is the partial fraction expansion of* $f(x)$*?*

Examples of Rule IV are always very long and we won't discuss them further in MS2002.

4. Write the partial fraction expansion as a single fraction "$f(x)$", and set it equal to

$f(x)$. Compare the numerators of $f(x)$, $u(x)$; and the numerator of "$f(x)$", $v(x)$; by setting them equal to each other:

Find the coefficients in the partial expansion using one of two methods:

(a) The coefficients of $u(x)$ must equal those of $v(x)$. Solve the resulting simultaneous equations. This is the best method if there are any quadratic terms but works equally well if there are none.

(b) If $u(x)$ and $v(x)$ agree on all points then $f(x)$="$f(x)$". Generate m simultaneous equations in m variables by plugging in m different values $x_1, x_2, \ldots, x_m$ and solving the equations:

This method works best when all the factors are linear.

Examples

1. Find the partial fraction expansion of $\dfrac{7}{2x^2 + 5x - 12}$.
Solution: First factor the denominator:

Now to each of these factors associate a term in the partial fraction expansion and write as a single fraction:

Now we want to compare (use method (b)):

2. Evaluate $\displaystyle\int \frac{6x^2-3x+1}{(4x+1)(x^2+1)}\,dx$.

   Solution: Firstly there is no direct integral but the method of partial fractions provides a manipulation as the integrand is a (proper) rational function. Let us find the partial fraction expansion of

   $$\frac{p(x)}{q(x)} = \frac{6x^2-3x+1}{(4x+1)(x^2+1)}.$$

   First we factor $q(x)$ as much as possible... Hence we have a partial fraction expansion:

   So we want $6x^2-3x+1 = A(x^2+1)+(Bx+C)(4x+1)$:

   We use method (a) to determine the coefficients as there was a quadratic in q. Hence we must solve the simultaneous equations:

So now we have

$$\frac{7}{2x^2-3x+1}=\frac{2}{4x+1}+\frac{x-1}{x^2+1}=2\frac{1}{4x+1}+\frac{x}{x^2+1}-\frac{1}{x^2+1}.$$

Hence to integrate (sum and scalar rules) we have

$$\int\frac{7}{2x^2-3x+1}\,dx=2\int\frac{1}{4x+1}\,dx+\int\frac{x}{x^2+1}\,dx-\int\frac{1}{x^2+1}\,dx.$$

Use the substitutions $u=4x+1$ and $v=x^2+1$ for the first two — the third is directly integrable as $\arctan x$:

3. Evaluate $\displaystyle\int\frac{dx}{x^5-x^2}$.

   Solution: Proceeding as above to partial fractions; factorise the denominator:

   Now you can factorise x^3-1 by the factor theorem or memory of a^3-b^3. I prefer to come up with the formula on the spot as follows:

In other words $x^3 - 1 = (x-1)(x^2+x+1)$ so we have

$$x^5 - x^2 = x^2(x-1)(x^2+x+1).$$

Here x^2 is not a quadratic factor (although if you carry on with case III it'll come out fine); it is the linear factor $(x-0)$ repeated twice (Case II). Can we factor x^2+x+1 any further... Hence we have partial fraction expansion:

$$\begin{aligned}\frac{1}{x^5-x^2} &= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{Dx+E}{x^2+x+1} \\ &= \frac{A}{x}.\frac{x}{x}.\frac{x-1}{x-1}.\frac{x^2+x+1}{x^2+x+1} + \frac{B}{x^2}.\frac{x-1}{x-1}.\frac{x^2+x+1}{x^2+x+1} \\ &+ \frac{C}{x-1}.\frac{x^2}{x^2}.\frac{x^2+x+1}{x^2+x+1} + \frac{Dx+E}{x^2+x+1}.\frac{x^2}{x^2}.\frac{x-1}{x-1}.\end{aligned}$$

Hence we want

$$1 = Ax(x-1)(x^2+x+1) + B(x-1)(x^2+x+1) + C(x^2)(x^2+x+1) + (Dx+E)(x^2)(x-1).$$

Normally with the quadratic I would suggest method (a) but here maybe (b) is easier (with so many terms to multiply out: we would have to solve a system of simultaneous equations in *five* unknowns); let $x = 0,\ 1$:

Now, to generate three equations, to solve for A, D and E try $x = -1,\ 2,\ -2$ on:

$$1 = Ax(x-1)(x^2+x+1) - (x-1)(x^2+x+1) + \frac{1}{3}(x^2)(x^2+x+1) + (Dx+E)(x^2)(x-1).$$

This yields

$$3A + 3D - 3E = -2. \tag{2.4}$$

$x = 2$,

Which yields

$$21A + 12D + 6E = -2 \tag{2.5}$$

[Ex]: Show that $x = -2$ yields:

$$3A + 4D - 2E = -2 \tag{2.6}$$

and show that the solution of the simultaneous equations is given by $A = 0$, $D = -1/3$ and $E = 1/3$. Complete the integral.

Exercises

Evaluate the following integrals.

1. $\displaystyle\int \frac{dt}{t^2 - 4}$ Ans: $\displaystyle\frac{1}{4}\ln\left|\frac{t-2}{t+2}\right| + C$

2. $\displaystyle\int \frac{dx}{(8-x)(6-x)}$ Ans: $\displaystyle\frac{1}{2}\ln\left|\frac{(8-x)}{6-x}\right| + C$

3. $\displaystyle\int \frac{5x-2}{x^2-4}\,dx$ Ans: $\ln\left|(x-2)^2(x+2)^3\right| + C$

4. $\displaystyle\int \frac{6x^2-2x-1}{4x^3-x}\,dx$ Ans: $\displaystyle\frac{1}{4}\ln\left|\frac{x^4(2x+1)^3}{2x-1}\right| + C$
   [Autumn 2011 Question 4(a)]

5. $\displaystyle\int \frac{(x-1)\,dx}{x^3-x^2-2x}$ Ans: $\displaystyle\frac{1}{6}\ln\left|\frac{x^3(x-2)}{(x+1)^4}\right| + C$

6. $\displaystyle\int \frac{dx}{x^3+3x^2}$ Ans: $\displaystyle\frac{1}{9}\ln\left|\frac{x+3}{x}\right| - \frac{1}{3x} + C$

7. $\displaystyle\int \frac{dx}{(x+2)^3}$ Ans: $\displaystyle-\frac{1}{2}(x+2)^{-2} + C$

8. $\displaystyle\int \frac{(x^3-1)\,dx}{x(x-2)^3}$ Ans: $\displaystyle\frac{-17x+27}{4(x-2)^2} + \frac{1}{8}\ln\left|x(x-2)^7\right| + C$

9. $\displaystyle\int \frac{dx}{2x^3+x}$ Ans: $\displaystyle\frac{1}{2}\ln\left|\frac{x^2}{2x^2+1}\right| + C$

10. $\displaystyle\int \frac{(x^2+x)\,dx}{x^3-x^2+x-1}$ Ans: $\ln|x-1| + \arctan x + C$

11. $\displaystyle\int \frac{x^2-2x-3}{(x-1)(x^2+2x+2)}\,dx$ Ans: $\displaystyle\frac{1}{10}\ln\left|\frac{(x^2+2x+2)^9}{(x-1)^8}\right| - 2\arctan(x+1) + C$

12. $\displaystyle\int \frac{dx}{16x^4-1}$ Ans: $\displaystyle\frac{1}{8}\ln\left|\frac{2x-1}{2x+1}\right| - \frac{1}{4}\arctan 2x + C$

13. $\displaystyle\int \frac{2x^2+x-8}{x^3+4x}$ Ans: $\displaystyle\frac{1}{2}\arctan\frac{x}{2} + 2\ln\left|\frac{x^2+4}{x}\right| + C$

2.10 Area

2.11 Area of a Plane Region

Recall from Chapter 1 that if $a < b$ and $f(x) \geq 0$ on $[a, b]$, then $\int_a^b f(x)\,dx$ gives the area enclosed between $y = f(x)$ and the x-axis on the interval $[a, b]$.

More generally, if $a < b$ and $g(x) \leq f(x)$ on $[a, b]$, then

$$\int_a^b [f(x) - g(x)]\,dx = \int_a^b \text{“upper curve} - \text{lower curve”}\,dx$$

measures the area enclosed between $y = f(x)$ and $y = g(x)$:

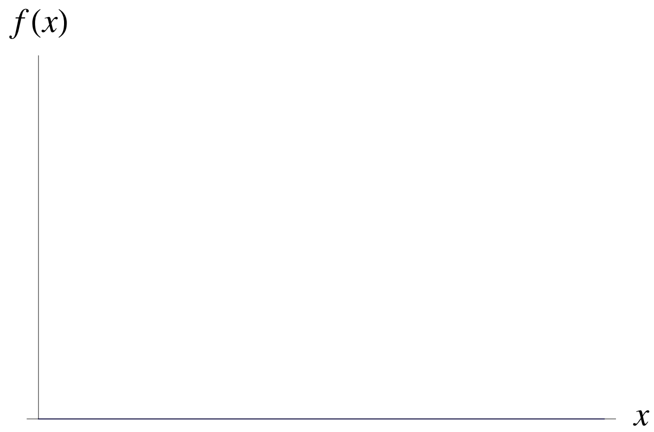


Figure 2.10: Whenever $f(x) \geq g(x)$, $\displaystyle\int_a^b (f(x) - g(x))\,dx$ measures the area between the two curves $y = f(x)$ and $y = g(x)$.

Examples

1. Find the area of the region bounded by the curves $y = x^2$ and $y = -x^2 + 4x$.
   Solution: It is usually advisable to draw a rough diagram when computing areas or volumes. We know that both functions go through the origin and we know that $y = x^2$ has its minimum there also. $y = -x^2 + 4x$ is a ‘$\bigcap$’ quadratic so has a maximum — either at the midpoint of the roots (due to symmetry) or where $dy/dx = 0$:

   At $x = -2$, $y = 4$ so we have an idea of the situation:

   The points of intersection is on both curves...

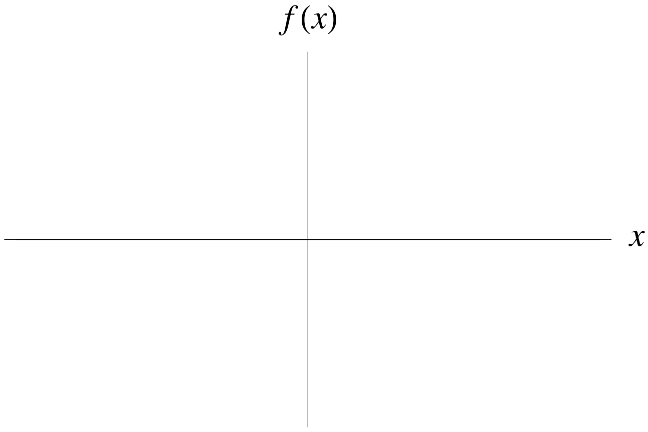


Figure 2.11: We must find the second point of intersection.

Hence we evaluate the integral:

2. Roughly sketch the region enclosed by the curves $x = 1 - y^2$ and $x = y^2 - 1$. By integrating with respect to y, find the area of the region enclosed by these two curves. *Solution*: Note that we know what $y = 1 - x^2$ and $y = x^2 - 1$ look like. These look exactly the same except the rôles of x and y are reversed:

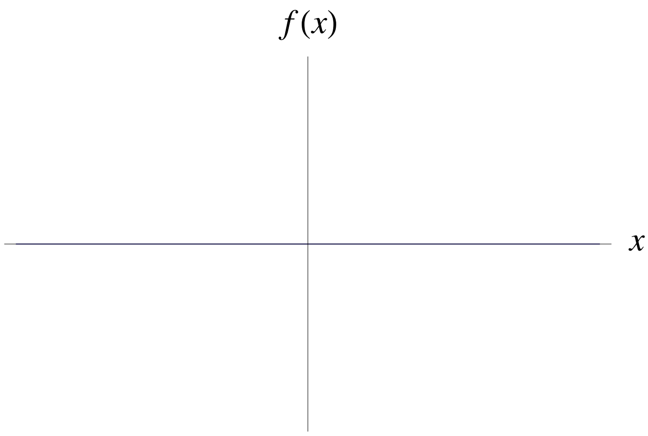


Figure 2.12: Once again we must find the points of intersection.

To find the points of intersection:

Hence we can integrate with respect to y from -1 to 1 — and $x = 1 - y^2$ is the 'upper curve':

3. Find the area enclosed by the curves $y = x$ and $x = y^2 - 12$.
Solution: Draw a picture

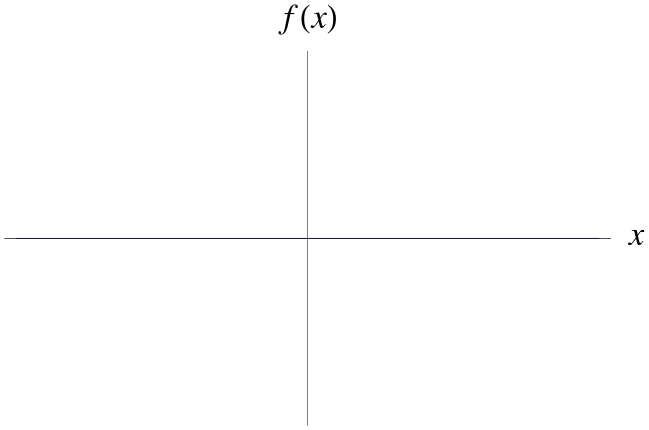


Figure 2.13: $y = x$ is a line while $x = y^2 - 12$ is a quadratic except that the ys are the inputs and xs are the outputs — hence the quadratic is on it's side. We need to find the points of intersection.

Find the points of intersection:

Hence the curves intersect at $(-3, -3)$ and $(4, 4)$. Note that the curve $x = y^2 - 12$ is actually composed of two functions glued together — namely $y = +\sqrt{x + 12}$ and $y = -\sqrt{x + 12}$ (which is which?). Note now that the area enclosed can be split into two separate integrals:

(a) the area between the curves $y = +\sqrt{x + 12}$ and $y = -\sqrt{x + 12}$ in the region $-12 \leq x \leq -3$.

(b) the area between the curves $y = +\sqrt{x + 12}$ and $y = x$ in the region $-3 \leq x \leq 4$.

Thus

$$\text{Area} = \int_{-12}^{-3} \left[\sqrt{x + 12} - (-\sqrt{x + 12})\right] dx + \int_{-3}^{4} \left[\sqrt{x + 12} - x\right] dx.$$

We will evaluate these separately:

Now write in terms of x and use the original limits:

Now the second integral:

Adding this up and adding it to the first answer gives the area as 343/6.

Exercises

1. Find the area of each of the following regions. In each question a rough diagram of the region will help.

   (a) Bounded by $y = -x^2$ and $y = -4$. Ans: 32/3

   (b) Bounded by $y = 2 - x^2$ and $y = -x$. Ans: 9/2

   (c) Bounded by $y = \sqrt{x}$ and $y = x^3$. Ans: 5/12

   (d) Bounded by $y^2 = 4x$ and $x^2 = 4y$. Ans: 16/3

   (e) Bounded by $y = x^2$, the x-axis and the lines $x = 1$ and $x = 2$. Ans: 7/3

   (f) Bounded by $y = x^2 - 4x$, the x-axis, and the lines $x = 1$ and $x = 3$. Ans: 22/3

   (g) Bounded by $y = x^3 - 2x^2 - 5x + 6$, the x-axis, and the lines $x = -1$ and $x = 2$. Hint: you will need to divide the region into two parts and compute an integral for each part. Ans: 157/12

   (h) Bounded by the curve $y = e^x$, the coordinate axes, and the line $x = 2$. Ans: $e^2 - 1$

2. The region bounded by the parabola $y^2 = 4x$ and the line $4x - 3y = 4$ is shown in the diagram.

Find the area of this region using integration. (Hint: it is easier if you integrate with respect to y.) Ans: 125/24

3. Find the area of the region bounded by the parabola $y^2 = 2x - 2$ and the line $y = x - 5$ using two different approaches: (i) integrate with respect to x (this will require you to break the region into two parts, each of which has its own integral) (ii) integrate with respect to y. Hint: this problem is similar to Example... Ans: 18

2.12 Further Applications

2.13 First Order Separable Differential Equations

A differential equation is an equation containing one or more derivatives, e.g.,

$$y' = x^2,$$
$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} = \sin x.$$

Most laws in physics and engineering are differential equations.

The *order* of a differential equation is the order of the highest derivative that appears;

$$y' = x^2 \quad \text{is first order}$$
$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 1 \quad \text{is second order}$$
$$(\cos x)\left(\frac{d^2y}{dx^2}\right)^3 + \frac{dy}{dx} = y \quad \text{is second order.}$$

A function $y = f(x)$ is a *solution* of a differential equation if when you substitute y and its derivatives into the differential equation, the differential equation is satisfied. For example, $y = \tan x$ is a solution of the differential equation $y' = 1 + y^2$ since if $y = \tan x$, then

A differential equation can have many solutions: $y' = 2$ has a solution $y = 2x + C$ for every constant C. The *general solution* of a differential equation is the set of all possible solutions. The differential equation $y' = 2$ has general solution $y = 2x + C$, where the constant C is arbitrary. It can be very difficult to find the general solution of a differential equation. We shall consider only certain first-order differential equations that can be solved fairly readily.

2.14 First-order Separable Differential Equations

A *separable* first-order differential equation is one that can be written in the form

In this situation we can *separate the variables*:

Each side can now be integrated:

The point of separating the variables is that we cannot usually integrate expressions like $\int y\,dx$ where both variables appear.

Example

Solve the separable first order differential equation:

$$y' = xy.$$

Solution: First separate the variables and integrate:

Usually we want to solve for y:

Here there are two infinite families of solutions. The solution of a first-order differential equation will always contain an unknown constant — and might have different families of solutions also (e.g. the solution $y^2 = x + C$ has the families $y = +\sqrt{x+C}$ and $-\sqrt{x+C}$). However an extra piece of numerical data such as "$y = 2$ when $x = 1$" sometimes reduces this to a unique solution. Note that this will usually be written as $y(1) = 2$ — for the input $x = 1$, the output is $y = 2$. This extra data is called an *initial condition* or *boundary condition* and the entire problem (differential equation and boundary condition) is often called an *initial-value problem* or *boundary-value problem.*

Example

Solve the initial-value problem

$$\frac{dy}{dx} = \frac{1+x}{xy} \quad \text{for } x > 0, \quad \text{where } y(1) = -4.$$

Solution: First separate the variables and integrate:

Now apply the boundary condition:

Now substitute in the constant and hopefully solve for $y(x)$:

Now the fact that $y = -4$ at $x = 1$ and that $\sqrt{x} > 0$ where defined implies that the solution is $y(x) = -\sqrt{2(\log_e x + x + 7)}$. [Ex:] Show that this solves the differential equation and satisfies the boundary condition.

Further Remarks: Picard's Existence Theorem

There is a theorem in the analysis of differential equations which states that if a differential equation is suitably *nice* in an interval about the boundary condition then not only does a solution exist but it is unique. This allows us to define functions as solutions to differential equations. For example, an alternate definition of the exponential function, e^x, is the unique solution to the differential equation:

$$\frac{dy}{dx} = y\,,\ \ y(0) = 1.$$

Exercises

1. Solve the following differential equations:

   (a) $y' = 3x^2 + 2x - 7$ Ans: $y = x^3 + x^2 - 7x + C$

(b) $y' = 3xy^2$ Ans: $3x^2y + Cy + 2 = 0$

(c) $\dfrac{dy}{dx} = \dfrac{3x\sqrt{1+y^2}}{y}$ Ans: $2\sqrt{1+y^2} = 3x^2 + C$

(d) $\dfrac{dy}{dx} = \dfrac{x}{4y}$, $y(4) = -2$ Ans: $x^2 = 4y^2$

2. The point $(3, 2)$ is on a curve, and at any point (x, y) on the curve the tangent line has slope $2x - 3$. Find the equation of the curve. Ans: $y = x^2 - 3x + 2$

3. The slope of the tangent line to a curve at any point (x, y) on the curve is equal to $3x^2y^2$. Find the equation of the curve, given that the point $(2, 1)$ lies on the curve. Ans: $-\dfrac{1}{y} = x^3 - 9$

2.14.1 Exponential growth and decay

Consider the differential equation

where k is a non-zero constant and $y > 0$. This equation models many natural processes where a quantity y increases $(k > 0)$ or decreases $(k < 0)$ at a rate proportional to its size and t is usually time. It is separable so we can solve it:

As $y = y(t)$ (i.e., y is a function of t), putting $t = 0$ in this equation yields $y(0) = e^{0+C} = e^C$. That is, the solution of is

$$y(t) = y(0)e^{kt}. \tag{2.7}$$

When $k > 0$ the function $y(t)$ grows exponentially (e.g., population growth), while if $k < 0$ the solution decays exponentially (e.g., radioactive decay). Graphs of these two situations:

Example

The growth of bacteria in a certain culture is proportional to the number of bacteria present. If initially there are $1,000$ bacteria, and the number doubles in 12 minutes, how long will it take before there are $1,000,000$ bacteria present?
Solution: Write y for the number of bacteria. Then $dy/dt = ky$ where k is for the moment unknown and t is time measured in minutes. From (2.7), the solution of this differential

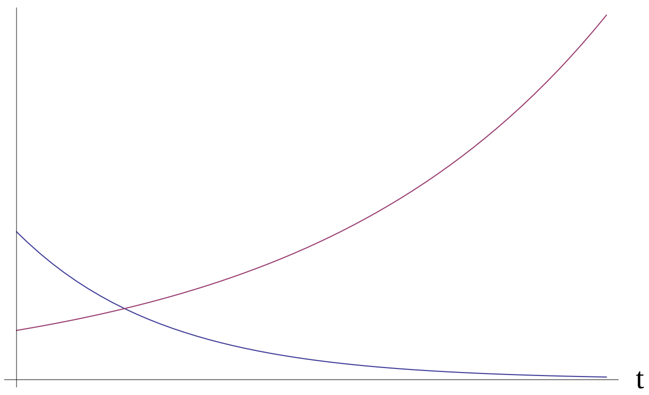


Figure 2.14: Examples of exponential growth and decay.

equation is $y(t) = y(0)e^{kt}$. We are told that $y(0) = 1000$, so $y(t) = 1000e^{kt}$. Note that this could be done by solving the differential equation but this would be acceptable in an exam situation. In this context, $y(0) = 1000$ would be a boundary condition.

To find k, we need another boundary condition. Luckily we know $y(12) = 2,000$:

We now have $y(t) = 1000e^{(t\ln 2)/12}$. To answer the question posed in this problem, we must solve for t in the equation $y(t) = 1,000,000$:

Now taking logs:

which on a calculator is approximately 119.6.

Exercises

1. Solve the initial-value problem $dy/dt = -6y, y(0) = 5$. Ans: $y(t) = 5e^{-6t}$
2. Bacteria grow in a culture at a rate proportional to the number present. If there are 1,000 bacteria initially and the number doubles in 1 hour, how many bacteria will there be in 3.5 hours? Ans: $8000\sqrt{2} \approx 11,300$
3. In a chemical reaction the rate of conversion of a substance is proportional to the amount of the substance still untransformed at that time. After 10 minutes, 1/3 of the original amount of the substance has been transformed. After 15 minutes (from the beginning of the experiment), 20g has been transformed. What was the original weight of the substance? Ans: 43.9g
4. The rate of natural increase of city's population is proportional to the population. If the population increases from 40,000 to 60,000 in 40 years, when will the population be 80,000? Ans: 68.4 years