Linear Systems: Summary of Methods

Gaussian Elimination Methods
Roughly,

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & \star & \star & \star \\ 0 & 1 & \star & \star \\ 0 & 0 & 1 & \star \end{array}\right]$$

We can use this at any time this is the full theory.

Gaussian Elimination with Partial Pivoting
Roughly,

$$\left[\begin{array}{ccc|c} 2 & a_{12} & a_{13} & b_1 \\ 4 & a_{22} & a_{23} & b_2 \\ 8 & a_{32} & a_{33} & b_3 \end{array}\right] \to \left[\begin{array}{ccc|c} 8 & a_{32} & a_{33} & b_3 \\ 4 & a_{22} & a_{23} & b_2 \\ 2 & a_{12} & a_{13} & b_1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & \star & \star & \star \\ 0 & 1 & \star & \star \\ 0 & 0 & 1 & \star \end{array}\right]$$

We use Gaussian Elimination with partial pivoting when we are using rounded-off decimals.

Cramer's Rule
Roughly, given the system

$$\left[\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array}\right],$$

we have

$$x = \frac{D_x}{D} = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}},$$

and $y = \dfrac{D_y}{D}$. We use Cramer's Rule when the solution is unique and we want to know only one of the variables.

Jacobi Method
Roughly, to solve a diagonally dominant system

$$\begin{aligned} 8x + 4y - 2z &= 16 \\ 3x + 18y - z &= -15 \\ x + y - 7z &= 21 \end{aligned}$$

we approximate by

$$\begin{aligned} 8x_0 &= 16 \\ 18y_0 &= -54 \\ -7z_0 &= 21 \end{aligned}$$

Now if $(x_0, y_0, z_0) = (2, -3, -3)$ is an approximate solution we get a better one with the *Jacobi equations*:

$$\begin{aligned} x_1 &= \frac{16 - 4y_0 + 2z_0}{8} \\ y_1 &= \frac{-15 - 3x_0 + z_0}{18} \\ z_1 &= \frac{21 - x_0 - y_0}{-7}. \end{aligned}$$

We use the Jacobi method when the system is *diagonally dominant*. Diagonally dominant linear systems always have unique solutions and the Jacobi Method is guaranteed to converge to the unique solution. It is an *iterative* method and so easy to implement on a computer.

Gauss-Siedel Method
Roughly, we follow the same steps as Jacobi's Method up to the Jacobi equations. However instead of using the previous approximation (x_0, y_0, z_0) to find (x_1, y_1, z_1), we use the very latest approximations. E.g we use y_0 and z_0 to find x_1. Now we use x_1 and z_0 to find y_1, and finally x_1 and y_1 to find z_1:

$$\begin{aligned} x_1 &= \frac{16 - 4y_0 + 2z_0}{8} \\ y_1 &= \frac{-15 - 3x_1 + z_0}{18} \\ z_1 &= \frac{21 - x_1 - y_1}{-7}. \end{aligned}$$

It does everything the Jacobi Method does but faster. Also it is moderately more difficult to implement on a computer.