MATH6000 — Essential Mathematical Skills

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Contents

0.1 Introduction

Lecturer

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This page will comprise the webpage for this module and as such shall be the venue for course announcements including definitive dates for the tests. This page shall also house such resources as links (such as to exam papers), as well supplementary material. Please note that not all items here are relevant to MATH6000; only those in the category 'MATH6000'. Feel free to use the comment function therein as a point of contact.

Module Objective

This module is about numeracy and basic algebraic competence. Its aim is to ensure that the first-year student acquires proficiency across the spectrum of numerical and algebraic skills needed for the study of science and engineering subjects. Continuous assessment is based on the principle of no compromise on minimal standards for essential skills.

Module Content

The Fundamentals of Arithmetic with Applications

The arithmetic of fractions. Decimal notation and calculations. Instruction on how to use a calculator. Ratio and proportion. Percentages. Tax calculations, simple and compound interest. Mensuration to include problems involving basic trigonometry. Approximation, error estimation, absolute, relative and relative percentage error. The calculation of statistical measures of location and dispersion to include arithmetic mean, median, mode, range, quartiles and standard deviation.

Indices and Logarithms

Indices with a discussion of scientific notation and orders of magnitude. Conversion of units. Logarithms and their use in the solution of exponential equations. Discussion of the number e and natural logarithms.

Basic Algebra

The laws of algebra expressed literally and illustrated both numerically and geometrically. Algebraic manipulation and simplification to include the factorisation of reducible quadratics. Transposition of formulae. Function notation with particular emphasis on functions of one variable.

Graphs

Graphs of quantities which are in direction proportion and indirect proportion. Graphs of simple linear, exponential and logarithmic functions. Reduction of non-linear relations to linear form to allow for the estimation of parameters.

Assessment

This module is assessed entirely by coursework. The pass mark for this module is 60%. All questions in each assessment to be answered. The assessment of this module is inextricably linked to the delivery. The student must reattend the module in its entirety in the case of a failure.

Continuous Assessment

There are five 20% short-answer question tests:

1. On the first portion of the indicative content described under heading one up to and including problems on mensuration.

2. On the second portion of the indicative content described under heading one to include approximation/error analysis and calculations of a statistical nature.

3. A general coverage of scientific notation, orders of magnitude and applied problems requiring a thorough knowledge of indices and logarithms for their solution.

4. Algebraic manipulation and simplification, transposition of formulae and effective use of function notation.

5. Plotting and analysis of graphs relating to quantities which are: in direct proportion and in inverse proportion; related linearly, exponentially or logarithmically.

Lectures

It will be vital to attend all lectures as many of the examples, explanations, etc. will be completed by us in class.

Tutorials

The aim of the tutorials will be to help you achieve your best performance in the tests and exam.

Exercises

There are many ways to learn maths. Two methods which arent going to work are

1. reading your notes and hoping it will all sink in

2. learning off a few key examples, solutions, etc.

By far and away the best way to learn maths is by doing exercises, and there are two main reasons for this. The best way to learn a mathematical fact/ theorem/ etc. is by using it in an exercise. Also the doing of maths is a skill as much as anything and requires practise.
There are exercises in the notes for your consumption. The webpage may contain a link to a set of additional exercises. Past exam papers are fair game. Also during lectures there will be some things that will be *left as an exercise*. How much time you can or should devote to doing exercises is a matter of personal taste but be certain that effort is rewarded in maths.

In this set of notes, some exercises have been taken from John Bird, Engineering Mathematics Fifth Edition and New Concise Maths 1 by George Humphrey.

Reading

Your primary study material shall be the material presented in the lectures; i.e. the lecture notes. Exercises done in tutorials may comprise further worked examples. While the lectures will present everything you need to know about MATH6000, they will not detail all there is to know. Further references are to be found in the library. Good references include:

- John Bird 2010, *Basic Engineering Mathematics*, Fourth Ed., Elsevier Science Ltd England
- Stroud, K.A.; Booth, Dexter J. 2009, *Foundation Mathematics*, Palgrave MacMillan England

The webpage may contain supplementary material, and contains links and pieces about topics that are at or beyond the scope of the course. Finally the internet provides yet another resource. Even Wikipedia isn't too bad for this area of mathematics! You are encouraged to exploit these resources; they will also be useful for further maths modules.

0.2 Motivation: Eight Applications of Mathematics in Industry

> *The essence of mathematics is not to make simple things complicated, but to make complicated things simple.*

S. Gudder

Figure 1: **Cryptography**: Internet security depends on *algebra and number theory*

Figure 2: **Aircraft and automobile design**: The behaviour of air flow and turbulence is modelled by *fluid dynamics* - a mathematical theory of fluids

Figure 3: **Finance**: Financial analysts attempt to predict trends in financial markets using *differential equations*.

Figure 4: **Scheduling Problems**: What is the best way to schedule a number of tasks? Who should a delivery company visit first, second, third,..? The answers to these questions and more like them are to be found using an area of maths called *discrete mathematics*.

Figure 5: **Card Shuffling**: If the casino doesn't shuffle the cards properly in Blackjack, tuned in players can gain a significant advantage against the house. Various branches of mathematics including *algebra*, *geometry* and *probability* can help answer the question: *how many shuffles to mix up a deck of cards?*

Figure 6: **Market Research**: Quite sophisticated mathematics underlies *statistics* - which are the basis on which many companies make very important decisions.

Figure 7: **Structural Design**: Materials technology is littered with equations, etc that then feeds into structural design. Flying buttresses were designed to balance these equations and hence the building itself.

Figure 8: **Medicine**: A load of modern medical equipment measures a patient indirectly by-way of measuring some signal that is emitted by the body. For example, a PET scan can tell a doctor how a sugar is going around your body: the patient takes ingests a mildly radioactive chemical that the body sees as sugar. As this chemical moves around the body it emits radiation which can be detected by the PET scanner. The mathematically-laden theory of *signal analysis* can be used to draw up a representation of what's going in the body.

Chapter 1

The Fundamentals of Arithmetic with Applications

He who refuses to do arithmetic is doomed to talk nonsense.

John McCarthy

1.1 Number

A *natural number* is an ordinary counting number 1,2,3,.... Here the dots signify that this list goes on forever.

The *set* or collection of all natural numbers $\{1, 2, 3, \dots\}$ we call $\mathbb{N}$ for short:

We might use the following notation to signify the sentence "*n is a natural number*":

said "*n in* $\mathbb{N}$" or "*n is an element of* $\mathbb{N}$". The symbol '$\in$' means 'in' or 'is an element of'.

The set of *integers* or *whole numbers* is the set of natural numbers together with 0 and all the negative numbers: $\dots, -3, -2, -1, 0, 1, 2, 3, \dots$. Once again the dots signify that this list continues indefinitely in both directions. We denote the set of "*integerZ*"" by:

The set of *fractions* (or *Quotients*) is the set of all ratios of the kind

Examples include $1/2, -4/3, 211/24$. We call the 'top' number the *numerator* and the 'bottom' number the *denominator*. We will see in Chapter 3 why we don't let the denominator equal zero here (it corresponds to division by zero, which is contradictory).

Like $\mathbb{N}$ and $\mathbb{Z}$ this is also an infinite set which we denote by:

$n \neq 0$ is shorthand for "*n is not zero*".

A *real number* is *any* number that can be written as a decimal. Examples:

- $1 = 1.0$
- $3 = 3.0$
- $-5 = -5.0$
- $1/2 = 0.5$
- $2/3 = 0.6666...$
- $\sqrt{2} = 1.41421356...$
- $\pi = 3.14159265...$

In the context of this module a real number is just any number at all be it a natural number, negative number, fraction, square root, etc. For those of you looking to jump the gun *complex numbers* are not real numbers...

Again we write $x \in \mathbb{R}$ to signify "x is a real number".

Evaluation of Formulae

The following question was asked on an Irish television quiz programm:

> *Fill in the missing number:* $2 + 4 \times \Box = 30$.

A contestant said the answer was 5. "Correct" said the quizmaster. But they were both wrong. The answer should have been 7.

In Mathematics, by convention alone, multiplications are done before additions, so that

$$2 + 4 \times 5 = 2 + 20 = 22 \text{ , but } 2 + 4 \times 7 = 2 + 28 = 30.$$

So the correct answer is 7.

More generally, the order in which mathematical operations should be done — the *Hierarchy of Mathematics* — is as follows:

1.

2.

3.

4.

Hence if we want to write add two and four, then multiply this by five we write:

Exercises

1. Calculate each of the following:

   $(i)\, 6 \times 2 + 3$ $(ii)\, 3 + 12 \div 6$ $(iii)\, 2(5) + 4$

   $(iv)\, 2(2 + 3)$ $(v)\, 4 \times 5 - 12$ $(vi)\, 36 \div 9 - 1$

   $(vii)\, 5 + 2 \times 3 + 12 \div 4$ $(viii)\, 4 - 18 \div 6$ $(ix)\, 30 \div (6 - 1) + 2$

   $(x)\, (8 - 2) \div (5 - 2)$

   Selected Answers: (iii) 14, (vi) 3, (ix) 8

2. In the following problems, calculate the top and bottom separately before doing the division. The bar, —, acts exactly like a bracket.

   $(i)\, \frac{2(5-1)+2}{3\times6-4\times2}$ $(ii)\, \frac{5^2-7}{3^2-7}$

   $(iii)\, \frac{6(5-8)}{-3(7-4)}$ $(iv)\, \frac{5(-1)-3(3)^2}{4(3-1)}$

   Selected Answer: (iii) 2

1.2 Fractions

1.2.1 Multiplication of Fractions

Ordinarily we would learn how to add numbers before multiplying them but this is not necessarily the case for fractions. We start here with so-called *Egyptian* fractions. When $n \in \mathbb{N}$ a fraction of the form

is what we call an Egyptian fraction. We first encountered fractions as proportions of a whole. For example $1/9$ is represented as the portion of area taken up by the slice as shown:

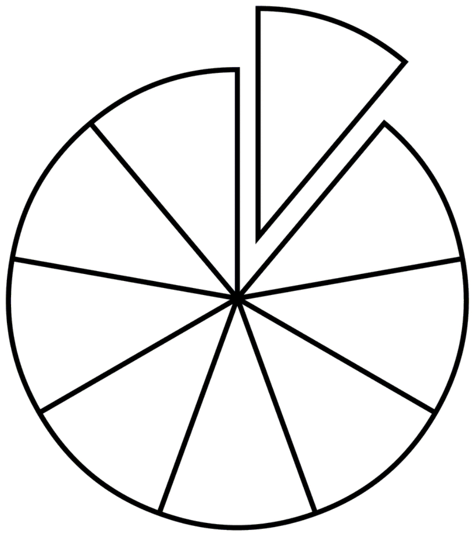

Figure 1.1: $1/9$ can be represented as *one out of nine*. $2/9$, $3/9$ then is similarly easy to visualise.

There is a slightly more sophisticated way of looking at fractions and it might just help those of us who have had problems with the arithmetic of fractions before. We start with the integers $\mathbb{Z}$. We know that we can multiply them together no problem:

Also we can solve some simple equations such as $4x = 20$ and $3x = -18$ pretty much in our head.

Now what about the equations $2x = 1$, $3x = 1$, $4x = 1$, $\dots$? Clearly there is no integer solution to these equations and we just define $1/n$ as the number or 'object' that solves the equation

Or in other words, $1/n$ *is the number that when multiplied by* n *gives 1.*

We could look at non-Egyptian fractions as solutions to these equations as well. For example, $2/3$ is the solution to $3x = 2$ but we won't do this exactly. We agree that we can multiply numbers?

We will define m/n as

In a more technical setting we would say that $1/3$ is the *multiplicative inverse* of 3 and even write $1/3 = 3^{-1}$. What happens when we multiply two Egyptian fractions? Well take say 3 and 7. What is

Now we note that $5 \times 6 = 6 \times 5$. That is the order of multiplication of numbers doesn't matter. Also recall that one times any number is the same number (e.g. $1 \times 7 = 7$):

That means that $\frac{1}{3} \cdot \frac{1}{7} = \frac{1}{21}$... This provides us a way of multiplying together two fractions:

$$\frac{5}{6} \times \frac{7}{4} =$$

This explains also how to properly cancel fractions. Take for example 15/12. A lot of us here would say something along the lines of, *oh, divide the top and bottom by* 3 *to get* $\frac{5}{4}$. Now why is this OK but say adding three above and below isn't? Well the actual calculations are

When the fraction cannot be simplified any more we might say that it is in 'reduced form'. Always try and write fractions in reduced form. More on this in Section 3. Note finally that we have $\frac{7}{7} = 7 \times \frac{1}{7} = 1$ so that $\frac{7}{7} = 1$. We might encounter so-called 'mixed fractions' such as

I think that you are better off converting these to so-called[1] 'improper fractions' as this makes the arithmetic easier.

Exercises:

1. Convert to improper form:

$$(i)\, 1\tfrac{1}{4} \quad (ii), 2\tfrac{3}{4} \quad (iii)\, 1\tfrac{3}{11} \quad (iv)\, 2\tfrac{1}{4}$$
$$(v)\, 1\tfrac{5}{11} \quad (vi)\, 1\tfrac{1}{2}$$

   Selected Answers: (iii) $\frac{14}{11}$ **(vi)** $\frac{3}{2}$.

2. Write each of the following as a one fraction:

$$(i)\, \tfrac{3}{4} \times \tfrac{1}{5} \quad (ii)\, \tfrac{3}{5} \times \tfrac{5}{9} \quad (iii)\, \tfrac{3}{2} \times \tfrac{5}{8}$$
$$(iv)\, \tfrac{7}{2} \times \tfrac{7}{6} \quad (v)\, \tfrac{8}{3} \times \tfrac{15}{4} \quad (vi)\, \tfrac{7}{6} \times \tfrac{2}{3} \times \tfrac{5}{8}$$

   Selected Answers: (iii) $\frac{15}{16}$ **(vi)** $\frac{35}{72}$

1.2.2 Division of Fractions

The way we looked at fractions in the last section conveniently allows us to redefine division:

[1] crap terminology: there is nothing wrong with 7/6. It just means seven sixths.

Definition

To divide by a number simply multiply by one over that number.

Examples

Now how about say $6 \div \frac{3}{7}$? Our new definition says that

But what the hell is $\frac{1}{3/7}$? Well according to the last section we can either look at is as the solution to

or, equivalently, the number that when we multiply it by $\frac{3}{7}$ we get one. It should be pretty clear at this point that

Hence $6 \div \frac{3}{7}$

Another example: $\frac{2}{3} \div \frac{6}{7}$

A Slightly Different Way

There is a slightly different way to approach division of fractions which uses the fact that multiplying by one doesn't change anything — and we can express one in many different ways (e.g. 121/121). This approach invites you to write your division of fractions as one big fraction:

Now what you want to do is identify what is making this 'big' fraction awkward. How about that 3 on the bottom? What would get rid of that? Well you could multiply by 3 on the top... however we can't just do that. BUT if we also multiply by 3 on the bottom we shall have multiplied by $3/3 = 1$...

Now the seven is in the way so repeat the trick:

Both methods are equivalent.

Exercise: Write each of the following as one fraction

$$(i)\ \tfrac{2}{3} \div \tfrac{5}{3} \qquad (ii)\ \tfrac{7}{9} \div \tfrac{14}{3} \qquad (iii)\ 1\tfrac{1}{5} \div 3\tfrac{3}{5} \qquad (iv)\ 1\tfrac{1}{2} \div 1\tfrac{3}{4}$$

$$(v)\ 5\tfrac{1}{4} \div 10\tfrac{1}{2} \qquad (vi)\ 4\tfrac{1}{3} \div 1\tfrac{6}{7} \qquad (vii)\ 4 \div \tfrac{1}{5} \qquad (viii)\ 6\tfrac{1}{4} \div 1\tfrac{7}{8}$$

Selected Answers: (iii) 1/3 **(vi)** 7/3

1.2.3 Addition & of Fractions

Just like we can add together two natural numbers to produce another natural number we should be able to add together two fractions to produce another. In this era of the calculator most of you would input sums such as

into a calculator and be done with it. Although you might disagree with me the introduction of the calculator to the Junior Cert did a lot to restrict your mathematical ability at a later date. For example, where x is *some*, unspecified real number how do you add

Here we will look at a method of adding fractions that is not the traditional method of adding fractions but will help you later with adding algebraic fractions such as above.

Adding Fractions: Traditional Method Examples

1. $\dfrac{3}{13} + \dfrac{7}{13}$

   When the denominators are equal this should be very straightforward. Recall that $\dfrac{3}{13}$ is just another way of writing

   So if you are adding $\dfrac{3}{13} + \dfrac{7}{13}$ all you are doing is adding three thirteenths to seven thirteenths:

2. When the denominators are unequal we cannot do this. Take for example $\dfrac{1}{6} + \dfrac{2}{3}$. Hence we find a way of writing the fractions over the same denominator as we know that $\dfrac{2}{4} = \dfrac{1}{2}$ etc. The way we learnt how to do this in school was to first

   (a) **Find the L.C.M. (lowest common multiple) of the two denominators**

   (b) **Write both fractions over the L.C.M. and add**

   Now there is nothing inherently bad about this method but finding the L.C.M. of the denominators is not really necessary as the next example might show.

3. $\dfrac{3}{4}+\dfrac{4}{5}$

   (a) **Find the L.C.M. (lowest common multiple) of the two denominators**

   (b) **Write both fractions over the L.C.M. and add**

In this last example, the denominators 4 and 5 did not share any factors[2], so the L.C.M. of 4 and 5 is just 4×5. What if we looked at $\dfrac{1}{6}+\dfrac{2}{3}$ and just looked at the 'L.C.M.' $6\times 3=18$?

This works perfectly well and this is the method that I would advocate. Now to go one step further I will show how to implement this method easily in a way which allows us to add algebraic fractions easily. The problem is that we can't really put, say $\dfrac{7}{41}$ in terms of 1/1763s very easily. What we are going to do is simply multiply by one! Subtraction works in the obvious way.

Examples

1. Write as a single fraction $\dfrac{3}{4}+\dfrac{5}{8}$.
   Solution:

2. Write as a single fraction $4\dfrac{1}{5}+1\dfrac{1}{4}$.
   Solution:

[2]we say that 4 is *co-prime* to 5

3. Write as a single fraction $\dfrac{2}{7} - \dfrac{3}{5}$
   Solution:

Exercises:

$(i)\ \frac{3}{4} + \frac{5}{8}$	$(ii)\ \frac{3}{5} + \frac{2}{3}$	$(iii)\ \frac{2}{3} - \frac{1}{2}$
$(iv)\ \frac{5}{8} - \frac{3}{4}$	$(v)\ \frac{2}{9} - \frac{3}{5}$	$(vi)\ 5\frac{1}{4} + 4\frac{1}{3}$
$(vii)\ 3\frac{1}{3} + 2\frac{4}{5}$	$(viii)\ 3\frac{1}{3} - 2\frac{5}{6}$	$(ix)\ 2\frac{1}{3} - 5\frac{3}{4}$
$(x)\ 3\frac{1}{4} - 2\frac{1}{8} + 2\frac{1}{2}$	$(xi)\ -2\frac{1}{5} + 3\frac{7}{10} - 4\frac{1}{15}$	$(xii)\ 1 - \frac{2}{3}$
$(xiii)\ 4 - 1\frac{1}{5}$	$(xiv)\ 2\frac{1}{2} - \frac{29}{8} - 3 + 1\frac{1}{2}$	

Selected Answers: (iii) 1/6 **(vi)** 115/12 **(ix)** −41/12 **(xii)** 1/3

1.3 Decimals: The Real Number System

In a lot of applied sciences the numbers we use tend to be decimals. The main advantage of decimals is that we have a good idea of their order of magnitude. For example, which is larger 4.6 or $\dfrac{943}{205}$...

One thing that we might not appreciate is what does say 5.67891 mean:

One fact that we might have been told in school is that every fraction can be written as a decimal:

A fact that we can probably appreciate right here is that any *terminating* decimal, e.g. 0.127 can also be written as a fraction:

A less well known fact is that every *repeating* decimal, for example 0.123211232112321...... can also be written as a decimal[3]. However there are decimals, real numbers, which cannot be written as a fraction. What is π? What is $\sqrt{2}$? Neither of these can be written as a fraction[4].

[3]using *infinite geometric series*. 0.123211232112321...... is in fact equal to 1369/11111 for example

[4]there is a way to show that the infinity of fractions is in a certain sense smaller than the infinity of decimals.

All the rules of arithmetic we learnt already (such as the Hierarchy of Mathematics) also apply to decimals. I would actually prefer ye to do a lot of arithmetic with fractions by hand but calculators are of course very useful when doing arithmetic with decimals. Let this exercise be an exercise in the Hierarchy of Mathematics and your use of the calculator also.

Exercise:
Find each of the following:

part	number	answer
(i)	$3(4.5) + 8.4$	23.7
(ii)	$12.88 \div 4.6 + 1.2$	4
(iii)	$3(8.2) + 2(5.4)$	35.4
(iv)	$2.7(7.3 - 5.1)$	5.94
(v)	$5.5 \div 0.25 + 4.5 \div 0.9$	27
(vi)	$50.46 \div (10.1 - 1.4)$	5.8
(vii)	$4.5(0.036 \div 0.06 + .4)$	20.7
$(viii)$	$(2.4)^2 + (0.3)^2$	5.85
(ix)	$6.4(1.2 + 1.3)^2$	40
(x)	$\frac{10(7.168+2.832)}{8(8.762-5.637)}$	4
(xi)	$\frac{(2.7-0.3)(2.7+0.3)}{(1.2)^2}$	5
(xii)	$\sqrt{22.4 \div 3.5 + 2.01}$	2.9
$(xiii)$	$\frac{3(5.6)+1.6}{\sqrt{37.21}+\sqrt{3.61}}$	2.3
(xiv)	$\sqrt{\frac{9}{25}} \times \sqrt{\frac{25}{16}}$	0.75
(xv)	$\frac{1}{(0.4)^2}$	6.25

1.4 Ratio & Proportion

Consider the following problem. Three individuals *Abe*, *Barney* and *Carl* are playing a dice game in an illegal casino for a total of E3,000 and the game has got to the point where there is only one throw of a die left such that

1. if the die shows 1, 2 or 3 then Abe wins
2. if the die shows 4 or 5 then Barney wins
3. if the die shows a 6 then Carl wins.

Suddenly, before the die is thrown, the Gardaí raid the gambling den and grab all the dice and equipment. Luckily for the gamblers they escape with the E3,000. How should they divide up the E3,000 now that they game has been finished prematurely?

A ratio is a comparison between two or more similar quantities measured in the same units. In this example the ratio was 3 : 2 : 1. Notice however how natural this problem was in comparison to the problem

> *Divide 3,000 in the ratio 3:2:1.*

We will use this problem as a reminder of how to do ratios. First note that 2 : 1 is the same as 4 : 2... multiplying across does not change the comparison[5]. We will thus endeavour to write our ratios in the *proportional form*:

Such that all the fractions add up to a whole. All we have to do is divide across by the sum of the numbers!

Examples

Express each of the following ratios in proportional form:

1. 12 : 15

2. 14 : 28 : 35

3. $2 : 1\frac{1}{2}$

[5]hence we can divide also!

4. 0.25 : 0.75

5. 800 m to 2 km

Writing ratios in this form makes it very straightforward to divide or share quantities as we have seen above.

Example

Divide

1. E28 in the ratio 2 : 5

2. 300 kg in the ratio 2 : 5 : 8

In some questions the ratios are given in disguise.

Example

E560 is shared between A, B and C so that A gets twice as much as B and B gets twice as much as C. How much does each receive?

The best approach is let the smallest share be one part. Therefore C receives one part, B receives two parts and C four parts. Now the requirement is divide E560 in the ratio $1:2:4$

Sometimes we are given the value of some of the parts.

Example

A and B share a sum of money in the ratio $4:3$. If B's share is E15, calculate:

1. the total amount shared between A and B
2. the amount that A received

Solution:

1. First we deal with the ratio:

   Now E15 represents 3/7 so that 1/7 is E5 and thus the whole is E35.
2. E20...

Applied Example

An alloy is made up of metals A and B in the ratio $2.5:1$ by mass. How much of A has to be added to 6 kg of B to make the alloy?
Solution: First of all deal with the ratio:

So we can say that 6 kg is 2/7 of the mass so that 1/7 of the mass is 3 kg so the total mass is 21 kg. Therefore there is 15 kg of metal A.

Exercises: Starred exercises should be of test standard.

1. Express each of the following ratios in proportional form:

$$(i)\,3:9 \qquad (ii)\,14:21 \qquad (iii)\,30:25$$
$$(iv)\,20:100 \qquad (v)\,88:77 \qquad (vi)\,15:20:25$$
$$(vii)\,1\tfrac{1}{4}:2 \qquad (viii)\,1\tfrac{3}{4}:2\tfrac{1}{4} \qquad (ix)\,2\tfrac{1}{2}:7\tfrac{1}{2}$$
$$(x)\,0.25:1.25$$

Selected Solutions: (iii) $\frac{6}{5}:\frac{1}{5}$ **(vi)** $\frac{1}{4}:\frac{1}{3}:\frac{5}{12}$ **(ix)** $\frac{1}{4}:\frac{3}{4}$

2. In a class of 30 pupils there are 20 girls. Find the ratio of
   (a) the number of boys to the number of girls
   (b) the number of boys to the number of pupils in the whole class

3. A football pitch has a length of 120 m and its width is 80 m. Calculate:
   (a) the ratio of its length to its width
   (b) the ratio of its length to its perimeter
   (c) the ratio its perimeter to its width

4. Divide each of the following quantities in the given ratio:

	Quantity	Ratio
(i)	E24	$1:3$
(ii)	50 cm	$3:2$
(iii)	88	$7:4$
(iv)	260 g	$6:7$
(v)	$E15.30$	$2:3$
(vi)	$E4,000$	$5:7:8$
(vii)	28 kg	$1\frac{1}{2}:2$
$(viii)$	24 kg	$0:25:1.25$
(ix)	$E44$	$1\frac{1}{3}:2\frac{1}{3}$

Selected Solutions (iii) $56:32$ **(vi) E1,000 : E1,400 : E1,600 (ix) E16 : E28**.

5. Divide E24 into three equal parts. Hence, or otherwise, divide E24 in the ratio 1:2.

6. E1,040 was divided in the ratio 6:7. The larger amount was given to charity. How much was this?

7. A man has two children, a boy aged eight years and a girl aged 12 years. If he divides E400 in the ratio of their ages, how much does each child get?

8. Two schools are to receive a grant from the Department of Education in proportion to their number of pupils. If one school has 450 pupils and the other has 720, how would a grant of E56,160 be divided between them.

9. Divide 60 in the ratio 1.5 : 2.5 : 3.5.

10. E120 is shared between P, Q and R so that P gets twice as much as Q and Q and R get equal shares. How much does P receive.

11. E300 is shared between X, Y and Z so that Y gets three times as much as X and Z gets twice as much as Y.

    (a) Who received the smallest share?

    (b) How much does each receive?

12. A prize fund was divided between two people in the ratio 2:3. If the larger prize was E120, calculate the total prize fund.

13. The ages of a father and daughter are in the ratio 8:3. If the father is 48 years old, how old is the daughter?

14. Two lengths are in the ratio 8 : 5. If the larger length is 120 cm, find the other length.

15. A, B and C share a sum of money in the ratio 2 : 3 : 4. If C's share is E48, find

    (a) the total sum of money shared and

    (b) the amount A and B received.

16. A woman gave some money to her four children in the ratio 2 : 3 : 5 : 9. If the difference between the largest and smallest share is E11.76, how much money did she give altogether?

17. * When mixing a quantity of paints, dyes of four different colours are used in the ratio of 7 : 3 : 19 : 5. If the mass of the first dye used is $3\frac{1}{2}$ g, determine the total mass of the dyes used.

18. * Determine how much copper and how much zinc is needed to make a 99 kg brass ingot if they have to be in the proportions copper : zinc to be :8 : 3 by mass.

1.5 Percentages

The word *per cent* means 'per 100' but we will be better off thinking of 1% as one hundredth of a whole. So what we can actually do is think

This allows us to pass easily from percentage to fraction to decimal:

Examples

1. Write as a fraction in its simplest form:

   (a) $15\% = 15\frac{1}{100} = \frac{15}{100} = \frac{3}{20}$

   (b) $37\frac{1}{2}\% = 37.5\frac{1}{100} = \frac{37.5}{100} = \frac{75}{200} = \frac{3}{8}$

2. Write as decimals:

   (a) $32\% = 32\frac{1}{100} = 0.32$

   (b) $27\frac{1}{2}\% = 27.5\frac{1}{100} = 275\frac{1}{1000} = 0.275$

Exercise:
Write each of the following percentages as: (i) a fraction in reduced form and (ii) a decimal:

$$(i)\,50\% \quad (ii)\,5\% \quad (iii)\,80\% \quad (iv)\,45\% \quad (v)\,55\%$$

$$(vi)\,15\% \quad (vii)\,120\% \quad (viii)\,180\% \quad (ix)\,62\tfrac{1}{2}\% \quad (x)\,18.4\%$$

$$(xi)\,16\tfrac{2}{3}\% \quad (xii)\,\tfrac{1}{2}\%$$

Selected Solutions: (iii) 4/5 **and** 0.8 **(vi)** 3/20 **and** 0.15 **(ix)** 5/8 **and** 0.625

We have already seen the power of multiplying by one. Converting from fractions and decimals to percentages is another case of these. If $\% = 1/100$ then

Example

Write as percentages:

1. $0.42 = 0.42(100\%) = 42\%$
2. $\frac{3}{5} = \frac{3}{5}(100\%) = 3(20)\% = 60\%$
3. $\frac{3}{40} = \frac{3}{40}(100\%) = 3\frac{5}{2}\% = 7.5\%$

Exercise

1. Write each of the following as a percentage

$$(i)\,0.18 \quad (ii)\,\tfrac{1}{2} \quad (iii)\,\tfrac{2}{5} \quad (iv)\,\tfrac{33}{50}$$
$$(v)\,\tfrac{5}{6} \quad (vi)\,\tfrac{27}{25} \quad (vii)\,\tfrac{1}{3} \quad (viii)\,\tfrac{2}{3}$$

2. Seán scored 63 out of 90 in a maths test and 18 out of 30 in an English test. In which test did he achieve the higher percentage?
3. A dealer bought a machine for E7,040 and immediately sold it for E8,000. Calculate his profit, as a percentage of the selling price.
4. * When 1600 bolts are manufactured, 36 are unsatisfactory. Determine the percentage unsatisfactory.

Selected Solutions: (iii) 40% (vi) 108%

We know that on a journey of, say 10 km, that after 2 km we have completed one fifth, or 0.2 or indeed 20% of the journey. All we did here is say

This means that we can quite easily express one quantity as a percentage of another.

Example

Express

1. 80 c as a percentage of E2.40
2. 400 m as a percentage of 2 km

Exercises

1. In each of the following, express the first quantity as a percentage of the second:

$$(i)\,3,5 \qquad (ii)\,36,300 \qquad (iii)\,15 \text{ cm, } 60 \text{ cm}$$

$$(iv)\,E1.50,\ E7.50 \quad (v)\,E.126,\ E8.40 \quad (vi)\,63 \text{ c, } E1.80$$

   Selected Answers: (iii) 25% (vi) 35%

2. There are 40 red, 60 green, 100 blue and 50 white marbles in a bag. What percentage of the total number of marbles is each colour?

3. An article that costs E240 is increased in price by E60. What is the increase as a percentage of the old price? What is the increase as a percentage of the new price?

4. * Two kilograms of a compound contains 30% of element A, 45% of element B and 25% of element C. Determine the masses of the three elements present.

5. * A concrete mixture contains seven parts by volume of ballast, four parts by volume of sand and two parts by volume of cement. Determine the percentage of each of these three constituents correct to the nearest 1% and the mass of cement in a two tonne dry mix, correct to 1 significant figure.

Often like the last exercise there we have to express a change in a quantity as a percentage change. We use the convention that we express the change in the quantity as a percentage of the original quantity. In terms of a formula

Examples

1. A woman's salary was increase from E40,000 to E44,800. Calculate the percentage increase in her salary.
   Solution:

2. A man went on a diet and reduced his weight from 120 kg to 111 kg. Calculate his percentage decrease in weight.
   Solution:

Exercises

1. Calculate the percentage increase or decrease in each of the following quantities:

Part	Original Quantity	New Quantity
(*i*)	60 marks	45 marks
(*ii*)	25 litres	22 litres
(*iii*)	250 m	215 m
(*iv*)	80 km/h	84 km/h
(*v*)	72 marks	81 marks
(*vi*)	120 litres	40 litres

2. A man's salary was increased from E30,900 to E33,063. Calculate his percentage increase in salary.
3. * A drilling machine should be set to 250 rev/min. The nearest speed available on the machine is 268 rev/min. Calculate the percentage 'over-speed'.

Our definition of $\% = \frac{1}{100}$ also makes finding a given percentage of a quantity straightforward because instead of 24% we can talk about $\dfrac{24}{100}$ of... all we have to do is multiply by $\dfrac{24}{100}$.

Example

Find 12% of E284.

If we use percentages the following become quite familiar

$$50\% = \frac{1}{2}$$

$$25\% =$$

$$10\% =$$

$$33\frac{1}{3}\% =$$

$$20\% =$$

$$75\% =$$

$$12\frac{1}{2}\% =$$

$$5\% =$$

$$66\frac{2}{3}\% =$$

$$2\frac{1}{2}\% =$$

Exercise
Find

(*i*) 25% of E150 (*ii*) 50% of 184 m (*iii*) 35% of E520.20

(*iv*) 13% of $2\frac{1}{2}$ km (*v*) 8% of E50 (*vi*) $2\frac{1}{2}$% of E80.40

We can now increase or decrease a quantity by a percentage. To increase by say 24% find 24% of the quantity and add it to the original. Similarly for decreases:

Here ΔQ means a change in the quantity Q.

Examples

1. In 1997 the population of a town was 5,400. In 1998 it increased by 12%. What was the population in 1998?
   Solution:

   Alternatively we can see that the ‘98 population is 112% (or 1.12) of the ‘97 population:

2. A machine was bought for E15,000. During the year its value depreciated by 15%. Calculate the value of the machine at the end of the year.
   Solution:

Alternatively we can see that the end of year price is 85% (or 0.85) of the start of year price:

Exercises

1. John weighs 125 kg before going on a diet. He sets himself a target of losing 8% of his original weight. What is his target weight?
2. Apples cost 60 c each. The reduces the price by 5%. Calculate the new price.
3. Because of low sales, a firm reduces its work force by (a) 18% of the factory workers and (b) 24% of the office staff. If 350 people work in the factory and 50 people work in the office, how many people will work in the firm after the reductions.
4. Express $\frac{2}{3}$ of 0.48 as a percentage of 2.56.
5. * A screws dimension is $12.5 \pm 8\%$ mm. Calculate the possible maximum and minimum length of the screw.

Sometimes we will be given a percentage of a quantity and have the find the whole.

Examples

1. 85% of a number is 153. Find the number.

   Alternatively using $\% = 1/100$

   The first method is probably easier.
2. $4\frac{1}{2}\%$ of a number is 36. Find 16% of the number.
3. The cost of a train ticket has risen by 5% to E15.96. What was the original price of the ticket.
   Solution: Now E15.96 is 105% of the original price:
4. A bicycle was sold for E384 at a loss of 20%. Find the original cost of the bicycle.
   Solution: Now E384 is 80% of the original cost:

Exercises.

1. 15% of a number is 90. Find the number.
2. $12\frac{1}{2}$% of a certain sum of money is E72. Find the sum of money.
3. 125% of a sum of money is E60. Find this sum
4. A bicycle was sold for E272 at a loss of 15%. Find the original cost and the amount of loss.
5. A piece of elastic was stretched by 15% to a length of 16.1 cm. Calculate its unstretched original length.
6. By selling a car for E8,840, the owner incurs a loss of 35% on the purchase price. Calculate the purchase price.
7. A sold and article to B for a gain of 10%. B sold it on to C for a gain of 15%. If C paid E23.50, how much did A and B pay for it?
8. 13% of a number is 156. Find 8% of the number.
9. When 9% of the pupils in a school are absent, 637 are present. How many students are on the school roll?
10. A youth club collects the same membership subscription from each of its 140 members. 15% of the club's members have still to pay their subscription. How many members have not paid their subscription? If the subscriptions not yet paid amount to E50.40, calculate the subscription for each member.
11. * A block of monel alloy consists of 70% nickel and 30% copper. If it contains 88.2 g of nickel, determine the mass of copper in the block.
12. * In a sample of iron ore, 18% is iron. How much ore is needed to produce 3600 kg of iron?
13. * The output power of an engine is 450 kW. If the efficiency of the engine is 75%, determine the power input.

1.6 Tax & Interest

1.6.1 VAT

Value-added tax (VAT) is a tax on goods and services. The rate of VAT is given in the form of a percentage, for example 23% VAT on alcohol. We need to be careful as customers, as some prices are inclusive of VAT, while other prices are exclusive of VAT.

Examples

1. VAT at 23% is added to a bill of E160. Calculate the total bill, including the VAT.
   Solution: The bill is now

2. A garage bill came to E120. When VAT was added to the bill it amounted to E138. Calculate the rate of VAT.
   Solution: Clearly the VAT is E18.

Exercises

1. Calculate the VAT payable on a garage bill of E380 if the rate of VAT is 18%.
2. A man bought a coat for E150 + VAT an electric razor for E80 + VAT. If VAT on clothes is 12% and VAT on electronic equipment is 21%, find how much he paid in total including the VAT,
3. A tanker delivered oil to a school. Before the delivery the meter showed 11,360 litres of oil in the tanker. After the delivery, the meter reading was 7,160 litres.
   (a) How many litres of oil were delivered to the school?
   (b) Calculate the cost of oil delivered if 1 litre of oil costs 41c.
   (c) When VAT was added to the price of the oil delivered, the bill to the school amounted to E2,083.62. Calculate the rate of VAT added.

Often a price includes VAT, and we have to work in reverse to calculate the VAT or the price before VAT was added on.

Example

A garage bill for repairs came to E295.20, including VAT at 23%. Calculate the bill before VAT was added.
Solution: Now E295.20 is 123% so we divide by 123 to find 1%:

Exercises:

1. Complete the following table:

	Price including VAT	VAT Rate	Price without VAT	VAT
(i)	E252	5%		
(ii)	E89.60	12%		
(iii)	E56.16	8%		

   Selected Answer: (iii) E52, E4.16

2. A table is priced at E767, which includes VAT at 18%. How much does the table cost without VAT?

3. A motorbike is priced at E1,086.40 which includes VAT at 12%. Calculate (i) the price without VAT and (ii) the VAT.

4. A boy bought a calculator for E73.80, which included VAT at 23%. Find the price of the calculator is VAT is reduced to 15%.

1.6.2 Income Tax

Here we present the PAYE tax system in Ireland[6]. In reality it doesn't happen exactly

1. For all your work over the year you are paid. This is your *gross income* Depending on your personal circumstances, you will be given a *standard rate cut-off point*. Income below this is considered low earnings and income above this considered high earnings.

[6] we're not going to include the Universal Social Charge here

2. Now to run public services the government comes along and takes taxation of you. They don't want too take much off low earners so what they do is take 20% of your low earnings. As high earners have more ability to pay they take 41% of your high earnings. 20% is called the *standard rate* and 41% is called the *higher rate.* What they take off you is your *gross tax.*

3. However is this a just system? Should a person supporting children have to pay as much tax as someone who doesn't have any? Should somebody who is working in retirement pay as much tax as a young person who has never paid a cent of tax? If we tax a family so much that they can't afford the mortgage social services might have to intervene etc. So depending on your circumstances they give you some of your gross tax back. The money they give you back are called your *tax credits* and the taxation they hold onto is called your *nett tax* or *tax payable.*

   What you have left over is your *nett income* or your *take home pay.*

In summary,

Examples

1. A man has a gross yearly income of E25,000. He has a standard rate cut-off point of E28,000 and a tax credit of E1,200. The standard rate of tax is 18% of income up to the standard rate cut-off point. Calculate:

   (a) the amount of gross tax for the year.

   (b) the amount of tax paid for the year.

   (c) the nett income for the year.

   Solution:

   (a) O.K. 18% on low earnings and what rate on high earnings...

   (b) We know the gross tax but we get tax credits back:

   (c) Nett income is just gross income less nett tax

2. A woman has a gross yearly income of E47,000. She has a standard rate cut-off point of E25,000 and a tax credit of E3,800. The standard rate of tax is 18% of income up to the standard rate cut-off point and 35% on all income above the standard rate cut-off point. Calculate:

   (a) the amount of gross tax for the year.

   (b) the amount of tax paid for the year.

   Solution:

   (a) Recall 18% on low earnings and 35% on the high earnings. Here we have E25k of low earnings and E22k of high earnings:

   This is her gross tax.

   (b) To find the tax payable we just give her back her tax credits:

3. A man has a gross yearly income of E24,000. He has a standard rate cut-off point of E27,200 and a tax credit of E2,850. If he pays tax of E1,470, calculate the standard rate of tax.
   Solution: If his nett tax is E1,470 and his tax credits are E2,850 then we know that

What percentage is this of his gross income?

Exercises:

1. Complete the following table. Note G.I. is gross income, C.O.P. is standard rate cut-off point, S.R. is standard rate, G.T. is gross tax, T.C. is tax credits, N.T. is nett tax and N.I. is nett income.

	G.I.	C.O.P.	S.R.	G.T.	T.C.	N.T.	N.I.
(i)	E17,500	E21,200	20%		E2,600		
(ii)	E19,400	E20,300	17%		E2,728		
(iii)	E19,570	E22,450	21%		E2,082		

   Selected Answer: (iii) E4,109.70, E2,027.70 E17,542.30

2. A man has a gross yearly income of E19,000. He has a standard rate cut-off point of E23,400 and a tax credit of E2,090. The standard rate of tax is 18% of income up to the standard rate cut-off point. Calculate:
   (a) the gross tax for the year.
   (b) the nett tax for the year.
   (c) the nett income for the year.

   Express nett tax for the year as a percentage of gross income for the year.

3. A man has a gross yearly income of E38,000. He has a standard rate cut-off point of E23,000 and a tax credit of E3,300. The standard rate of tax is 20% of income up to the standard rate cut-off point and 38% on all income above the standard rate cut-off point. Calculate:
   (a) the gross tax for the year.
   (b) the tax payable for the year.

4. A woman has a gross yearly income of E28,500. She has a standard rate cut-off point of E30,100 and a tax credit of E2,240. If she pays tax of E2,890, calculate the standard rate of tax.

1.6.3 Interest

How do banks make money?

This E100 is called the principal. We will denote it by P later when we want to write down a formula. The E1 that the saver gets back is interest on the savings/investment and the extra E2 that the borrower pays back is interest on the loan. The E1 leftover is the bank's profit. Interest is a payment made that is a percentage of the principal saved or borrowed. In this example the interest on savings/investment is 1% on the interest on borrowings is 2%. The interest rate or percentage is usually yearly but could be quarterly, monthly, weekly, daily, etc.

Examples

1. E250 was invested for a year at 2.5% per annum. Calculate the interest earned.

2. E540 is borrowed at 8% per annum. Calculate the amount owed after one year.
   Solution: The amount after one year will be

   So we simply find the interest

Exercises:

1. E540 was invested for a year at 6% per annum. Calculate the interest.
2. E580 is borrowed at $3\frac{1}{2}\%$ per annum. Calculate the amount owed after one year.
3. A man borrowed E5,000 from a bank at a rate of 4% per annum. He agreed to pay off the loan at the end of one year with one payment. How much did he need to pay to clear the loan?

Usually when people save they don't save for just one year but longer. If we want we can analyse this year by year. For example, 1.2% on E1,000 over five years will accrue as

$$E1012,\ E1024.14,\ E1,036.43,\ E1,048.87,\ E1061.46.$$

If we were naïve we might think that because the interest earned in the first year is E12 the interest after five years should be E60...

Having to calculate the interest year by year is a bit tedious so we come up with a formula/equation instead. Formulae express a relationship between various quantities no matter what their values are; for example $E = mc^2$. We use letters to stand for quantities. Here E is rest-energy, m is mass and c is the speed of light. If we want to calculate the rest-energy of 3 kg of a substance we simply substitute in $m = 3$ in this equation. So we need notation.

Let i be the interest rate. This will be a percentage, for example 3%, so will usually be a number between 0 and 1. For example $3\% = 3\frac{1}{100} = 0.03$. Now let P_n be the value of the investment after n years. We will call $P_0 = P$ the principal. Now we want analyse how P_1 is related to P. Just for now we will denote interest by I. Well

So to get P_1 from P we simply multiply by $(1 + i)$. How about how do we get from P_1 to P_2?

There is clearly nothing special about years zero, one or two here. Clearly for any year n

Now we can ask what is P_n:

$$\begin{aligned} P_n &= (1+i)P_{n-1} \\ &= (1+i)[(1+i)P_{n-2}] = (1+i)^2 P_{n-2} \\ &= (1+i)[(1+i)P_{n-3}] = (1+i)^3 P_{n-3} \\ &= \vdots \\ &= (1+i)^n P \end{aligned}$$

This is our first formula.

Formula: Compound Interest

The value of an investment of P after n years at an annual interest rate of i is given by

$$P_n = P(1+i)^n \tag{1.1}$$

Note that this formula does not apply when the interest rate changes or where repayments are made.

Example

Calculate the compound interest on E2,500 for three years at 6% per annum.
Solution: We have a principal $P = 2,500$, invested at a rate of $i = 6\% = 0.06$ for $n = 3$ years:

Exercises:

1. Calculate the compound interest on each of the following investments.

   $(i)\ E2,500$ for 2 years at 8% per annum $(ii)\ E1,200$ for 8 years at 3% per annum

   $(iii)\ E350$ for 12 years at 10% per annum $(iv)\ E10,000$ for 15 years at 6% per annum

   $(v)\ E15,000$ for 18 years at 12% per annum

   Selected Answer: (iii) E1098.45

2. A man borrowed E7,500 at 8% per annum compound interest. He agreed to pay back the entire loan including interest after three years. How much did he pay back?

3. Post Office savings certificates pay 3% per annum compound interest if the money is invested for three years. How much would E20,000 amount to if invested with the Post Office for three years?

1.7 Mensuration

In this section we are interested in geometric problems of length, area and volume.

1.7.1 Area

A way to start thinking about area is in terms of tiling. We take a unit of length, form a square and call this one unit of area. Now we can find the area of a rectangle:

If we accept the axiom (assumption) that areas are *additive* and are preserved under translations then we can find the area of a triangle:

This means that, theoretically, we can find the area of *polygons*[7] by using triangulation:

One particular class of polygons are *quadrilaterals* — which are polygons with four sides. A parallelogram is an example of a quadrilateral:

[7] a figure formed by three or more points in the plane joined by line segments

We can see that its area is its base by its perpendicular height. We can find the area of a circle by using integration — basically forming an infinite triangulation of very small triangles:

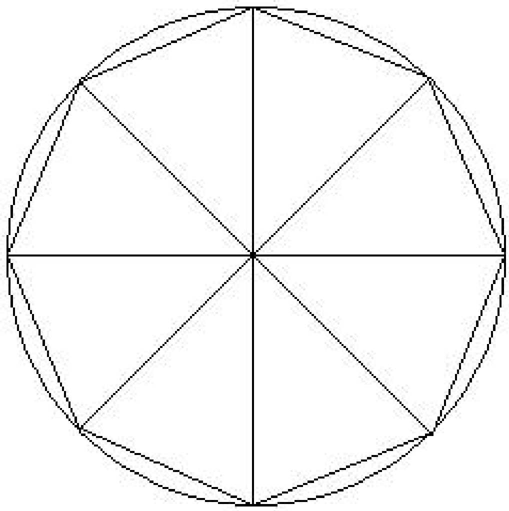

Figure 1.2: By following out this procedure we can show that the area of a circle is given by πr^2 — where r is the radius. We define π here by length/diameter: this gives the circumference as $2\pi r$.

Examples

1. Find the cross-sectional area of the girder (a) and the area of the perimeter path in (b).

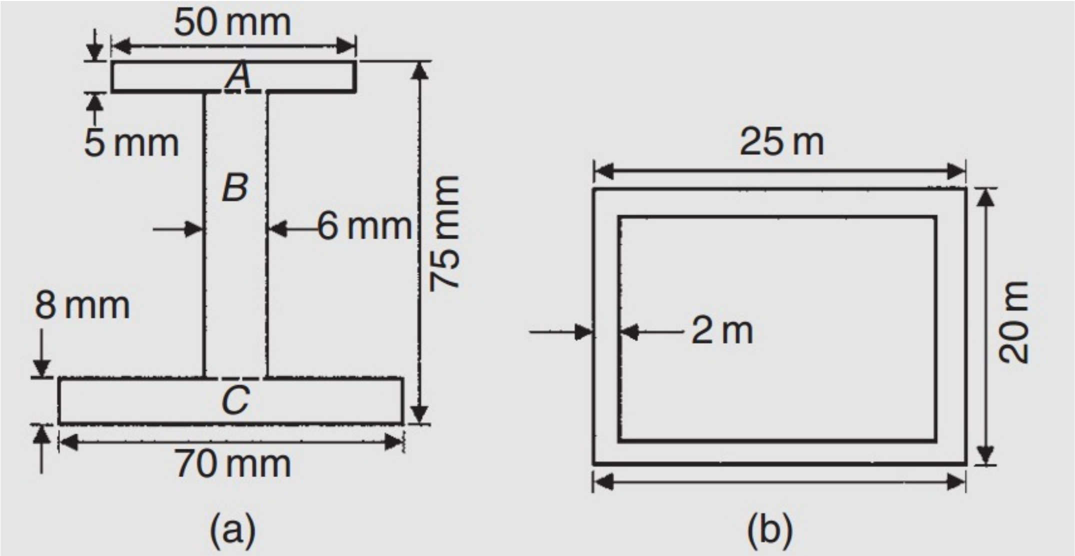

Figure 1.3: Diagram taken from Engineering Mathematics by Bird.

Solution:

(a) Using the additivity of area we can find the areas of the three rectangles A, B and C separately and add them up:

(b) We can use additivity of area to get the answer by looking at

2. This diagram shows the gable end of a building. Determine the area of brickwork in the gable end.

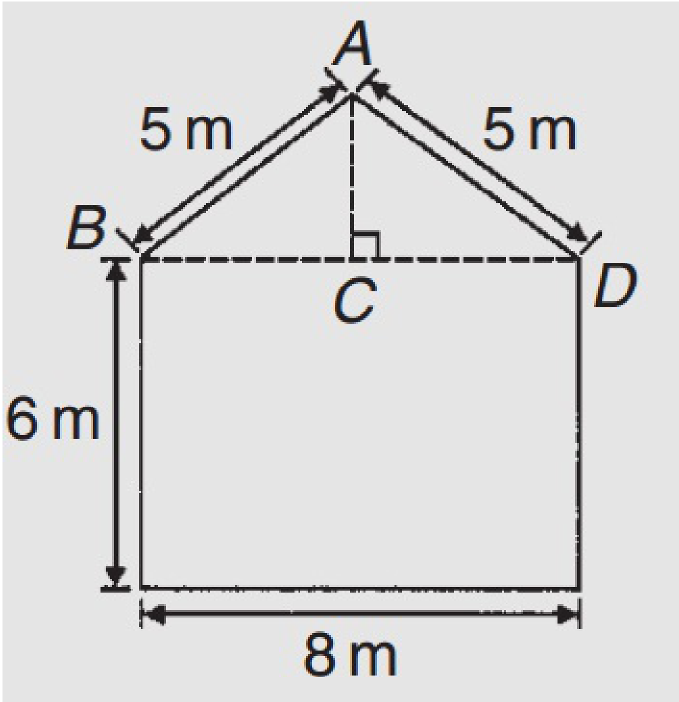

Figure 1.4: Diagram taken from Engineering Mathematics by Bird.

Exercises

1. A rectangular plate is 85 mm long and 42 mm wide. Find its area in square centimetres. **Answer** 35.7 **cm**^{2}

2. * A rectangular garden measures 40 m by 15 m. A 1 m flower border is made round the two shorter sides and one long side. A circular swimming pool of diameter 8 m is constructed in the middle of the garden. Find, correct to the nearest square metre, the area remaining. **Answer 482 m**^{2}

3. Determine the area of each of the angle iron sections shown in the diagram.

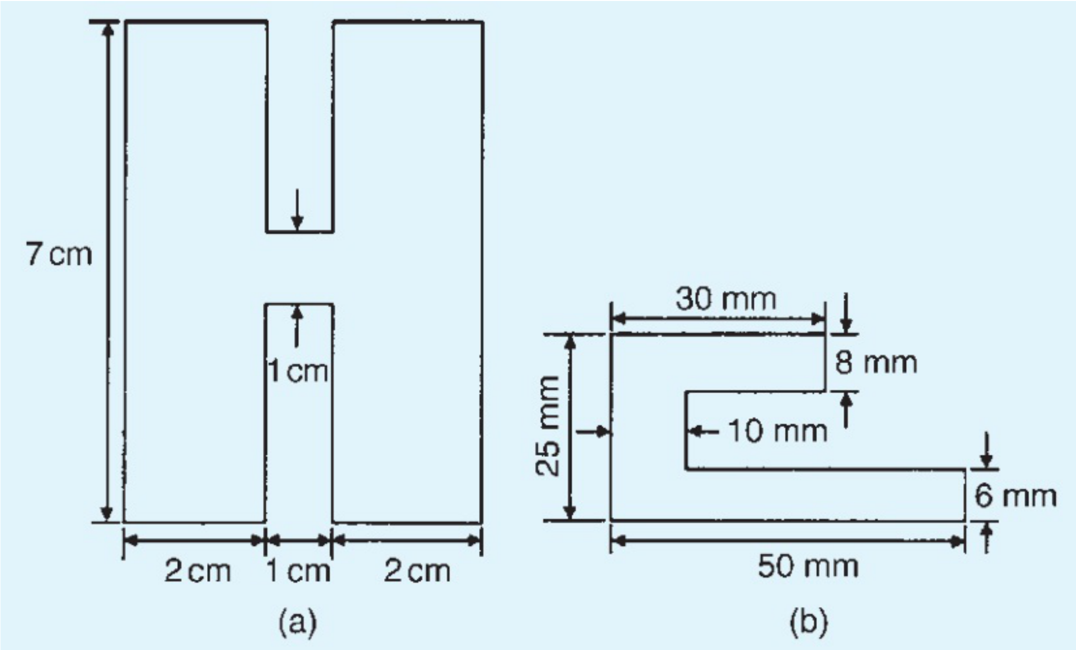

Figure 1.5: Diagram taken from Engineering Mathematics by Bird.

Answers (a) 29 cm^2 (b) 650 mm^2

Examples

1. Find the areas of the circles having
   (a) a radius of 5 cm
   (b) a diameter of 15 mm
   (c) a circumference of 70 mm

   Solution:

   (a) Just using the formula $A = \pi r^2$:

   (b) If the diameter is 15 mm then the radius is 7.5 mm:

   (c) The circumference is given by $2\pi r$. So $70 = 2\pi r$. If we divide by 2π we get the radius:

2. A metal pipe has an outside diameter of 5.45 cm and an inside diameter of 2.25 cm. Calculate the cross-sectional area of the shaft

   Solution: First a diagram:

   This shape is known as an *annulus* Now we use the additivity of area:

Exercises:

1. Determine the area of circles having a (a) radius of 4 cm (b) diameter of 30 mm (c) circumference of 200 mm. **(Answers: (a) 50.27 cm^2) (b) 706.9 mm^2 (c) 3183 mm^2**
2. An annulus has an outside diameter of 60 mm and an inside diameter of 20 mm. Determine its area. **(2513 mm^2)**
3. * If the area of a circle is 320 mm2^2 find (a) its diameter, and (b) its circumference. **((a) 20.19 mm (b) 63.41 mm)**
4. Determine the area of the shaded area **(5773 mm^2)**:

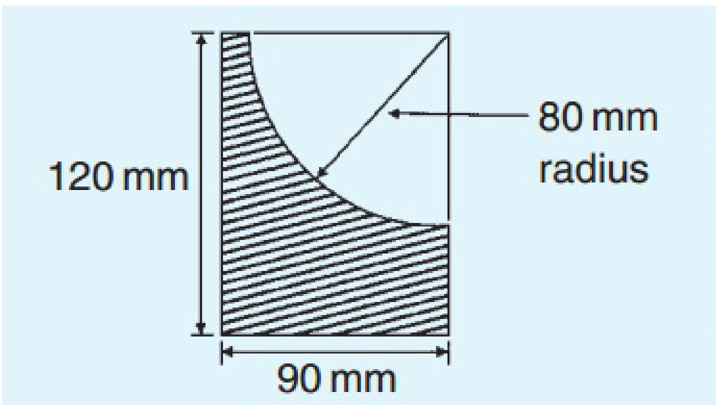

Figure 1.6: Diagram taken from Engineering Mathematics by Bird.

5. An archway consists of a rectangular opening topped by a semi-circular arch as shown in the Fig 1.7. Determine the area of the opening if the width is 1 m and the greatest height is 2 m. **(1.89 m^2)**

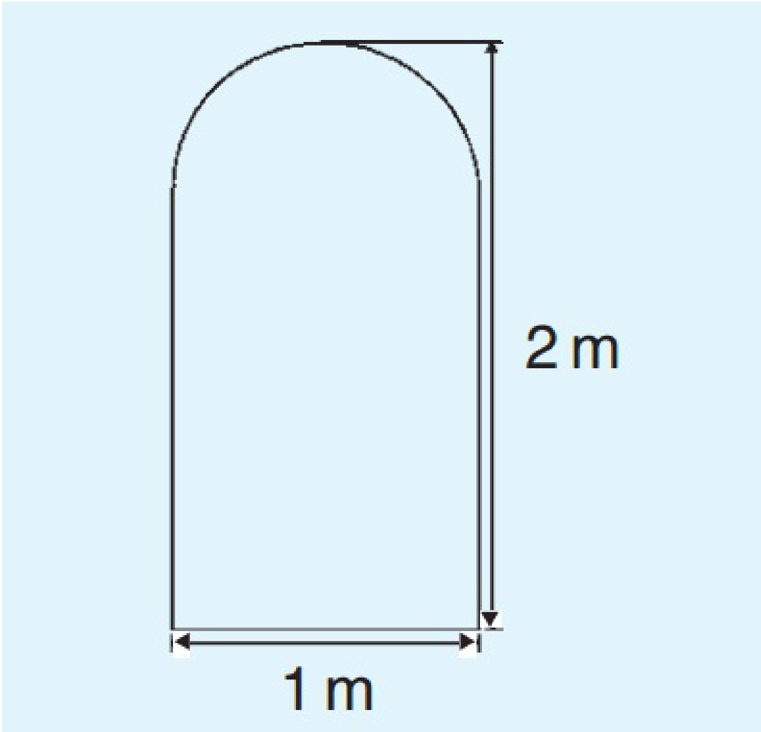

Figure 1.7: Diagram taken from Engineering Mathematics by Bird.

1.7.2 Trigonometric Ratios

Triangle Facts

1. We usually refer to a triangle with 'points' A, B and C. In this case the angle 'at' C is denoted by $\angle BCB$ and the side opposite C is called AB:

   Although we might also refer to a triangle with angles A, B and C; and, perhaps, opposite sides a, b and c:

2. The angles in a triangle add up to 180°.

3. If the triangle is right-angled we have Pythagoras Theorem:

4. A triangle with two equal sides will also have their opposite angles equal. The converse statement is also true: if two angles in a triangle have equal measure then the length of the sides opposite them will also be equal. Such a triangle is called *isosceles*:

Example

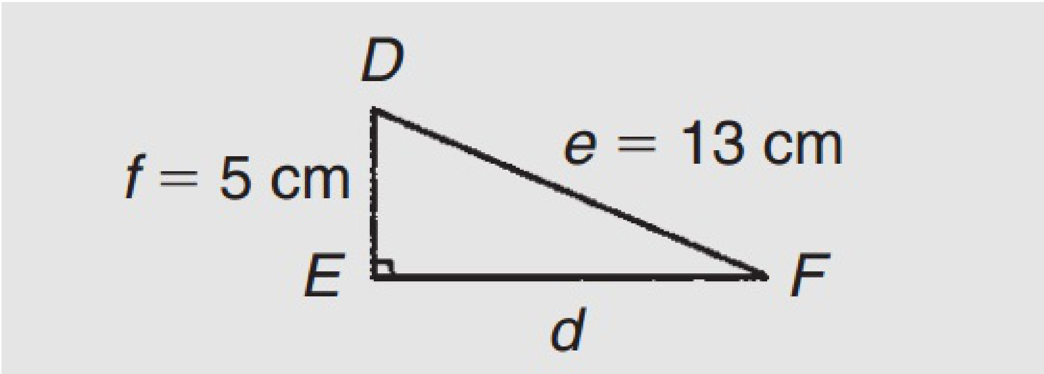

Figure 1.8: Find the length of EF (Diagram taken from Engineering Mathematics by Bird).

Solution: As we have a right-angled triangle (R.A.T.), Pythagoras Theorem applies:

Exercises:

1. * In a triangle CDE, $D = 90°$, $CD = 14.83$ mm and $CE = 28.31$ mm. Determine the length of DE. [24.11 mm]

2. Triangle PQR is isosceles, Q being a right angle. If the hypotenuse is 38.47 cm find (a) the lengths of sides PQ and QR, and (b) the value of $\angle QPR$. [(a) 27.20 cm each (b) 45°]

3. A man cycles 24 km due south and then 20 km due east. Another man, starting at the same time as the first man, cycles 32 km due east and then 7 km due south. Find the distance between the two men. [20.81 km]

4. * A ladder 3.5 m long is placed against a perpendicular wall with its foot 1.0 m from the wall. How far up the wall (to the nearest centimetre) does the ladder reach? If the foot of the ladder is now moved 30 cm further away from the wall, how far does the top of the ladder fall? [3.35 m, 10 cm]

5. Two ships leave a port at the same time. One travels due west at 18.4 km/h and the other due south at 27.6 km/h. Calculate how far apart the two ships are after 4 hours. [132.7 km]

In a right-angled triangle, special ratios exist between the angles and the lengths of the sides. We look at three of these ratios. Consider the right-angled triangle below:

We can define three trigonometric ratios for this triangle in terms of the side lengths a, b and c:

We can show that the trigonometric ratios are independent of the size of the triangle[8]

Example

If $\cos X = \frac{9}{41}$, determine the value of the other trigonometric ratios.

Solution: First we draw the situation in a *model triangle*[9]:

Now we use Pythagoras to calculate the length of the other side:

[8]briefly, *similar pairs triangles* — that is those which have the same angles — have sides that are in proportion. For example, if ΔABC is similar to ΔPQR then $|AB|/|BC| = |PQ|/|QR|$ if the angle opposite AB is equal to the angle opposite PQ, etc.

[9]the trigonometric ratios are equal for similar triangles so we can choose whatever size triangle we want to figure out the other ratios.

Now we have

Exercises

1. In triangle ABC shown below, find $\sin A$, $\cos A$, $\tan A$, $\sin B$, $\cos B$ and $\tan B$

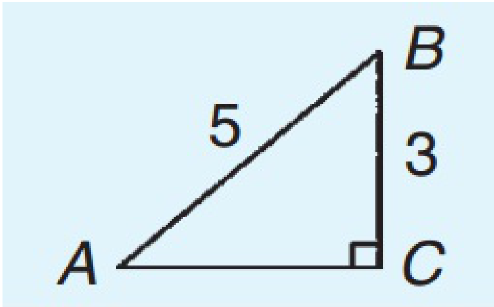

Figure 1.9: Diagram taken from Engineering Mathematics by Bird.

2. For the R.A.T. shown below please find $\sin \alpha$, $\cos \theta$ and $\tan \theta$:

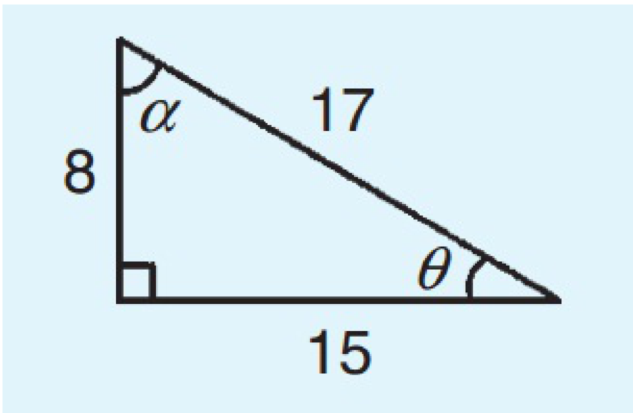

Figure 1.10: Diagram taken from Engineering Mathematics by Bird.

3. * If $\cos A = 12/13$ please find $\sin A$ and $\tan A$.

Three Special Triangles

There are three particular triangles that occur very often:

Exercises

1. Using a 30-60 triangle, calculate $\cos 60°$.
2. Show that $\tan 80 \neq 2\tan 40$ and that $\cos 50 \neq 2\cos 25$.
3. Find a θ in each case: (i) $\tan\theta = \sqrt{3}$ (ii) $\sin\theta = 1/\sqrt{2}$.

'Solving Right-Angled Triangles'

In the same way as giving two distinct points defines a line, there are a number of ways of 'defining' a triangle.

(SSS) The lengths of the three sides

(SAS) The lengths of two sides and the angle between them

(ASA) The length of one side and two angles

(RSS) In a R.A.T., any two sides.

This means that if we have enough information about a triangle, then there is only one triangle with those properties, and we can use both Pythagoras Theorem and the trigonometric ratios to find *all* the sides and angles. We will have to have enough information to define a triangle however. Note that AAA and SS is *not* enough to define a triangle:

Examples

1. Suppose Δ is a right-angled triangle with an angle $A = 32°$ and a hypotenuse of length 8 cm. How long is the side opposite $32°$?

   Solution: First draw a drawing of the situation:

   Now choose the trigonometric ratio that links the required side with the known angle and known side and write down this equation and solve.

2. In Δabc, $|\langle bca| = 90°$, $|\langle abc| = 34°$, and $|ac| = 20$ m. Calculate $|bc|$, correct to two decimal places.

3. Calculate the length of the hypotenuse of a triangle with an acute angle $31°$ with adjacent side of length 10.

   Solution:

Practical Applications

Many practical problems in navigation, surveying, engineering and geography involve solving a triangle. Mark on your triangle the angles and lengths you know, and label what you need to calculate, using the correct ratio to link the angle or length required with the known angle or length. Angles of elevation occur quite often in problems that can be solved with trigonometry.

Examples

1. A ladder, of length 5 m, rests against a vertical wall so that the base of the ladder is 1.5 m from the foot wall. Calculate the angle between the ladder and the ground, to the nearest degree.

   Solution:

2. When the angle of elevation of the sun is 28°, an upright flagpole casts a shadow of length 6 m. Calculate the height of the pole, correct to one decimal place.

   Solution:

Exercises

1. * In Δxyz, $|\langle xyz| = 90°$, $|xz| = 200$ cm, and $|yz| = 150$ cm. Write down the ratio $\sin\langle yxz$, in it's simplest form. Calculate, to the nearest degree $|\langle yxz|$ and $|\langle yzx|$.

2. Solve the following triangles:

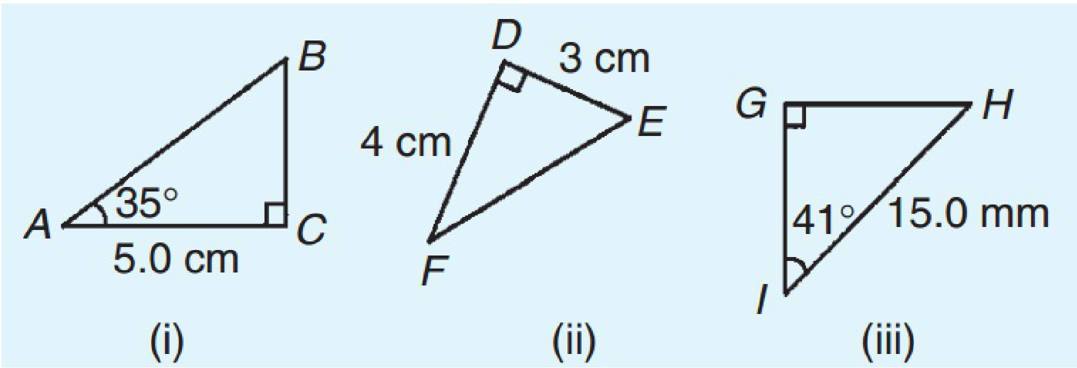

Figure 1.11: Diagram taken from Engineering Mathematics by Bird.

   (i) $BC = 3.50$ **cm,** $AB = 6.10$ **cm,** $\angle B = 55°$ **(ii)** $FE = 5$ **cm,** $\angle E = 53.13°$**,** $\angle F = 36.87°$ **(iii)** $GH = 9.841$ **mm,** $GI = 11.32$ **mm,** $\angle H = 49°$

3. * A ladder rests against the top of the perpendicular wall of a building and makes an angle of 73° with the ground. If the foot of the ladder is 2 m from the wall, calculate the height of the building.

4. * An electricity pylon stands on horizontal ground. At a point 80 m from the base of the pylon, the angle of elevation of the top of the pylon is 23°. Calculate the height of the pylon to the nearest metre.

5. * If the angle of elevation of the top of a vertical 30 m high aerial is 32°, how far is it to the aerial?

1.8 Approximation*

Being able to estimate an answer to a problem before making an exact calculation can be an important skill. One example where an estimate can help is in eliminating arithmetic slips. For example if you hope to find 13% of 801 and your answer is 10.413 then you have clearly made a mistake because

Another reason why we might make an approximation would be to make a forecast. For example, suppose that you spend 5 minutes every night writing a novel. Can you write a 1,000 page novel done in ten years in this fashion? Well, approximately:

It happens quite frequently in both pure and applied mathematics that there are problems which can't be solved exactly. An example from pure mathematics is *Abel's Impossibility Theorem* which states that

but we still might want to solve such an equation. We have numerical techniques to find approximate solutions.

A very simple example from 'applied mathematics' is the question

Theoretically we could answer this but...

In practise we will be approximating quantities by numbers and we will write

read *the quantity Q is approximated by x*, or *the quantity Q is approximately equal to x.*

Examples

1. We can approximate quantities such as

$$\frac{48.27 + 12.146}{14.82 - 3.02}.$$

by replacing the numbers by whole numbers:

We can compare this with the exact answer which turns out to be 5.12. Here $5 \approx 5.12$.

2. The average height of the front row $\bar{x}$ approximates the average height of the entire class μ.

3. $\pi \approx 3.14$ and $\pi \approx \dfrac{22}{7}$.

4. When an angle A is measured in *radians* we have

Now what are radians?

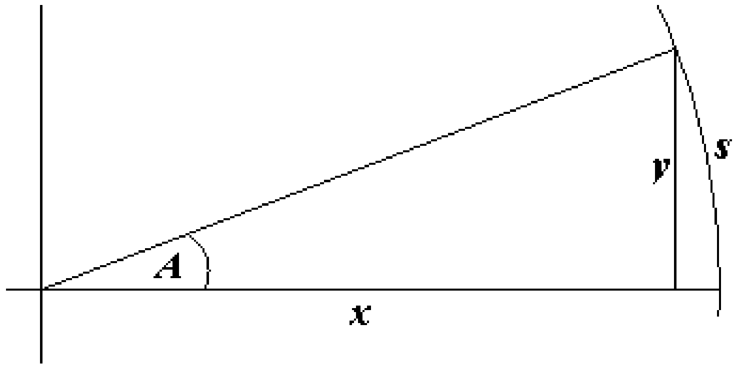

Figure 1.12: An angle A at the centre of the unit circle

A measured in radians is given by:

$\sin A$ is given by:

Now suppose A is small, that is $A \approx 0$;

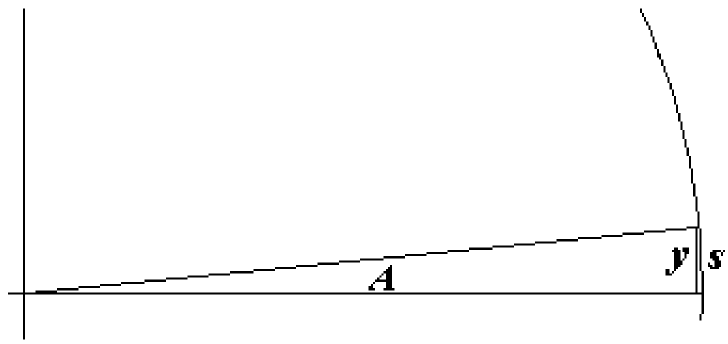

Figure 1.13: When $A \approx 0$, $s \approx y$ hence $A \approx \sin A$

5. Given a curve, what is the slope of the tangent at a point?

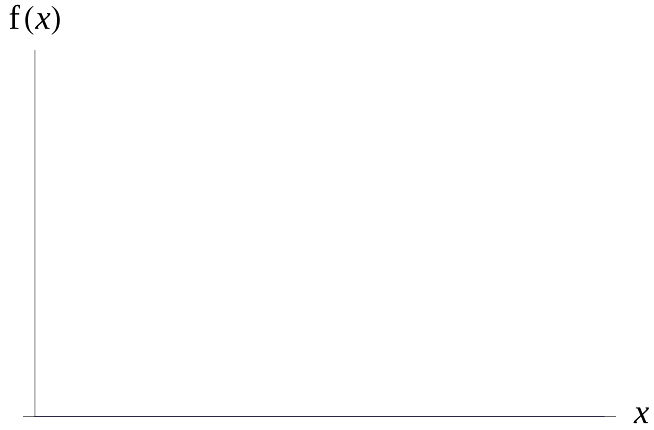

Figure 1.14: We approximate the slope of the tangent to the curve at the point $(3, f(3))$ by the slope of the line passing through $(3, f(3))$ and a point close to $(3, f(3))$.

6. Recall our approximation of the area of a circle by the area of a polygon.

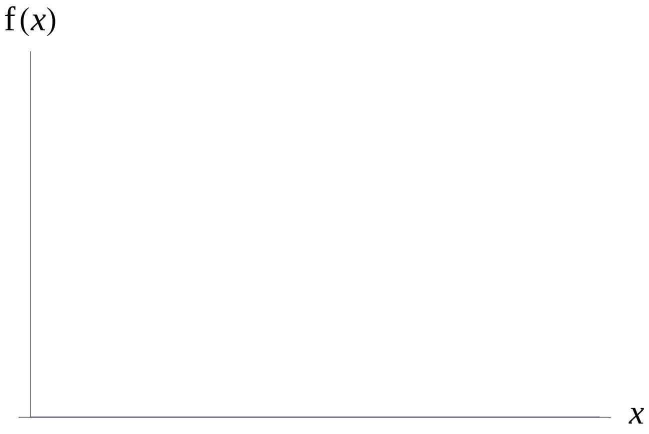

Figure 1.15: We approximate the area under the curve using rectangles.

1.8.1 Error

A good question is when we have made an approximation is

Usually a branch of mathematics called *numerical methods* comes up with methods of approximation and the branch of mathematics called *numerical methods* answers the above question. Remarkably in a lot of cases we can know in advance that an approximation can't be *too bad*. All of this

Examples

1. We can approximate the area below by using 'midpoint' rectangles.

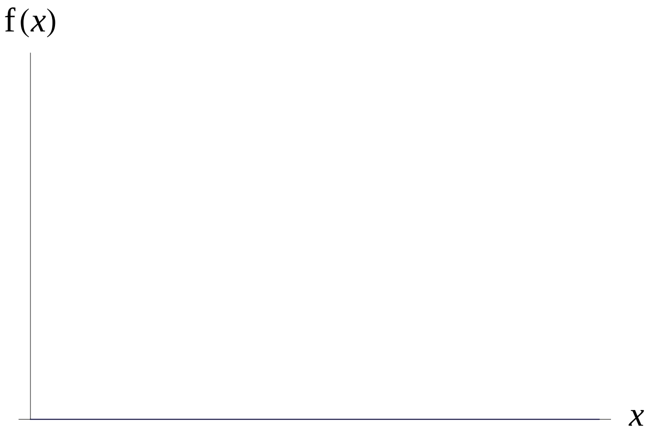

Figure 1.16: This approximation submits readily to numerical analysis.

We can show (and know in advance in fact) that if 29 rectangles are used, then the difference between the area and the approximation is at least less than 0.0001.

2. When doing mortgage payment calculations quantities such as

occur. By rounding off the $i/12$ term, say using the approximation

$$0.006 \approx 0.0058\dot{3} = \frac{0.07}{12}$$

the errors can 'propagate' in significant ways. For example

$$\left(1+\frac{0.07}{12}\right)^{360} \approx 8.1165$$

$$(1+0.006)^{360} \approx 8.61535$$

It doesn't sound too much but for the example of a thirty year mortgage for E300,000 this little but manifests as a difference of more than E40 a month, that is over E485 per year and more than E14,500 over the lifetime of the mortgage. All over a fourth decimal place!! Beware of this so called 'round-off error' and where possible use exact numbers.

For us here in Essential Maths we are just going to look at round-off error. To talk about this we need a number of definitions.

Absolute Error

Suppose that A is an approximation to a quantity Q. Then the *absolute error* of the approximation is given by

Example

Find the absolute error when we round-off to approximate $4.81 + 7.25$.

Solution: Our round-off gives the approximation $5 + 7 = 12$. The exact value of $4.81 + 7.25 = 12.06$ so the absolute error is

Suppose that you are approximating the heights of a statue and a cliff:

In both cases the absolute error is the same but clearly we feel that the statue approximation is worse. We believe this because our statue approximation overshot the true value by 50% and the cliff approximation was under by only 2%.

As another example, note that whenever you use a ruler that there is a possible error of 1 mm in your measurement as the ruler is only graduated to millimetres. Using a ruler to measure the width of a iPhone is clearly worse than using it to measure the dimension of a television. To capture this we define the relative error (which is usually converted to a percentage).

Relative Error

Suppose that A is an approximation to a quantity Q with absolute error ΔQ. Then the *relative error* of the approximation is given by

Example

Use rounding to approximate

$$\frac{2.9 \times 11.4 + 2.196}{21.018 \div 3.1}.$$

Also calculate the absolute and relative error.

Solution: First off we implement the rounding:

Now using a calculator we can see that the exact value is 5.2. Therefore the absolute error is 0.2. We find the relative error:

which is approximately (!) 3.846%.

Exercise: In each case, use rounding to calculate an approximate answer, find the absolute error and the relative percentage error.

$(i)\, 8.73 - 5.82$	$(ii)\, 3.8 \times 5.3$
$(iii)\, 48.36 \div 7.8$	$(iv)\, 4.15(11.1 - 4.3)$
$(v)\, 2.9 \times 4.1 + 3.04$	$(vi)\, \frac{15.332+8.94}{9.1-3.18}$
$(vii)\, \frac{3.95\times 8.42+3.953}{1.8\times 4.3+1.2}$	$(viii)\, \frac{30.317}{\sqrt{24.7009}}$

Selected Answers: (iii) 6, 0.2, 3.23% (vi) 4, 0.1, 2.44%

1.9 Statistics

Statistics is the science of collecting, studying, analysing and making judgements based on data. The subject divides broadly into two branches: *descriptive* and *inferential statistics*.

Descriptive statistics involves describing the main features of a collection of data. Descriptive statistics are distinguished from inferential statistics in that descriptive statistics aim to summarize a data set, rather than use the data to learn about the population that the data are thought to represent. Activities include graphing the data (putting a spin on things) and calculating key summary statistics such as the *average* or the *standard deviation*. The aim of descriptive statistics is to summarise the data. We could also include methods of collection of data.

Inferential statistics is the process of drawing conclusions from recorded data. Typically the data is not complete in that there are measurement errors or the data is just a sample from a much larger population. The outcome of statistical inference may be an answer to the question "what should be done next?", where this might be a decision about making further experiments or surveys, or about drawing a conclusion before implementing some organizational or governmental policy. Note that descriptive statistics precedes inferential statistics in the sense that data is necessary for inferential statistics and it is descriptive statistics that provides this. Therefore, for MATH6000, we are interested in Descriptive statistics.

There are many different types of data and it is useful to be aware of this. As a quick example, the heights of the MATH6000 class is numerical and hence ordered data. However, the hometowns of the MATH6000 class is not numerical and not ordered so that these are fundamentally different types of data that require different presentations and summary statistics in order to summarise them.

Nominal Data — is data that cannot be ordered, for example the eye colours of ten children or the martial status of groups of individuals as single, married, widowed or divorced.

Ordinal Data — is data that can be ordered, for example satisfaction levels in a consumer survey: *very happy, happy, indifferent, unhappy, very unhappy.*

Numerical Data — is exactly what you think it is. Numerical data has the advantage of having a natural ordering. More 'mathematical' methods of data presentation can be used to demonstrate numerical data. An example: the heights of the MATH6038 class.

Whole Number Data — again, exactly what you think it is. Numerical data where the only possibilities are whole numbers; for example, the number of employees in a business.

Continuous Data — data that can take on a infinite number of values that are arbitrarily close to each other; for example, the time taken to serve customers could be anything from zero to an infinite number of seconds.

1.9.1 Average

When you hear the word average we often think of the mean. For example, if Ann is 28, Betty is 31 and Carolone is 31 we say that their average age is 30. However this is not the only 'average', To be more careful, an average is any 'measure of central tendency' — or if you will the 'middle value'. We might want to talk about the average of nominal data: for example who does the average Irish male support in soccer?

Mean

The mean is the 'usual' average that we usually use. It is used for numerical data only. It is calculated by adding up the all the data points and dividing by the number of data points:

Let $x = \{x_1, \dots, x_n\}$ be a collection of data. The *mean-average* of x, $\bar{x}$, is given by

$$\begin{aligned}\bar{x} &= \frac{x_1 + \cdots + x_n}{n} \\ &= \frac{\sum_{i=1}^{n} x_i}{n}.\end{aligned}$$

Consider the expression $x_1 + x_2 + \cdots + x_n$. In statistics and in mathematics more generally we will often like to add up a number of numbers; quite often in fact. To handle this we use what is called *sigma*-notation. The symbol 'Σ' means add up. So for example

means add up all the ages in this class. Sometimes there will be a natural order on what we want to count up. For example the sum of the first five square numbers:

If the i-th thing that we want up can be given in terms of i, say a_i, then for the sum we can write something like

This means 'add up the pattern a_i, starting at $i = 1$ and up to $i = n$. For example the sum of the first five square numbers could be denoted by

In the case where we want to find the mean-average of the numbers $x = \{x_1, x_2, \dots, x_n\}$, we add them up and divide by the number of numbers — in this case n. Now we could write this as

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}, \qquad (1.2)$$

where $\displaystyle\sum_{i=1}^{n} x_i$ means add up all the x_i from x_1 to x_n.

Mode

The mode(s) of a data set is (are) those data points which occur most often and is often the best average to use for nominal data. What does 'a la mode' mean? As an example which county do you 'live in':

Median

The median is an average that is used for ordinal data. It is primarily used for a finite set of discrete data, it is that data point which divides the data into a 'bottom half' and 'lower half'.

Example

Find the median of the following data:

$$15, 4, 46, 23, 57, 3, 5, 34, 57, 243, 5.$$

Solution: First we order the data:

Hence 23 is the median.

When the number of data points is even, the median is given as the midpoint of the two 'middle' elements.

Example

Find the median of

$$2, 344, 23, 555, 643, 2, 542, 57$$

Solution:

Example

Determine the mean, median and mode for the set $\{2, 3, 7, 5, 5, 13, 1, 7, 4, 8, 3, 4, 3\}$.

Solution: The mean is found by adding the numbers together and dividing by the number of numbers:

To find the median first write the numbers in ascending order. The middle number is the median.

The mode is the element which occurs most often:

Which average should you use? By and large you have a choice but hard and fast rules are:

1. Use the mean unless there are large *outliers* in the data set.
2. If there are large outliers in the data set use the median.

When the data is symmetric about its mode all three will agree.

Example: Average Industrial Wage

The distribution of income looks like

A few millionaires will skew the mean to the right — so that more than half the population don't then make "the average industrial wage". So depending on what your vested interest is, you may use the mean or the median (or the mode). This is a demonstration of the famous quip "*There are only three kinds of lies. Lies, damned lies and statistics*". If you are disingenuous with statistics you can support a lot of marginal arguments.

Exercises: In Problems 1 to 4, determine the mean, median and modal values for the sets given.

1. $\{3, 8, 10, 7, 5, 14, 2, 9, 8\}$ **[mean 7.33, median 8, mode 8]**

2. $\{26, 31, 21, 29, 32, 26, 25, 28\}$ **[mean 27.25, median 27, mode 26]**

3. $\{4.72, 4.71, 4.74, 4.73, 4.72, 4.71, 4.73, 4.72\}$ **[mean 4.7225, median 4.72, mode 4.72]**

4. $\{73.8, 126.4, 40.7, 141.7, 28.5, 237.4, 157.9\}$ **[mean 115.2, median 126.4, no mode]**

1.9.2 Deviation

Note that data can be spread out or quite concentrated (around the average). For example, consider soccer players vs rugby players:

Range

The *range* is simply the difference between the largest and smallest elements in the data set. For example, Munster have Paul O'Connell (198 cm) and Peter Stringer (170 cm) — a range of at least 28 cm.

Interquartile Range

In the same way that the median divides the data into the bottom 50% and the top 50%, the interquartile range is the *middle* 50%. To find the interquartile range we first divide the data into four parts. For example: $\{2, 3, 4, 5, 5, 7, 9, 11, 13, 14, 17\}$ is divided into four by $4, 7$ and 13. We call 4 the first quartile, Q_1, 7 the second quartile AKA the median, Q_2 and 14 the third quartile, Q_3. The interquartile range is $Q_3 - Q_1$. Alternatively specify $Q_1 \to Q_3$. There are slightly different conventions for choosing the quartiles so we will just introduce the idea of the 'middle 50%' and leave it at that.

Standard Deviation

The *standard deviation* is a measure the of spread of the data. Standard Deviation is a kind of average deviation/distance from the mean. A small standard deviation means that the data is concentrated around the 'average' (soccer), while a large standard deviation means that the data is more spread out (rugby).

Let $S = \{x_1, \dots, x_n\}$ be numerical data with a mean-average of $\bar{x}$. Then the *standard deviation*, σ, is given by:

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n}(x-\bar{x})^2}{n}} \tag{1.3}$$

The Greek letter is pronounced *sigma*. Σ is the capital Greek S and σ is the small s. We haven't actually said what the square root of a number is even though a lot of people have a good idea. Similarly to how we defined $1/n$ as the solution of the equation $nx = 1$, we could define $\sqrt{n}$ as the *positive* solution to the equation $x^2 = n$ — that is the square root of n is the positive number that when squared equals n; e.g. $\sqrt{49} = 7$ as $7^2 = 49$.

In terms of actually calculating this object by hand let us dissect this formula

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n}}.$$

It looks like a bit of a mouthful but let's take our time.

1. If we know that the data is $\{x_1, x_2, \ldots, x_n\}$ we can at least calculate the mean-average, $\bar{x}$. It appears in the formula so we will need it.

2. Next we could calculate the *deviations*, $(x_i - \bar{x})$ from the mean

3. We are going to add up all the $(x_i - \bar{x})^2$ so we may as well square these deviations:

4. Now we need to add them up — $\sum(x_i - \bar{x})^2$:

   Note $\sum(x_i - \bar{x})^2$ does *not* mean add up all the deviations $(x_i - \bar{x})$ and *then* square them. If we wanted to do that we would have written $\left(\sum(x_i - \bar{x})\right)^2$.

5. Now divide by the number of elements in the data set to get the $\dfrac{\sum(x_i - \bar{x})^2}{n}$.

6. Finally take the square root of this using a calculator.

We don't have to follow these exact six steps but when we use the formula they should happen naturally.

Example

Determine the standard deviation from the mean of the set of numbers: $\{5, 6, 8, 4, 10, 3\}$, correct to 4 significant figures.

Solution: First we must calculate the mean-average

Now we calculate the deviations... and square them

Now we can use the formula:

Exercises: Questions 1-3 taken from Engineering Math by Bird.

1. Determine the standard deviation from the mean of the set of numbers: $\{35, 22, 25, 23, 28, 33, 30\}$ correct to 3 significant figures. [**4.60**].

2. The values of capacitances, in microfarads, of ten capacitors selected at random from a large batch of similar capacitors are: $\{34.3, 25.0, 30.4, 34.6, 29.6, 28.7, 33.4, 32.7, 29.0, 31.3\}$. Determine the standard deviation from the mean for these capacitors, correct to 3 significant figures. [**2.83 microfarads**].

3. The tensile strength in megapascals for 15 samples of tin were determined and found to be $\{34.61, 34.57, 34.40, 34.63, 34.63, 34.51, 34.49, 34.61, 34.52, 34.55, 34.58, 34.53, 34.44, 34.48, 34.40\}$. Calculate the mean and standard deviation from the mean for these 15 values, correct to 4 significant figures. **[mean 34.53 MPa, standard deviation 0.07474 MPa]**

Chapter 2

Indices and Logarithms

By relieving the brain of all unnecessary work, a good notation sets it free to concentrate on more advanced problems, and in effect increases the mental power of the race.

Alfred North Whitehead

2.1 Indices

Definition

Let $a \in \mathbb{R}$ and $n \in \mathbb{N}$. Then

$$a^n = \underbrace{a \times a \times \cdots \times a}_{n \text{ multiplicands}} \tag{2.1}$$

We can refer to a^n as "*a to the power of n*" and say a^n is "*a power of a*".
Finally a^2 is a *square* (*a-squared*), a^3 is a cube (*a-cubed*) and a^n is an *n-th power*.
For example, 3^7 is an example of a power of 3.

Basic Properties of Powers

Now that we have a shorthand way of writing repeated multiplication we must investigate how they combine together? What happens when we multiply together two powers? If we divide one power by another? What about a power of a power? What if we took two numbers, multiplied them together and took a power of that? We will see that answering these questions will raise more questions.
Before we start our investigation I want you to be aware of the following:

Definition

Let $a \in \mathbb{R}$ and $m, n \in \mathbb{N}$ (both m and n are natural numbers). Then a^m and a^n are *like powers*.
Now nobody apart from me in a second calls this the Golden Rule of Powers, but it'll become clear very soon why it is indeed the case:

(The Golden Rule of Powers)

Multiplication of Like Powers

Consider 7^2 and 7^3. What is really happening when we multiply them together?

There is nothing special about 7^2 and 7^3 here. Indeed any real number a and natural numbers m, n could replace the roles of 7, 2 and 3:

Proof.

Try to get any kind of similar rule for unlike powers and you'll see what I mean by the Golden Rule.

Example

Write $4 \times 4 \times 16$ as a power of 4.

Solution:

Division of Like Powers

Consider 10^5 and 10^2. What is really happening when we divide, say, 10^5 by 10^2:

Again there is nothing special about 10^5 and 10^2 here. Again let $a \in \mathbb{R}$ and $m, n \in \mathbb{N}$:

Proof.

Example

Write

$$\frac{10^2 \times 10^5}{100}$$

in the form 10^n for $n \in \mathbb{N}$.

Solution:

Repeated Powers?

What is $(9^3)^4$?

Again this is a general result. Let $a \in \mathbb{R}$ and $m, n \in \mathbb{N}$:

Proof.

Example

Write $(4^3)^5$ in the form 2^n for $n \in \mathbb{N}$.

Solution: First write 4 as a power of 2:

Now we can write $(4^3)^5$ as a power of two:

Powers of a Product

What about $(2 \times 5)^3$?

Again this is a general result. Let $a \in \mathbb{R}$ and $m, n \in \mathbb{N}$:

Proof.

Example

Let $x \in \mathbb{R}$. Write $4x^2$ as a square.

Solution:

Another similar case is that of a power of a fraction. Now our work from the first chapter should pay dividends. We know how to multiply fractions together so we have

Zero Powers, Negative Powers & Fractional Powers

You may have noticed that our definition for indices/ powers has only defined powers when the exponent is a natural number. What about the following:

(P1) 2^0?

(P2) 3^{-8}??

(P3) $4^{1/2}$???

(P4) $5^{\sqrt{2}}$?????

Note that every natural number is an integer and that every integer is a fraction (e.g. $-11 = -11/1$). What we will do is construct definitions for zero, negative numbers and fractions that *extend the definition for natural numbers* and *make the laws of indices consistent.*

In other words we will choose a definition for fractional powers[1] such that for all $a \in \mathbb{R}$, $m, n \in \mathbb{Q}$ the following make sense:

If it makes sense for fractions it will also make sense for integers and natural numbers as these are fractions also. Note at this point we will not define all real powers. To define, for example, $5^{\sqrt{2}}$, in such a way that the definition extends that of natural numbers and the laws of indices still hold, is a bit trickier. The first odd thing about this is that $\sqrt{2} \notin \mathbb{Q}$ - that is $\sqrt{2}$ *cannot be written as a ratio of whole numbers*. The two ways that I know how to define $5^{\sqrt{2}}$ properly involve quite a bit of calculus or quite a bit of sequences and series. We don't worry too much about this - we know it can be done - you can work away with any real exponent as long as you follow the laws of indices!

Zero Powers

We have already stated the 'fact' about all non-zero numbers that:

Now consider $2^5/2^5$, using the second law of indices:

Once again this is a result independent of 2 (but note $0^n/0^n = 0/0$ which doesn't make sense... we don't define 0^0 - it won't come up anywhere either). Let $a \in \mathbb{R}$, $a \neq 0$:

Example

Suppose that $a, b \in \mathbb{R}$ such that $a = b^3$. Simplify

$$\frac{a^2}{(b^2)^3}$$

Solution: First we replace $a = b^3$:

Negative Powers

What is 3^{-8}? Using the first law of indices and the definition of 3^0:

[1]will be for $a > 0$ only

We know the story by now. Let $a \in \mathbb{R}$, $a \neq 0$. Let $n \in \mathbb{N}$:

Example

Evaluate $10^{-1} \times 100$. What does this tell you about x^{-1}?

Solution:

This tells us a few things. Firstly if you want to divide by x you can multiply by x^{-1}. We already saw that if you want to divide by x multiply by $\frac{1}{x}$... but then of course $x^{-1} = \frac{1}{x^1} = \frac{1}{x}$.

There is actually another way of explaining why zero powers are one that might also sheds light on how we do negative powers. We start with the fact that multiplying by one doesn't change anything. So given any number you want, say 8 we can write

In fact we can think of *always starting at one*. Now we can make products or powers of eight as follows:

So to increase powers you multiply by 8. Now coming back in the other direction from 8^5.

To go decrease powers you divide by 8. Hence if we want to go from $8^1 \rightarrow 8^0$ we divide by eight a final time:

If we go back to the idea that we should *always starting at one* we see that 8^0 starts at 1 and... well we don't do anything there are no eights to multiply it and we are stuck at one! This is called an *empty product*. Utterly bizarre examples include

> *Let S be the set of all students who are at least ten foot tall. Now take the ages of all these students and multiply them together. The answer is one.*

Such is the mystery of the so-called *empty-set*!

If we continue this policy of dividing reduces powers we can see why the negative powers make sense:

Fractional Powers

What is $4^{1/2}$? Now using the third law of indices:

So we see that $4^{1/2}$ is the number that when squared gives 4... Now to keep things easy for us, we will only consider fractional powers of *positive real numbers* (i.e. numbers bigger than zero. We write $a > 0$ for "*a is bigger than zero*"). Try and figure out what $(-1)^{1/2}$ if you want to see what I mean. Hence let $a \in \mathbb{R}$, $a > 0$:

What about $5^{1/6}$? Similar story, using the third law of indices:

So we see that $5^{1/6}$ is the number when brought to the power of 6 gives you 5. This is known as the *sixth root* of 5 which we denote by $\sqrt[6]{5}$, and extends naturally from "*the*" square root (e.g. Let $x \in \mathbb{R}$, $x > 0$. Then $\sqrt{x} = \sqrt[2]{x}$). Again, let $a \in \mathbb{R}$, $a > 0$ and $n \in \mathbb{N}$:

What about $6^{3/5}$? If we agree that whatever it is, it has to agree with, say the third law of indices, so that:

A subtlety of this is that we have $3 \times (1/5) = (1/5) \times 3$:

Finally, let $a \in \mathbb{R}$, $a > 0$ and $m/n \in \mathbb{Q}$:

Example

Putting our skills together!! Evaluate

$$\sqrt{\left(\frac{9^2 9^3}{3^4}\right)^5 3^{-6}}$$

Solution:

Exercises

1. Evaluate:

$$(i)\, 36^{1/2} \quad (ii)\, 125^{1/3} \quad (iii)\, 16^{1/4} \quad (iv)\, 1000^{1/3} \quad (v)\, 1000^{2/3}$$
$$(vi)\, 2^{-5} \quad (vii)\, 5^{-2} \quad (viii)\, 8^{2/3} \quad (ix)\, 4^{-1/2} \quad (x)\, 4^{-1}$$

2. Write these as a/b, where $a, b \in \mathbb{Z}$:

$$(i)\, 2^{-2} \;\; (ii)\, (1/4)^{1/2} \;\; (iii)\, 32^{-3/5} \;\; (iv)\, 16^{-1/4} \;\; (v)\, 27^{-2/3}$$

3. Which of each pair is greater?:

$$(i)\, 2^5 \text{ or } 5^2 \qquad (ii)\, 4^{1/2} \text{ or } (1/2)^4 \qquad (iii)\, 2^{-1/2} \text{ or } (-1/2)^2$$
$$(iv)\, (1/2)^6 \text{ or } (1/2)^7 \qquad (v)\, 7^2 \times 7^3 \text{ or } (7^2)^3$$

4. Solve for k:

$$(i)\, 2^k = 4 \qquad (ii)\, 4^k = 64$$
$$(iii)\, 8^k = 64 \qquad (iv)\, 2^k = 128$$
$$(v)\, 4^k = 2 \qquad (vi)\, 25^k = 5$$
$$(vii)\, 8^k = 4 \qquad (viii)\, 1000^k = 100$$
$$(ix)\, 32^k = 16 \qquad (x)\, 8^k = 1/2$$

5. Write each of these in the form a^p, where $p \in \mathbb{Q}$:

$$(i)\, a^7 \div a^2 \quad (ii)\, a^7 \times a^2$$
$$(iii)\, (a^7)^2 \quad (iv)\, \sqrt{a}$$
$$(v)\, \sqrt[3]{a} \quad (vi)\, \sqrt{a^7}$$
$$(vii)\, 1/a^3 \quad (viii)\, 1/\sqrt{a}$$
$$(ix)\, (\sqrt{a})^3 \quad (x)\, 1/(a\sqrt{a})$$

6. Find the value(s) of k in each case.

$$(i)\, 2^{3k}.2^k = 16 \quad (ii)\, 2^{2k+1} = 8^2 \quad (iii) 16^4.8 = 2^k$$
$$(iv)\, 5^{2k+1} = 125 \quad (v)\, 3^{k+1} = 9^{k-1} \quad (vi) 10^k.10^{2k} = 1,000,000$$
$$(vii)\, 2^{(k^2)}.2^k = 64 \quad (viii)\, 10^{(k^2)} \div 10^{2k} = 1,000$$
$$(ix) 5^k = 1/125 \quad (x)\, 2^k = 1/(4\sqrt{2})$$

7. Write these as 10^p:

$$(i)\, (10^9)^2 \quad (ii)\, 10^9 \times 10^2 \quad (iii)\, (10^5.10^3)^2 \quad (iv)\, (\sqrt{10})^{100}$$

8. State if these are true or false:

$$(i)\, 2^3.5^3 = 10^3 \quad (ii)\, (x^6)^7 = x^{13} \quad (iii)\, (3\sqrt{3})^3 = 3^{4.5}$$
$$(iv)\, (10\sqrt{10})^4 = 10^6 \quad (v)\, 2^7 \times 3^7 = 5^7 \quad (vi)\, (-3)^4 = -3^4$$

9. Write $\sqrt[3]{\sqrt{2}}$ as 2^p for $p \in \mathbb{Q}$.

10. If

$$\frac{((2\sqrt{2})^2)^3}{(2\sqrt{2})^2(2\sqrt{2})^3} = 2^n,$$

find the value of n.

2.1.1 Indices: Harder Examples

Here we present some more problems on indices to get us ready for the more difficult problems on the exercise sheet and test 3. In reality this will be the first time that we do algebra. These are *not* letters. We don't specify whether they are real numbers, fractions, integers or just natural numbers. To avoid problems with division by zero and finding roots of negative numbers we will declare at this point that

Algebraic Notation

We often use x and y to stand for some real numbers ($x,\, y \in \mathbb{R}$). We are inclined to multiply real numbers together. How do we represent x multiplied by y:

As you can imagine this can lead to all kinds of confusion. Instead we use xy to signify 'x multiplied by y'. This is exactly why I don't like so-called 'mixed fractions'.

1. Simplify $a^3b^2c \times ab^3$.

   Solution: The first thing to remark here is that we have a number of numbers multiplied together:

   We know that the order of multiplication of numbers is not important ($6\times 5 = 5\times 6$), therefore we may write:

   How many as, bs, cs?

2. Simplify
$$a^{1/3}b^2c^{-2} \times a^{1/2}b^{1/2}c^3.$$

   Solution: Once again we must recognise this as

   Now we have seen that two multiply like powers we add the powers. For the $c^{-2}c^3$ we can either think in terms of adding the powers or else thinking $c^{-2} = \dfrac{1}{c^2}$:

3. Simplify

$$\frac{p^3q^2r^5}{pqr^{-1}}.$$

Also evaluate for $p = 3$, $q = \frac{1}{8}$ and $r = 2$.

Solution: Now recalling how we defined fraction multiplication:

Handling the ps and qs isn't too hard. However the $\frac{r^5}{r^{-1}}$ isn't very natural so we go back to thinking that to divide powers you subtract:

Now, as the question requests, evaluate at $p = 3$, $q = \frac{1}{8}$ and $r = 2$:

4. Simplify

$$\frac{a^{1/2}b^3c^{2/3}}{a^{1/4}b^{3/2}c^{1/6}}.$$

Also evaluate when $a = 16$, $b = 9$ and $c = 4$.

Solution: Again recalling fraction multiplication:

Now none of these are natural so we go back the abstraction $\frac{x^m}{x^n} = x^{m-n}$:

Now we know that $x^{1/2} = \sqrt{x}$ but what about $b^{3/2}$:

5. Simplify

$$(a^6)^{1/2}(b^3)^3.$$

Solution: Now we know that $(b^3)^3$ is three copies of three copies of bs multiplied together so equal to b^9. We will have to use the abstraction of this $(x^m)^n = x^{mn}$ to calculate $(a^6)^{1/2} = a^{6\times\frac{1}{2}} = a^3$. So our answer is a^3b^9.

6. Simplify

$$\frac{(at^2)^3}{(at^{1/4})^4}.$$

Solution: For this we should look at the top and bottom separately first like we do whenever we have a fraction of complicated numbers. The $(at^2)^3$ is

Similarly the $(at^{1/4})^4$ is

Now $t^{1/4} = \sqrt[4]{t}$ so by definition $(t^{1/4})^4 = t$. Putting this together we have

7. Simplify

$$\frac{x^2y^2z^{1/2}}{(x^{3/2}yz^{5/2})^2},$$

but express in positive powers only.

Solution: Firstly we deal with the denominator:

Now separate using the multiplication of fractions:

However the question asks for positive powers only

8. Simplify

$$\frac{(a^3b^{1/3})(\sqrt{a}\sqrt[3]{b^2})}{(a^5b^3)^{1/2}}.$$

Solution: Again simplifying and below separately:

Now putting the as with the as and the bs with the bs:

Now looking at the bottom, and using the abstraction $(ab)^n = a^nb^n$:

Now putting everything back together:

Of course there is nothing stopping us simplifying the top and the bottom at the same time.

Exercises: These exercises are taken from page 41 of Engineering Mathematics (Fifth Ed.) by John Bird.

1. Simplify $(x^2y^3z)(x^3yz^2)$ and evaluate when $x = \frac{1}{2}$, $y = 2$ and $z = 3$.

   Answer: $x^5y^4z^3$ **and** $\frac{27}{2} = 13.5 = 13\frac{1}{2}$**.**

2. Simplify $(a^{1.5}b^1c^{-3})(\sqrt{a}bc^{-1/2}c)$ and evaluate when $a = 3$, $b = 4$ and $c = 2$.

   Answer: $(a^2b^{1/2}c^{-2})$ **and** $\frac{9}{2} = 4.5 = 4\frac{1}{2}$

3. Simplify
$$\frac{a^5bc^3}{a^2b^3c^2},$$
   and evaluate when $a = 1.5$, $b = 0.5$ and $c = \frac{2}{3}$.

   Answer: $a^3b^{-2}c$ **and 9**

4. Simplify
$$\frac{x^{1/5}y^{1/2}z^{1/3}}{x^{-1/2}y^{1/3}z^{-1/6}}.$$

   Answer: $x^{7/10}y^{1/6}z^{1/2}$

5. Simplify
$$(a^2)^{1/2}(b^2)^3(c^{1/2})^3.$$

   Answer: $ab^6c^{3/2}$

6. Simplify
$$\frac{(abc)^2}{(a^2b^{-1}c^{-3})^3}.$$

   Answer: $a^{-4}b^5c^{11}$

7. Simplify
$$(\sqrt{x}\sqrt{y^3}\sqrt[3]{z^2})(\sqrt{x}\sqrt{y^3}\sqrt{z^3}).$$

   Answer: $xy^3\sqrt[6]{z^{13}}$

8. Simplify

$$\frac{(a^3b^{1/2}c^{-1/2})(ab)^{1/3}}{(\sqrt{a^3}\sqrt{b}c)}.$$

Answer: $a^{11/6}b^{1/3}c^{-3/2} = \dfrac{\sqrt[6]{a^{11}}\,\sqrt[3]{b}}{\sqrt{c^3}}$

2.2 Scientific Notation

2.2.1 Powers of Ten

Consider the number 3,487. What *is* this?

Three thousand, four hundred, and eighty seven

How about 0.123?

Any non-zero number at all may be written in what's called *scientific notation.* This notation handles extremely large and small numbers and negates the need for too many decimal points. For science and engineering at least, scientific notation greatly eases calculations and data can be presented in a much nicer manner. In scientific notation, real numbers are written in the form:

with $a \in [1, 10)$ and $n \in \mathbb{Z}$. Writing numbers in scientific notation is also called writing them in *standard form.*

Examples

1. $300 = 3 \times 10^2$.
2. $4,000 = 4 \times 10^4$.
3. $5,720,000,000 = 5.72 \times 10^9$.
4. $0.0000000061 = 61 \times 10^{-9}$.

Scientific notation allows us to get a rough order on the magnitude of very small and very large numbers. How do we write a decimal x in scientific notation? We'll break it up into three cases.

Numbers Between 1 and 10

Write 4.345 in scientific notation.

Abstraction: If $1 \leq x < 10$:

Numbers Bigger than 10

What about 3187.2?

$$3,187.2 = 3.1872 \times 10^3$$

Why is the case? Well to go from $3,187.2$ we can go in three steps:

Now to go from 3,187.2 to 318.72 we divide by ten. To go to 31.872 we divide by ten again and finally we divide by ten a final time. To balance the books as it were we must account for these three divisions by ten. If we multiply now multiply by ten three times: 10^3 we haven't changed the original number and we are happy to write $3,187.2 = 3.1872 \times 10^3$.

Examples

1. 4342.43

2. 35.76

3. 1423344.243

Numbers Less than 1

What about 0.022?

$$0.022 \to 0.022 \times 10^2$$

Moving the decimal spot one to the right is equivalent to multiplying by ten — for every spot you move to the right you must multiply by ten — and if you multiply by 10 n times then you must divide by 10 n times: 10^{-n}.

Examples

1. 0.00243

2. 0.0076

3. 0.000000243

Examples

1. An electron's mass is about 0.00000000000000000000000000000091093822 kg. In scientific notation, this is

2. The Earth's mass is about 5973600000000000000000000 kg. In scientific notation, this is

3. The Earth's circumference is approximately 40000000 m. In scientific notation, this is

Examples

1. Express the following numbers, which are in standard form, as decimal numbers:

$$(i)\, 2.67 \times 10^{-2}\; (ii)\, 5.487 \times 10^5\; (iii)\, 3.44 \times 10^0.$$

   Solution: Firstly

   Also

2. Express in standard form, correct to three significant figures:

$$(i)\,\frac{3}{8}\quad (ii)\,19\frac{2}{3}\quad (iii)\,741\frac{9}{16}.$$

Solution: Use the calculator to find these as decimals:

$$(i)\,0.375\quad (ii)\,19.\dot{6}\quad (iii)\,741.5625.$$

2.2.2 Orders of Magnitude

Scientific notation also enables simpler order-of-magnitude comparisons. A proton's mass is 0.0000000000000000
kg. If this is written as

it is easier to compare this mass with that of the electron, given above. The order of magnitude of the ratio of the masses can be obtained by comparing the exponents instead of the more error-prone task of counting the leading zeros. In this case, −27 is larger than −31 and therefore the proton is roughly four orders of magnitude (about 10000 times) more massive than the electron.

Scientific notation also avoids misunderstandings due to regional differences in certain quantifiers, such as billion, which might indicate either 10^9 or 10^{12} depending on where in the world you are.

2.2.3 Manipulation of Scientific Notation

Multiplication

What about $(9.2 \times 10^8) \times (4.3 \times 10^{10})$:

Hence

$$(a \times 10^n) \times (b \times 10^m) = ab \times 10^{mn} \tag{2.2}$$

Division

What about

$$\frac{1.256 \times 10^{-13}}{8.976 \times 10^{-14}}$$

Hence

$$\frac{a \times 10^n}{b \times 10^m} = \frac{a}{b} \times 10^{n-m} \tag{2.3}$$

Powers

What about $(4.55 \times 10^{-13})^5$?

Hence

$$(a \times 10^n)^m = a^m 10^{nm} \tag{2.4}$$

Example

Write the following in scientific notation:

$$(i)\ (3.75 \times 10^4)(5.6 \times 10^5) \quad (ii)\ \frac{3.5 \times 10^6}{7 \times 3}.$$

Solution: For the first one we have

And the second

Exercises:

1. By converting each element to scientific notation, calculate each of the following and write your answer in the form $a \times 10^n$, where $1 \le a < 10$ and $n \in \mathbb{Z}$:

   $(i)\ 400,000 \times 2,000$ $\quad (ii)\ 6,000 \times 1,400$

   $(iii)\ 25,000 \times 0.0018$ $\quad (iv)\ 4,500 \times 1.5 \times 10^{-4}$

   $(v)\ 6,000,100/(3 \times 10^5)$ $\quad (vi)\ 8,888 \times 10^{-4}/0.000432$

   $(vii)\ (10,000)^4$ $\quad (viii)\ (1.8 \times 10^{-6})^{-2}$

   $(ix)\ (2.4 \times 10^{-31}) \times (4.123 \times 10^3)/(10^{-18})^3$

2. Multiply 3,700 by 0.2 and express your answer in the form $a \times 10^n$, where $1 \le a < 10$ and $n \in \mathbb{Z}$.

3. Write 2.8×10^3 as a natural number.

4. Express $(10^5) \times (1.8 \times 10^{-4})^{-4}$ in scientific notation.

2.3 Conversion of Units*

2.3.1 SI Units

The central concept of a physics theory is a physical quantity. A physical quantity is any property of matter, time and space that can be measured. Everyday physical quantities include:

time	length	area
volume	distance	speed
acceleration	mass	temperature

When a physical quantity is measured, it is compared with a standard amount, or *unit*, of the same quantity. For example, to say that a length of wire is twelve metres means that the piece of wire is twelve times longer than the metre. The metre is the unit of length. The result of a measurement is always a multiple of a unit. To say that a length is twelve is meaningless.

There are many different units of length: there is the centimetre, the inch, the foot, the yard, the metre, the furlong, the kilometre, the mile, the parsec, etc. In 1960 physicists agreed to use a particular system of units; the *Systéme Internationale*, or the SI.

To relate physical quantities easily, the SI defines *seven* basic quantities and associated units. To ease presentation of laws, for example, a symbol or letter is assigned to stand for physical quantities. Furthermore, the associated units have associated letter symbols. For example, the physical quantity is length, denoted by l (or s often); and the associated SI unit is the metre, denoted by m. Hence the physical quantity twelve metres may be denoted $l = 12$ m. The five basic quantities, their symbol, associated unit and unit symbol are as follows:

Basic Quantity	**Symbol**	**SI Unit**	**Symbol**
length			
time			
mass			
electric current			
temperature			

The unit of every other quantity is called a *derived unit* because it can be expressed as the product or quotient of one or more of the basic units. For example, density is kilogram per cubic metre. The symbol for density is ρ (the Greek letter *rho*). The unit is the kilogram per cubic metre: kg m^{-3}. A few examples of derived units are:

Physical Quantity	**Symbol**	**SI Unit**	**Symbol**
area			
volume			
speed			

Sometimes a derived unit can become quite complex. For example the unit of energy, E, in basic units is kg m^2 s^{-2}. This is given the special name to simplify matters, namely the *joule*, J.

Here are a number of common quantities with derived units with their own name:

Physical Quantity	**Symbol**	**SI Unit**	**Derived Unit**	**Symbol**
force				
pressure	P	kg m^{-1} s^{-2}		
power	P	kg m^2 s^{-3}		W
frequency				
magnetic flux density	B		tesla	T
magnetic flux	Φ		weber	Wb
activity of a radioactive source	A		becquerel	Bq

2.3.2 Conventions

1. When writing a unit in terms of basic units, a space is left between the symbol of each basic unit. For example, the unit of density is abbreviated kg m^{-3} not kgm^{-3}.

2. Sometimes the standard SI units are too large or too small to be used easily. Therefore multiples of the standard units are often used. The most common multiples used are:

Multiple	**Prefix**	**Symbol**
10^9	giga-	G
10^6		
10^3		
10^{-2}		
10^{-3}		
10^{-6}		μ
10^{-9}	nano-	n
10^{-12}	pico-	p

3. The name of the unit has the prefix written before it; e.g. 10 kilometres, 5 millinewtons.

4. There is no space between the prefix and the symbol for the unit: e.g. 60 millimetres is written as 60 mm; 20 kilowatts is written as 20 kW.

5. The kilogram *is* the SI unit of mass - not the gram.

2.3.3 Converting Units

What is 100 kilometres an hour in metres per second? The way to do this is to not worry about the '100' but just change the kilometres and the hour.

So now:
To convert from one unit to another, substitute each unit on the left-hand-side into terms of another on the right-hand side.

Example

Express 100 kilometres per hour in furlongs per fortnight!

Solution: First of all what is a furlong?

Now what is a fortnight?

Now we can replace ‘kilometres’ and ‘hours’:

Exercises

1. Show that the SI unit of area is the square metre.
2. If a body of mass m kilograms is moving with a velocity of v metres per second its momentum P is defined by the equation $P = mv$. Find the SI unit of momentum in terms of basic units.
3. When the velocity of a body changes its average acceleration a is given by:
$$a = \frac{\text{change in velocity}}{\text{time taken for change}} \tag{2.5}$$
Find the SI unit of acceleration.
4. What is the SI unit of density?
5. Pressure P is defined as force per unit area, i.e. $P = F/A$. If the SI unit of force is the newton (N), find the SI unit of pressure in terms of the newton and the metre.
6. How many square centimetres in one square metre?
How many cubic centimetres in one cubic metre?
How many grams in a kilgram?
7. Convert each of the following to standard SI units:
(i) 5 cm^2 (ii) 40 cm^2 (iii) 1 cm^3
(iv) 456 cm^3 (v) 1,000,000,000 cm^3
8. Convert each of the following to standard SI units. Please write your final answer in scientific notation:
(i) 105 km (ii) 57 mm (iii) 6.67×10^{-11} cm
(iv) 6×10^{27} grams (v) 9 grams per cubic centimetre (vi) 100 km h^{-1}
(vii) 5 nN $(viii)$ 10 μW (ix) 5 Gm

9. The unit of acceleration a is the metre per second squared (m s^{-2}). Force is equal to mass by acceleration, i.e. $F = ma$. The unit of force is the newton. Express the newton in terms of basic units.

10. By definition work W is equal to force F multiplied by distance s travelled, i.e. $W = Fs$. The unit of work is the joule (J). Express the Joule in terms of the newton.

11. Use the results of the previous two questions to write the unit of work, the joule, in terms of basic units.

12. By definition power P is equal to work done divided by the time taken, i.e. $P = W/t$. The unit of power is called the watt (W). Express the watt in terms of the joule and the second. Then express the watt in basic units.

2.3.4 Angle Measure

We have seen this already. We can show via the definition of radians that

$$\pi \text{ rad} = 180^\circ.$$

Examples

1. Convert $46^\circ\,26'$ to radians.

   Solution: First of all $46^\circ\,26'$ is in mixed units, like hours and minutes. Either using the calculator directly or by writing

   write in terms of degrees only. Now we can replace the $^\circ$ by $\pi/180$ rad:

2. Convert 2 radians to degrees.

   Solution: Simply replace 'radian' by its degree equivalent:

Note that angles measured in radians are dimensionless:

Also sines and cosines are dimensionless also:

2.3.5 Currency Conversion

This method can also be used when doing currency conversion. For example what is \$500 in euros? The conversion rate on 21 October 2012 is €1=\$1.302. Simply solve for \$ in terms of euros:

Example

If €1=¥320 (Japanese yen), convert ¥27,200 to euros.

Solution: Solve for yen in terms of euros:

Now write yen in terms of euros:

Exercises:

1. If €1=\$1.50, find the value of (i) €100 in dollars (ii) \$180 in euros.
2. A train ticket cost €40. How much would the ticket cost in Canadian dollars if €1=\$2.25.
3. A ticket to a football game costs €25. How much does the ticket cost in US dollars if €1=\$1.60.
4. An airline ticket costs \$416. If \$1=\$1.30, calculate the cost of the ticket in euros.

5. If €1=$1.20 (US), €1=$2.76 (Canadian), and €1=¥331.2,
   (a) how many US dollars would you get for €40?
   (b) how many Canadian dollars would you get for €350?
   (c) how many Japanese yen would you get for €180?
   (d) how many euros would you get for 90 US dollars?
   (e) how many euros would you get for 1,380 Canadian dollars?
   (f) how many euros would you get for 238,464 Japanese yen?
6. calculate the value of p, q and r, given the following exchange rates:

   (a) $1 (US)=$ p (Canadian) **(b)** $1 (US)=¥$q$ **(c)** $1 (Canadian)=¥r.
7. Calculate the exchange rate for $1 in each case, given the following transactions:

   (a) €80=$112 (US) **(b)** €75=$165 (Canadian)
8. A tourist changed €800 on board ship into South American rand, at a rate of €1=R2.4. How many rand did she receive? When she came ashore she found that the rate was €1=R2.48. How much did she lose, in rands, by not changing her money ashore?

2.3.6 Temperature Scales

Most of us would think that temperature is defined linearly with the melting/freezing point of water at $0\,^\circ C$ and the vaporisation/condensation point at $100\,^\circ$ and $50\,^\circ$ 'half-way as hot' as $100\,^\circ$. While this is an O.K. to think in everyday terms it won't do for a precise scientific definition.

It is actually non-trivial to define temperature and most people don't know it is actually done. The precise definition started coming about in the 19th century when physicists began to get a handle on thermal physics.

The physicists began to realise that the temperature of a body was related to the average speed of the molecules in it:

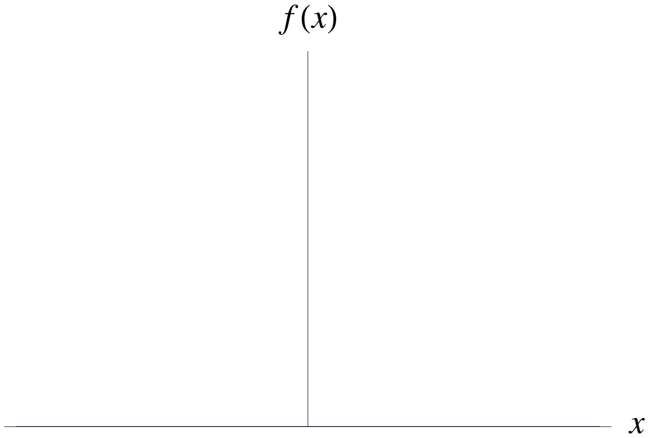

Figure 2.1: For $T\,°C$, the average speed of the molecules is approximately directly proportional to the temperature.

When people *extrapolated* this data, they predicted that the *absolute zero* temperature was $-273.16\,°C$. This was a temperature at which every molecule in a body would cease to move. More more in thermal physics showed that in fact

This means that $T \propto \overline{K.E.}$ (temperature is *proportional* to average kinetic energy). When we plot $\overline{K.E.}$ vs temperature (in Kelvins):

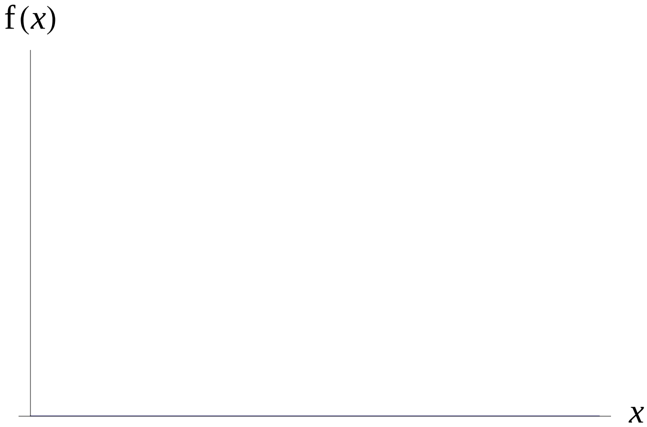

Figure 2.2: We have the conversion $T°C = T\,K + 273.16$.

Fahrenheit is another relative scale, defined similarly to Celsius except that the freezing point of water is taken as $32\,°F$ and the boiling point is taken as $212\,°F$. The conversion between them is given by

$$T\,°F = \frac{9}{5}T\,°C + 32, \tag{2.6}$$

Exercise: Find the formula that converts (i) Kelvin the Celsius (ii) Celsius to Fahrenheit (iii) Fahrenheit to Kelvin (iv) Kelvin to Fahrenheit

2.3.7 The Richter Scale

Richter Scales measure earthquakes. How? Has anyone seen a seismograph in action?

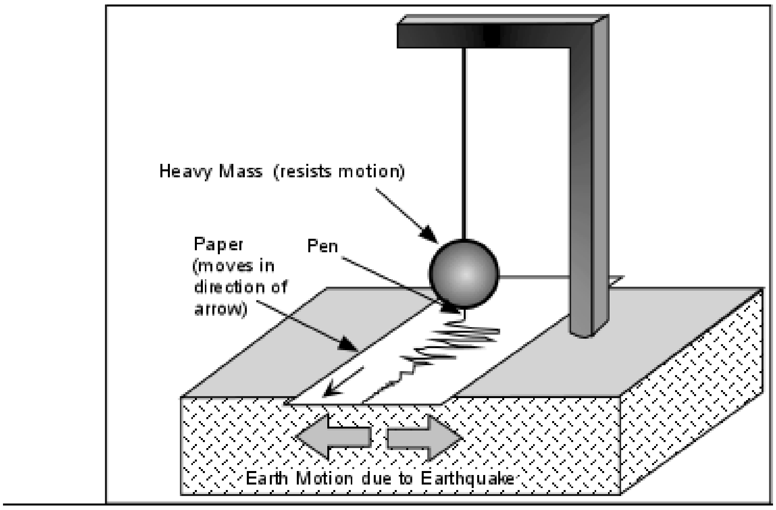

Figure 2.3: Earthquakes are measured by the amplitude of the waves (picture from earthsci.org).

The way Richter Scales work is that they assign a small amplitude of waves, say A_1, to one on the scale. A force two Earthquake is then defined as one whose wave amplitude is ten times as large as a force one (Two=$10A_1$). A force three earthquake has waves whose amplitude is ten times as large as a force two (Three=$10A_2$=$100A_3$) and a tenth as large as a force four earthquake. Such a scale is called *logarithmic*.

2.3.8 Rates of Change

Consider the rate of change of a quantity Q with respect to time. How do we define the average rate of change:

Therefore all rates of change (with respect to time) are measured in [units of Q] per second.

Examples

1. Speed, $v(t)$, is the rate of change of distance, $s(t)$, with respect to time, t. It is measured in

2. *Acceleration*, $a(t)$, is the rate of change of speed/velocity, $v(t)$, with respect to time, t. It is measured in

   If a particle has a negative acceleration it is often said to have a *retardation*. In other words an acceleration of -10 m s^{-2} is the same as a retardation of +10 m s^{-2}.

3. The *angular velocity*, $\omega(t)$, of a particle in circular motion is the rate of change of angle, $\theta(t)$, with respect to time, t:

4. The *angular acceleration* of a particle in circular motion is the rate of change of angular velocity, $\omega(t)$, with respect to time, t. It is measured in rad s^{-2}

5. The *linear density* of a rod, $\rho(x)$, is the rate of change of mass, $m(x)$, with respect to length, x. It is measured in

6. *Current*, $I(t)$, is the rate of change (or flow) of *charge*, $Q(t)$, with respect to time, t. It is measured in

7. The *growth rate of a population* is the rate of change of population, $N(t)$, with respect to time, t. It is measured in

8. The *marginal cost of production* is the rate of change of the cost of producing x items, $C(x)$, with respect to x. It is measured in

Economies of Scale occur when the marginal cost $C'(x)$ decreases.

2.3.9 Percentage & Errors

Percentages of a quantity are measured in the same units as the quantity:

Absolute errors are measured in the same units as the quantity:

However relative or percentage errors are dimensionless:

2.4 Functions

Functions and Graphs

A function like a process that produces an output when given an input. Suppose A is one collection of objects, and B is another collection. A function is like a mapping between A and B:

We can think of A as the collection of inputs, and B the collection of outputs. When we write $f : A \to B$ we mean that f is a function from A to B. A is called the *domain of the function* and is the set of (meaningful) inputs. The *range* of a function is the set of all outputs when we input all of the set A into f.

In this module the collections we are interested in are collections of numbers. For example the real numbers (i.e. the numbers on the numberline) — and we are interested in functions — or maps — that send real numbers to real numbers. As an example, the function $f(x) = x^2$ takes a real number input — and the output is that real number squared. This is a purely algebraic picture, but we can also consider it in the geometric picture[2].
We can look at the *graph* of a function:

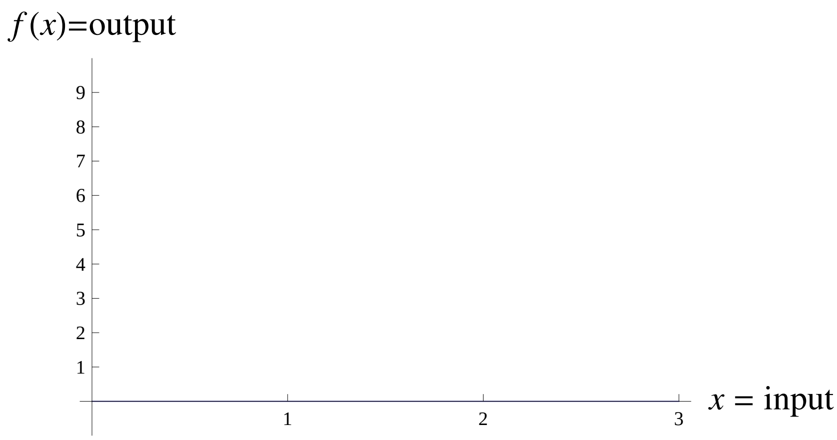

Figure 2.4: To plot the graph of a function — the collection of pairs $(x, f(x))$ as x runs over all the real numbers — you examine the outputs for various inputs.

[2]and one of the major themes of modern mathematics is thus. See more: http://jpmccarthymaths.com/2011/03/14/my-understanding-of-non-commutative-geometry/

Most if not all of the functions that we will encounter will be of the form:

What this means is that whatever number we feed into the function, x, the expression on the right gives the recipe for calculating $f(x)$.

Examples

1. $f(x) = 3x + 2$.
2. $g(x) = x^2$.
3. $h(x) = 1/(3x + 5)$.
4. $i(x) = \sqrt{x^2 + 1}$.
5. $j(x) = \sin x$.
6. $k(x) = 2^x$.
7. The volume V of a sphere depends on the radius r of the circle (input data). The relationship that connects r and V is

   With each positive $r \in \mathbb{R}$ there is associated one value of V and we say that V *is a function of* r; and we may write $V = V(r)$ to signify this.

Occasionally we will only take some of the real numbers to be our input. For example:

defines a function on the degree-measure of the acute angles in a right-angled triangle. Now consider the function $p : \mathbb{R} \to \mathbb{R}$, $p(x) = 10^x$. We can tabulate some of the values that $p(x)$ takes:

Input	-4	-3	-2	-1	$-1/2$	0	$1/2$	1	$4/3$
Output					0.316...		3.162...		21.554...

Recalling that the *graph* of a function is the set of all points/coordinates of the form (input, output). Take a pair of axes labeled x and $f(x)$ (or x and y). Now x is the input while $f(x)$ is the output:

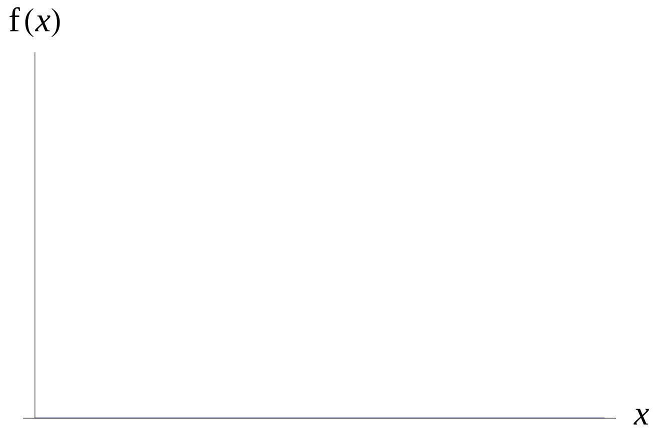

Figure 2.5: The graph of a function $g(x)$.

We could plot the points of $p(x) = 10^x$ on a graph:

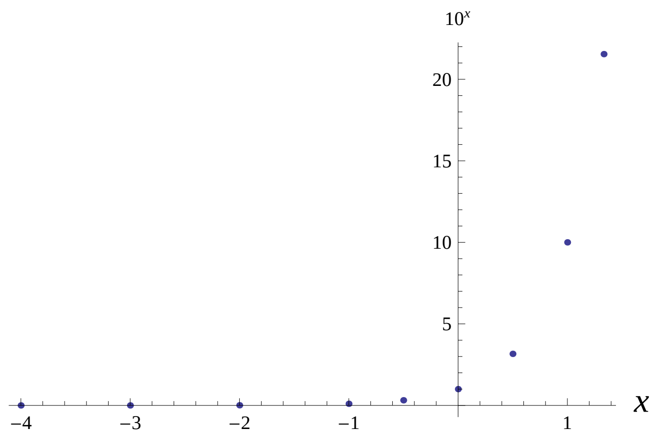

Figure 2.6: The graph of $f(x) = p^x$.

Now a feature of the graph of 10^x is that it is always increasing and always positive (this will be true for any a^x as long as $a > 1$):

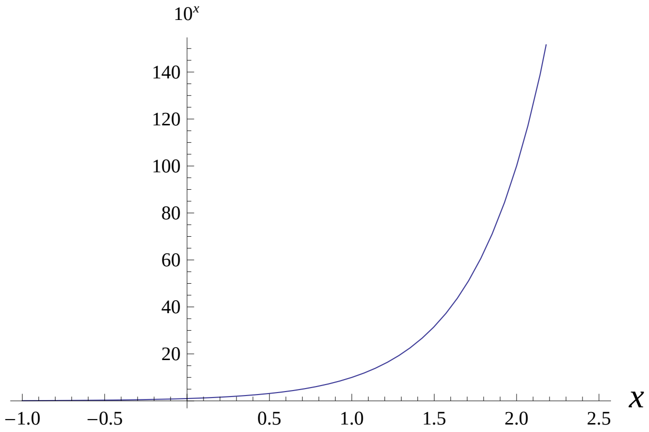

Figure 2.7: To every output we can associate a unique input.

This suggests that we could define a function going in the opposite direction, where "y" would be the input. Now this other function, which we call '$l(x)$' only takes *positive* numbers as input. So

where $\mathbb{R}^+$ is the set of positive real numbers. We define $l(x)$ as:

The terminology here is that $l(x)$ is the *inverse* of $p(x) = 10^x$. Again as we did in the last chapter, we can generalise this discussion - i.e there is nothing special here about 10.

2.5 Logarithms*

Definition

Let $a > 1$ and let $p(x) = a^x$. The *logarithm to base* a is a function $\log_a : \mathbb{R}^+ \to \mathbb{R}$ that is the inverse function of $p(x)$ so that

In other words, $\log_a y$ asks the question:

Examples

Evaluate

1. $\log_2 8$

   Solution

2. $\log_3 81$

   Solution

3. $\log_{25} 5$

   Solution

4. $\log_{10} 0.001$

   Solution

Properties of Logarithms

Now that we have defined what a logarithm is we must investigate common properties of them. What happens when we take the log of a product? If we take the log of a fraction? What about the log of a power?

The Log of a Product?

What is the log of 8×16 to the base 2:

$$\log_2(8 \times 16) = ?$$

As we've seen time and again, a process like this can be abstracted to prove a general rule for *all* numbers:

Example

Evaluate $\log_4 2 + \log_4 32$.

Solution:

The Log of a Power

What is $\log_2 8^4$?

So...

(Note we've only stated this for $n \in \mathbb{N}$ but in fact this is true for all $n \in \mathbb{Q}$.)

Example

Find $\log_5 25^{60}$.

Solution:

The Log of a Fraction

What about $\log_3(9/27)$?

So

Example

Evaluate $\log_2 80 - \log_2 5$.

Solution:

Remark

The best way to remember these formulae/facts/rules —

$$\log(xy) = \log x + \log y, \tag{2.7}$$

$$\log\left(\frac{x}{y}\right) = \log x - \log y, \tag{2.8}$$

$$\log(x^n) = n \log x. \tag{2.9}$$

is to think of the $\log x$ function as a *transform* from the positive real numbers to the real numbers that simplifies operations: multiplication → addition, division → subtraction, indices → multiplication:

The Change of Base Rule - Why Can't I Find $\log_3 12$ on My Calculator?

What is $\log_3 12$ actually equal to as a decimal...

So if we want to change from base a to base b:

Example

Evaluate $\log_8 5$ correct to 5 decimal places.

Solution:

Further Examples

1. What is $\log_3 3^x$?

   Solution:

2. What is $4^{\log_4 69}$?

   Solution:

3. What is $\log_a 1$?

   Solution:

4. What is $\log_a \sqrt{x}$?

   Solution:

5. What is $\log_a(1/x)$?

   Solution:

6. Express as a single logarithm:
$$\log_a x + \frac{1}{2}\log_a y :$$

   Solution:

7. Evaluate (for a general base)
$$\frac{\log 25 - \log 125 + \frac{1}{2}\log 625}{\log 5^3}.$$

Solution: First of all we should look at the top and bottom separately. The powers of a half jumps out at me:

Now we use the laws of addition and subtraction of logs:

Now use the fact that $\log 5^3 = 3\log 5$:

8. Solve the equation $4^x = 100$.

   Solution: Take the log of both sides:

2.6 The Natural Exponential & Logarithm Functions*

Two Distinguished Bases

For good reason the two bases that 'turn-up' most frequently are 10 and the special real number $e \approx 2.718$. As we have seen in a previous chapter, any non-zero real number at all may be expressed in the form:

for some $a \in [1, 10)$ (means "*a is a (real) number between* 1 *and* 10*, possibly equal to* 1 *but not equal to* 10*.*) For example,

Now taking the log to the base 10 of both sides:

Hence to find the base-10 logarithm of any number at all we must simply know all the logarithms (to base 10) of all the numbers between 1 and 10.

The importance of the base e comes from following observation:

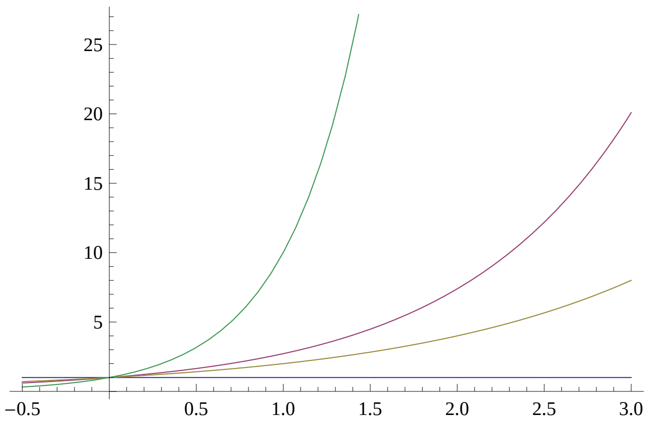

Figure 2.8: The slope of the curve of a^x is similar to a^x itself.

As it turns out, e^x is the unique function of the form $f(x) = a^x$ for which the slope of the curve of a^x is equal to a^x. We will write $\ln x$ for $\log_e x$, where 'ln' is somehow meant to stand for natural logarithm.

2.6.1 Exponential & Logarithmic Equations

An exponential equation is one of the form

We can usually takes an appropriate logarithm of both sides to simplify these — the Golden Rule of Like Powers applies though.

Another way is via the follow fact:

Fact/ Theorem

If $a > 1$ **and**

$$a^x = a^y,$$

then $x = y$.

Proof. All functions of the form $p(x) = a^x$ are increasing for $a > 1$:

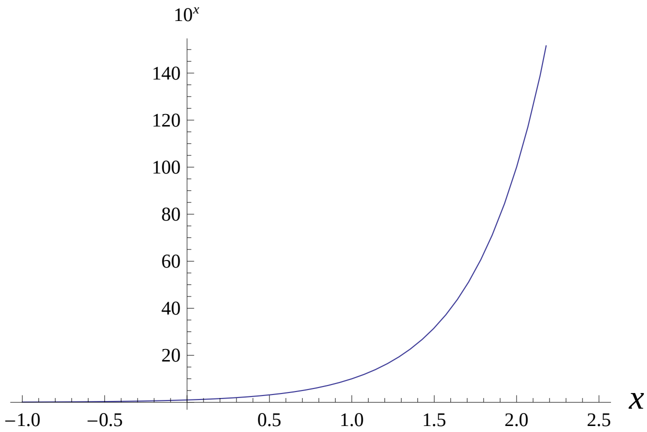

Figure 2.9: If $a^x = a^y$ then in fact $x = y$.

This fact is equivalent to the uniqueness of logarithms. The same fact also holds for logarithms:

Fact/Theorem

If $a > 1$ **and**

$$\log_a x = \log_a y,$$

then $x = y$.

Therefore if we can write an exponential equation in the form

we can then write $f(x) = g(x)$. Similarly if we rearrange a logarithmic equation into the form

we can also conclude that $f(x) = g(x)$. This should be your first attempt at solving an exponential and logarithmic equation. However this may not be always possible as the first example here shows.

Examples

1. If $4^n = 2000$, find n correct to 4 significant figures.

   Solution: We cannot write 2000 as a power of four so take the logarithm of both sides:

2. Solve
$$\log(x^4) - \log(x^3) = \log(5x) - \log(2x).$$

   Solution: Can we write this as $\log(f(x)) = \log(g(x))$?

Applied Examples

1. Suppose that the population of a town is 10k and the population increases every year by 4%. After how many years will the population have doubled?

   Solution: Let $P(n)$ be the population after n years. We have $P(0) = 10$ k. The population after one year is given by:

   After two years the population is:

   Hence after n years the population is given by

   Now the question is when is this equal to (or greater than 20 k)?

   To solve this we take the log of both sides:

   So the answer is 18 years.

2. Suppose that 1% of the radioactive material in a body decays every year. If there is initially 2 g of the material initially, after how long is only half of the radioactive material remaining?

   Solution: Let $N(t)$ be the population after t years. We have $N(0) = 2$ g. The population after one year is given by:

   After two years the population is:

Hence after n years the population is given by

Now the question is when is this equal to (or greater than 1 g)?

To solve this we take the log of both sides:

So the answer is 69 years.

Exercises

1. Solve for x:

$$(i)\ \log_2 8 = x \qquad (ii)\ \log_{10} x = 2 \qquad (iii)\ \log_3 81 = x$$

$$(iv)\ \log_5 x = 3 \qquad (v)\ \log_2 x = 10 \qquad (vi)\ \log_x 64 = 3$$

$$(vii)\ \log_{10} 10 = x \quad (viii)\ \log_{10} 1 = x \quad (ix)\ \log_{25} x = 1/2$$

2. Evaluate $\log_7 35 - \log_7 5$.
3. Solve for n (to 2 decimal places): $2^n = 20$.
4. Use a calculator to evaluate to two decimal places:

$$(i)\ \log_{10} 3 \ \ (ii)\ \log_{10} 7 \ \ (iii)\ \log_3 7 \ \ (iv)\ \log_7 3$$

5. Using the fact that 3^n is increasing, find the least value of $n \in \mathbb{N}$ such that $3^n > 1{,}000{,}000$.
6. If $p = \log_3 10$ and $q = \log_3 7$, write $\log_3 700$ in terms of p and q.
7. (a) Find the least value of $n \in \mathbb{N}$ such that $1.03^n > 2$.
   (b) Suppose that the population of a town increases by 3% every year. Using the previous result, write down how many years it takes to double.
8. If $p = \log_6 5$ and $q = \log_6 10$, write $\log_6(2\sqrt{6})$ in terms of p and q.
9. Expand:

$$\log_a \sqrt{\frac{ab}{c}}$$

10. Solve

$$\log 2t^3 - \log t = \log 16 + \log t.$$

Ans: 8

11. Solve
$$2\log b^2 - 3\log b = \log 8b - \log 4b.$$

    Ans: 2

12. Solve the equation $2^x = 3$ correct to three decimal places.

    Ans: 1.585

Chapter 3

Basic Algebra

Algebra is the metaphysics of arithmetic.

John Ray

These are not letters, they are numbers.

Me, just there.

3.1 Introduction: What the Hell *is* x??

Consider an equation: a mathematical statement expressing that two objects are equal, equivalent and one and the same. As an example;

In mathematics there are a number of uses for this = sign. There is the common;

which merely asserts that the sum of 1 and 2 is 3. Also there is the definition-type =;

which defines 3^3 for example, and by extension all these positive integer powers. Finally there is the equation or formula type =, the most famous of which is probably

Energy=(mass)×(speed of light)×(speed of light) $(E = mc^2)$.

Hence an equation is an expression that one object is equal to the other. Suppose now I told you that the difference between two numbers squared was the product of their respective sum and difference, and I stacked up the evidence in front of you as so you would believe me

in particular negatives aren't a problem because for example $(-4)^2 = (+4)^2$; the square of a negative number is positive. You could say though how do you know this is true of every number, how can you *prove* this is true? By going away and thinking I can demonstrate why it's true.

Theorem

The difference between two numbers squared is the product of their respective sum and difference.

Proof. Consider the diagram:

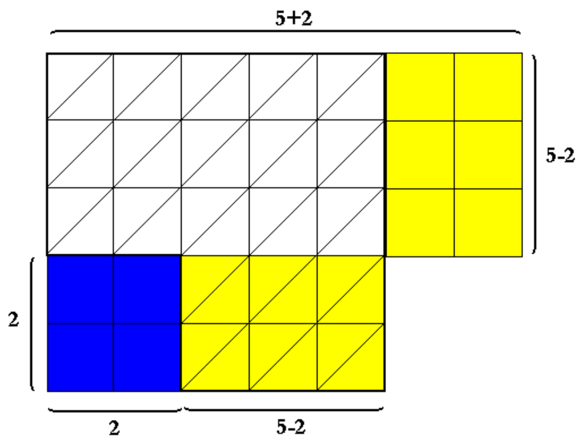

Figure 3.1: A representation of $5^2 - 2^2$.

The yellow $2 \times (5-2)$ segment at the bottom of the diagram can always be moved to the top right as shown. Therefore the difference between two numbers squared is the product of their respective sum and difference •

However quite rightly you could point out that this proof only proves the case for integers, $\mathbb{Z}$. What about fractions, $\mathbb{Q}$? Now things are getting decidedly shaky without xs. A proof I could use is illuminating in the sense that it uses the previous theorem to prove the case for fractions but I don't know how to construct a proof for *all* real numbers. You may refute my claim therefore for real numbers. This is when x kicks in. I do know that real numbers have certain properties however. Perhaps I can use the properties of real numbers to show that the difference between two numbers squared is the product of their respective sum and difference — even if it is the picture which gives us the "why". These properties are listed as an appendix at the end of this chapter. These properties are ones that we assume without proof. Special properties like this that we assume rather than prove are called *axioms*.

3.1.1 Laws of Algebra

Five facts about numbers[1]:

1. **Subtraction:** For all real numbers x:

   Example: $5 - 5 = 0$.

2. **Identity:** For all real numbers x:

   Example: $\pi \times 1 = \pi$

[1]four axioms and one theorem

3. **Distributivity** For any three real numbers x, y, z:

   This is called the *distributive law.* Note that this works both ways and explains how to group terms and factorise. One way says how to write a product as a sum; the other how to write a sum as a product.

4. **Division:** For all $x \neq 0$,

   Example: $-123/-123 = 1$.

5. **No Zero Divisors:** Suppose that $a, b \in \mathbb{R}$ such that $a \times b = 0$:

   Example: *Which number multiplied by* 5 *gives you* 0*?*

The use of x is a sort of every-number. It can now be said that the difference between two real numbers squared is the product of their respective sum and difference because the theorem can be proved algebraically. The proof below is incredibly thorough but shows the use of the axioms. Note the notation:

$$x \times y = x(y) = xy.$$

Theorem

If x**,** $y \in \mathbb{R}$**;**, *then*

$$x^2 - y^2 = (x - y)(x + y).$$

Proof.

$$\begin{aligned}
(x-y)(x+y) &= (x + (-y))(x + y) && \text{(subtraction)} \\
&= x(x + y) + (-y)(x + y) && \text{(subtraction)} \\
&= x^2 + xy + (-yx) + (-y)(y) && \text{(subtraction)} \\
&= x^2 + xy + (-xy) + (-1)(y)(y) && \text{(subtraction)} \\
&= x^2 + (xy + (-xy)) + (-1)(y^2) && \text{(subtraction)} \\
&= x^2 - y^2 && \text{(subtraction)}
\end{aligned}$$

Hence using x and letting it stand for *some* real number, and manipulating it according to the axioms of the real numbers, a result that is true of *all* real numbers is proven. Again this is more than anything an exposition of proof from axioms.

Examples

1. Write $x^2 - 16y^2$ as a product.

   Solution:

2. Write $x^2y^2 - 16^2$ as a product.

   Solution:

3.1.2 Why Bother?

Consider the following two problems:

1. What real number, when added to its square, equals 0?
2. Show that $n^2 + n + 41$ is always a prime number when n is a natural number.

Here there are two types of problem. One is a puzzle — what is x? Another is asking us to *prove* that an identity holds. *Solutions*:

1. Now first off, can anyone guess... How do you that there are no more solutions??? This is how we do algebra: suppose that x has the property that

   Now we use some of our facts in a logical way. Well $x^2 = x \times x$ and $x = x \times 1$. Also using distributivity we can write this sum as a product:

   Now using 'No Zero Divisors':

   Now considering $x + 1 = 0$ separately. I'm happy enough that $x = -1$ but if we still weren't convinced we could add -1 to both sides:

   We set up x as the fall guy — and by a series of logical deductions, show what x must be.
2. There is great numerical evidence for this 'theorem':

   Who believes it's true?

3.1.3 Some More Facts (!)

Multiplication by Zero

For all $x \in \mathbb{R}$**,** $0 \times x = 0$.

Division by Zero is Contradictory

There does not equal a real number equal to $1/0$*. (Division by any number* y *may is realised as multiplication by* $1/y$*.)*

Four More Facts

1. **For all real numbers** $x \in \mathbb{R}$ $-1 \times x = -x$.
2. $-1 \times -1 = +1$.
3. $(-a) \times (-b) = ab$.
4. $ax + bx = (a+b)x$.

Before doing any serious mathematics, it is important that this knowledge is understood to ensure competency at algebra, and prevent many of the common pitfalls faced by students who are not algebraically aware. Now whenever x or y is seen it is understood it stands for a real number.

3.1.4 Seven Deadly Sins

One of the biggest mistakes that people make is to assume that everything is *linear*. A linear function is a function $T : A \to B$ which has the property that

for all a_1, $a_2 \in A$ and $\lambda \in \mathbb{R}$. The *only* linear real functions are functions of the form $f(x) = ax$ for a constant $a \in \mathbb{R}$. Everything else is not linear.

Squaring is NOT Linear

Is $(x+y)^2 = x^2 + y^2$? NO:

The correct thing is to know what squaring means: $(x+y)^2 = (x+y)(x+y) = x^2 + 2xy + y^2$ when we multiply out (later).

The Square Root is NOT Linear

Is $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$? NO:

You cannot simplify $\sqrt{x+y}$ any further without knowing what x and y are.

Inverting/Taking the Reciprocal is NOT Linear

Is $\dfrac{1}{x+y} = \dfrac{1}{x} + \dfrac{1}{y}$? No:

Again there is nothing which you can do to simplify $\dfrac{1}{x+y}$ without knowing more about x and y.

Logs are NOT Linear

Is $\log_a(x+y) = \log_a(x) + \log_a(y)$? NO:

Again there is nothing you can do without more information.

The Exponential Functions are NOT Linear

Is $a^{x+y} = a^x + a^y$? NO:

We should notice the pattern by now:

Sine & Cosine are NOT Linear

Is $\cos(A+B) = \cos(A) + \cos B$? NO:

In fact we can show that

3.2 Algebraic Manipulation & Simplification

3.2.1 Simplifying Expressions

An expression in real numbers xs, ys, zs, etc. may often be simplified greatly. Because of commutativity and the distributive law of multiplication over addition; if a, b are real constants;

$$ax + bx = x(a+b) = (a+b)x$$

This means common terms can be added together to simplify an expression.

Example

Simplify $3x - 2 + 2x - 4$.

Solution:

3.2.2 Expanding Linear Products

Suppose $x \in \mathbb{R}$ and a, b, c, d are real constants. Then using the distributive law twice:

$$\begin{aligned}(ax+b)(cx+d) &= ax(cx+d) + b(cx+d) \\ &= acx^2 + adx + bcx + bd \\ &= acx^2 + (ad+bc)x + bd\end{aligned}$$

Examples

1. Expand $(2x+4)(x+3)$

   Solution:

2. Multiply out $(x+1)(3x^2+x+1)$.

   Solution:

3.2.3 Addition of Fractions

Suppose that we have real numbers[2] a and b and form the fraction $\dfrac{a}{b}$. How do we add fractions like this? Recall that we can add fractions using the lowest common multiple. Take $\dfrac{5}{12}+\dfrac{3}{8}$:

but this is not strictly necessary:

All that we need is a common denominator. The second approach to adding 5/12 and 3/8 can be used here. The general method of writing a general $\dfrac{a}{b}+\dfrac{c}{d}$ as a single fraction is the same:

After practise some do it in one line

but you are probably better off taking your time.

[2] b non-zero ✓

Examples

Write as a single fraction:

1. $\dfrac{x+2}{3} + \dfrac{x+5}{4}$

   Solution:

2. $\dfrac{1}{x+2} + \dfrac{1}{x+3}$

   Solution:

3. $\dfrac{t^2}{t-2} - \dfrac{t^2}{(t+2)}$

   Solution:

3.2.4 Factorising

Factorising is taking a sum or difference and writing it as a product. There are a multitude of reasons for factoring. For example,

1. Simplifying expressions. Often after a factorisation a simplification of a supposedly complicated object may be made.
   Example:
   $$\frac{x^2-9}{x-3} = \frac{(x-3)(x+3)}{x-3} = x+3\,, \text{ if } x \neq 3$$
2. Solving quadratic equations. To solve a quadratic equation

   means to find numbers that when plugged in for x make the equation true. We use the no-zero divisors theorem to do this: if a and b are real numbers and $a.b = 0$ then either $a = 0$ or $b = 0$. i.e. we simplify by writing the quadratic in the form:

Factoring is based on the *distributive law for multiplication over addition*; for all $x, y, z \in \mathbb{R}$:

If there is a sum like the right-hand side of this, with a factor common to both sides, it can be taken out as shown.

Examples

Write as a product (i.e. factorise)

1. $5ab + 10ac$

   Solution:

2. $6xy - 3y^2$

   Solution:

3. $pq - pr + p$

   Solution:

4. $2\pi r^2 + 2\pi rh$

   Solution:

5. $x^3 + x^2 + x$

   Solution:

6. $2a(x + y) + 3(x + y)$

   Solution:

7. $16ax^2 - 12a - 8x^2 + 6$

   Solution: It's not actually clear that we can factorise this. Let us start however by taking out what is common out of the first two terms... and something out of the second two that will leave something common:

8. $3abx^2 - 5axy - 3bxy + 5y^2$

   Solution: The same strategy again:

Dividing above and below

Suppose $a, b, c \in \mathbb{R}$, $b, c \neq 0$. *Then*

$$\frac{ac}{bc} = \frac{a}{b}$$

We have done this before! Remember these 'letters' are only numbers. Therefore be careful not to manipulate 'symbolically'!! Let $a, b, c, d \in \mathbb{R}$, $b, c \neq 0$. The following move is nonsense (unless $d = 0$):

Nowhere does it say that this should make sense. Don't ever forget that we are dealing with numbers here. Therefore we will try not to use the word *cancel*.

Examples

1. If $x = \dfrac{2t}{1+t^2}$ and $y = \dfrac{1-t^2}{1+t^2}$, write $\dfrac{x}{1-y}$ in terms of t.

   Solution:

2. Simplify $\dfrac{10ab}{2b}$.

   Solution:

3. Simplify

$$\left[\frac{1+x}{1-x} - 1\right] \div \frac{1}{1-x}.$$

   Solution: We start by noting that division is just multiplication by $1/(1-x)$:

   Now how about multiply above and below by $(1-x)$?

   Piece of cake!

3.2.5 Factorisation for Simplification

If we have a fraction then it might not be initially clear if a simplification can be found. However because we will need the numerator and denominator to be both products if we want to divide out a common term, the first thing to do should be to factorise above and below to see if these will be possible.

Examples

Simplify

1. $\dfrac{4x-8}{x^2-4}$.

   Solution: Hmmm, factorise above and below:

2. $\dfrac{3x}{3xy+6xz}$.

   Solution: Factorise the bottom:

3. Simplify

$$\frac{a^2-b^2}{5(a-b)+6(a-b)+7(a-b)}.$$

   Solution: Dealing with the top and bottom separately for now:

   If $a \neq b$ we can divide above and below by $a-b$:

Exercises

1. When $a=1$, $b=2$, $c=3$, $x=4$, $y=5$ and $z=6$, find the value of:

$$(i)\, 2a+3b+4c \qquad (ii)\, abc$$
$$(iii)\, 3xz+5bx \qquad (iv)\, 3(2b+a)+2(3x+2y)$$
$$(v)\, (a+b)^b \qquad (vi)\, (2abc/z)^c$$
$$(vii)\, (5(y+a))/(3(a^2+b^2)) \qquad (viii)\, 2(b^2+c+bc^3)$$

   Selected Answers: (iii) 112 **(vi)** 8

2. Remove the brackets in each of the following:

$$(i)\, 2(x+4) \qquad (ii)\, 5(2x^2+3x+4)$$
$$(iii)\, -3(x-4) \qquad (iv)\, -4(x^2-3x+4)$$

   Selected Answers: (iii) $12-3x$.

3. Remove the brackets and then simplify each of the following:

$(i)\, 3(2x+4)+2(5x+3)$ $(ii)\, 4(2x+3)+2(4x+6)$

$(iii)\, 3(2x+4)+7-3(x+5)-2x-4$ $(iv)\, 11x-3(6-x)+13-5(2x-1)$

$(v)\, 7(6-x)+21+5(x-7)-3(8-x)$ $(vi)\, 3p+2(4-p)+3(p-5)-2p+6$

$(vii)\, 5(q+4)-3q-29+3(q+3)$ $(viii)\, 5(2x^2-3x+2)-3(3x^2-6x+2)$

$(ix)\, 5(1-x+2x^2)-5(1-x-2x^2-18x^2)$ $(x)\, 2(x+y)+3(2x+3y)$

$(xi)\, 5(2x-y)-4(x-3)+3(y-5)$ $(xii)\, 4x-[4x-2(2x-2)]$

Selected Answers: (iii) x **(vi)** $2p-1$ **(ix)** $2x^2$ **(xii)** $4x-4$

4. Simplify each of the following:

$(i)\, 2x.5x$ $(ii)\, x.x^2$ $(iii)\, 4x^2(2x)$

$(iv)\, (5a^2)(-3a)$ $(v)\, 2p.2p.2p$ $(vi)\, (-x)(-x)$

$(vii)\, 4ab(3a^2b^2)$ $(viii)\, (3xy)(xy)$ $(ix)\, p.3p(-3p^2)$

$(x)\, (8xy)(xy)(4x)$

Selected Answers: (iii) $8x^3$ **(vi)** x^2 **(ix)** $-9p^4$

5. If $6x(4x^2)=kx^3$, find the value of k.

6. Remove the brackets and then simplify each of the following:

$(i)\, 2x(x+5)+3(x+5)$ $(ii)\, 2x(x-1)-3(x-1)$

$(iii)\, 2x(2x+3)+3(2x+3)$ $(iv)\, 2x(x^2+5x+3)+3(x^2+5x+3)$

$(v)\, 4x(2x^2-3x-6)+6(2x^2-4x)$ $(vi)\, a(a+1)+2a(a-3)+6a-3a^2$

$(vii)\, 5x(2x-y)-2y(2x-y)$ $(viii)\, 3x(2x+4y)+6y^2-6y(x+y)$

$(ix)\, a(b+c)-b(c-a)-c(a-b)$ $(x)\, 2[a(a+b)+b(b-a)]$

Selected Answers: (iii) $4x^2+12x+9$ **(vi)** a **(ix)** $2ab$

7. Multiply $2x^2-2x+1$ by $x+1$.

8. Write as a single fraction:

$(i)\, \frac{1}{5}+\frac{3}{4}$ $(ii)\, \frac{7}{5}-\frac{9}{10}+\frac{11}{15}$

$(iii)\, \frac{x}{5}+\frac{x}{4}$ $(iv)\, \frac{x}{2}+\frac{3x}{4}-\frac{5x}{3}$

$(v)\, \frac{x+2}{5}+\frac{x+7}{10}$ $(vi)\, \frac{2x+3}{7}+\frac{x+1}{3}$

$(vii)\, \frac{5x-3}{2}-\frac{3x-4}{3}$ $(viii)\, \frac{2x+5}{3}-\frac{4x-3}{2}$

$(ix)\, \frac{5x-1}{4}+\frac{x}{3}-\frac{5}{6}$ $(x)\, \frac{5x}{3}-\frac{1}{6}+\frac{2-3x}{2}$

$(xi)\, \frac{4x-3}{5}-\frac{x}{2}+\frac{1}{10}$

Selected Answers:

$$(iii)\ \frac{9x}{20} \quad (vi)\ \frac{13x+16}{21} \quad (ix)\ \frac{19x-13}{12}$$

9. Simplify each of the following:

$$(i)\ 20ab/5b \qquad (ii)\ 21pq/7q \qquad (iii)\ 28p/7$$

$$(iv)\ x^2/x \qquad (v)\ 3pq/3pq \qquad (vi)\ 6x^2y/3xy$$

$$(vii)\ 30p^2q/15p^2$$

Note: When we write a/b we mean:

$$\frac{a}{b} = a \times \frac{1}{b} = a \div b = ab^{-1}$$

Selected Answers: (iii) $4p$ **(vi)** $2x$

10. Simplify (i) $(x-1)^2 - (x+1)^2$ (ii) $2(x+2)^2 - 8(x+1)$.

11. Simplify:

$$\left(1 - \frac{1}{y}\right)\left(\frac{y}{y-1}\right)$$

12. Factorise

$$(i)\ 4a^2 - 2a \qquad (ii)\ 2pq - 5r - 5q + 2pr$$

$$(iii)\ 25a^2 - 16b^2 \qquad (iv)\ 3x^2 - 12x$$

$$(v)\ 16a^2 - 81b^4$$

13. By factorising, simplify

$$\frac{4x^2 - 64}{x - 4}$$

14. Simplify

$$\left(\frac{1}{x} - \frac{1}{x+h}\right) \div h$$

15. Simplify

$$\frac{(2x+3)^2 - 6(2x+3)}{2x-3}.$$

16. Simplify

$$\left(p - \frac{q^2}{p}\right) \div \left(1 + \frac{q}{p}\right)$$

17. The sides of a triangle have lengths $m^2 + n^2$, $m^2 - n^2$ and $2mn$. Prove that the triangle is right-angled.

18. These statements are either true or false. If true, provide a proof. If false provide a counter-example (i.e. an example where the left-hand side is not equal to the right-hand side). Ignore or account for division by zero.

(a)

$$\frac{1}{a+b} = \frac{1}{a} + \frac{1}{b}$$

(b)

$$\frac{a+b}{a} = 1 + b$$

(c)

$$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$$

(d)

$$\frac{a^2+a}{a} = a + 1$$

(e)

$$\frac{a^2+2a}{2a} = a^2 + 1$$

(f)

$$\frac{a(b+c)+3a}{a} = b + c + 3a$$

(g)

$$\frac{a(b+c)+3a}{a} = b + c + 3$$

(h)

$$\frac{m^2-n^2}{m-n} = m + n$$

(i)

$$\frac{m^2+n^2}{m+n} = m + n$$

(j)

$$\frac{m^2-n^2}{n-m} = -n - m$$

3.3 Equations

3.3.1 What is an Equation

An equation is a mathematical statement, in symbols, that two things are exactly the same (or equivalent). Equations are written with an equal sign, as in

The equations above are examples of an equality: a proposition which states that two constants are equal. Equalities may be true or false. Equations are often used to state the equality of two expressions containing one or more variables. In the real numbers it can be said, for example, that for any given $x \in \mathbb{R}$ it is true that:

The equation above is an example of an identity, that is, an equation that is true regardless of the values of any variables that appear in it. The following equation is not an identity:

It is false for an infinite number of values of x, and true for only two, the roots or solutions of the equation, $x = 0$ and $x = 1$. Therefore, if the equation is known to be true, it carries information about the value of x. To solve an equation means to find its solutions. Many authors reserve the term equation for an equality which is not an identity. The distinction between the two concepts can be subtle; for example:

$$(x+1)^2 = x^2 + 2x + 1$$

is an identity, while:

$$(x+1)^2 = 2x^2 + x + 1$$

is an equation, whose roots are $x = 0$ and $x = 1$. Whether a statement is meant to be an identity or an equation, carrying information about its variables can usually be determined from its context; or by making a distinction between the equality sign (=) for a statement not true except perhaps in particular situations, and the equivalence symbol ($\equiv$) for statements know to be true without further specification. Letters from the beginning of the alphabet like a, b, $c \ldots$ often denote constants in the context of the discussion at hand, while letters from end of the alphabet, like x, y, $z \ldots$, are usually reserved for the variables.

Figure 3.2: An equation or formula is like a weighing scales — when balanced.

Famous Equations

$$\pi = \frac{c}{d}$$

$$a^2 = b^2 + c^2$$

$$E = mc^2$$

$$F = \frac{Gm_1m_2}{r^2}$$

$$A = \pi r^2$$

$$1 + 1 = 2$$

$$y = mx + c$$

$$x_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\cos^2\theta + \sin^2\theta = 1$$

$$e^{i\pi} + 1 = 0$$

Once we know the basic algebra facts we can solve almost all equations. To solve an equation in x; e.g.

$$x^2 + x = 0,$$

we implicitly suppose that x solves the equation. What we do then is apply a series of moves until we get $x =$ 'something'. What moves?

Fundamental Principle of Equations

Your old school books used to classify equations into different types which needed various series of manipulations in order to solve them. This is ridiculous. By and large, all we need is this simple thought:

> *Identify what is difficult or troublemsome about the equation and get rid of it. As long as you do the same thing to both sides the equation will be replaced by a simpler equation with a simpler solution.*

We reacted to each equation: we see something we don't like, perhaps something that is getting in the way of us isolating x on the left-hand side and we choose to do something to get rid of the problem. As long as we the same thing to the other side we are O.K. It really is as simple as that!

Well nearly. We must always beware of little subtleties. Sometimes when we something to both sides we get extra solutions that do not solve the original problem and sometimes we lose solutions. I wouldn't think of these as exceptions to the principle, just subtleties that we might watch out for... In reality we should always take a second when we get any answer... does this answer make sense?

Too Many Examples

Solve each of the following for x:

1. $x - 3 = -10$.

   Solution:

2. $x/5 = 1$.

   Solution:

3. $\sqrt{x} = 2$.

   Solution:

4. $-3x = 12$.

   Solution:

5. $2x + 3 = 11$.

   Solution:

6. $6x - 7 = 2x + 13$.

   Solution:

7. $\dfrac{2x}{3} + \dfrac{x}{4} = \dfrac{11}{6}$.

   Solution:

8. $x^2 = 35$.

   Solution:

9. $4(x - 3) = 4$.

   Solution:

10. $3 - \sqrt{x} = -6$.

    Solution:

11. $4 = \dfrac{5}{x}$.

    Solution:

12. $\dfrac{48}{\sqrt{x}} = 3$.

    Solution:

13. $\dfrac{1}{x - 3} = 17$.

    Solution:

14. $\sqrt{x+4} = 10.$

    Solution:

15. $\sqrt{\dfrac{x}{4}} = 2.$

    Solution:

16. $\dfrac{1}{x-3} + \dfrac{5}{4} = 1.$

    Solution: O.K this is a bit harder... or is it?

17. $4 = \sqrt{\dfrac{x^2}{81}}.$

    Solution:

18. $3(2x+1) - 3(x+4) = 0.$

    Solution:

19. $\frac{1}{3}(x-1) = x - \frac{3}{5}x.$

    Solution:

20. $5e^{3x} = 90.$

    Solution:

Exercises

1. Solve each of the following equations:

$$(i)\,2x = 6 \qquad (ii)\,6x = 30 \qquad (iii)\,-2x = -20$$
$$(iv)\,5x = -15 \qquad (v)\,5x = 0 \qquad (vi)\,3x - 7 = 8$$
$$(vii)\,5x - 2 = -12 \qquad (viii)\,5x = 12 + 8x \qquad (ix)\,7x + 40 = 2x - 10$$
$$(x)\,2x - 5 = 1 - x$$

   Selected answers: (iii) $x = 10$ **(vi)** $x = 5$ **(ix)** $x = -10$

2. Solve each of the following equations:

$$(i)\,3(x+4) = 2(x+8) \qquad (ii)\,4(x+4) = 2(x+3)$$
$$(iii)\,3(x-1) + 5(x+1) = 18 \qquad (iv)\,4(x+5) - 2(x+3) = 12$$
$$(v)\,10(x+4) - 3(2x+5) - 1 = 0 \qquad (vi)\,3(2x+1) - 3(x+4) = 0$$
$$(vii)\,7(x-6) + 2(x-7) = 5(x-4) \qquad (viii)\,5 - 4(x-3) = x - 2(x-1)$$
$$(ix)\,11 + 4(3x-1) = 5(2x+1) + 2(2x-5) \qquad (x)\,2 + 5(3x-1) = 4(2x-3) + 2(x-3)$$

   Selected answers: (iii) $x = 2$ **(vi)** $x = 3$ **(ix)** $x = 6$

3. Solve each of the following equations:

$$(i)\ \frac{x}{3} + \frac{x}{4} = \frac{7}{12}$$

$$(ii)\ \frac{x}{3} - \frac{x}{5} = \frac{27}{20}$$

$$(iii)\ \frac{2x}{5} = \frac{3}{2} + \frac{x}{4}$$

$$(iv)\ \frac{x+4}{3} = \frac{x+1}{4} + \frac{1}{6}$$

$$(v)\ \frac{x+4}{3} - \frac{x+2}{4} = \frac{7}{6}$$

$$(vi)\ \frac{x-5}{3} + \frac{1}{15} = \frac{x-2}{5}$$

$$(vii)\ \frac{3x-1}{2} = \frac{x+8}{3} + \frac{x-1}{6}$$

$$(viii)\ \frac{x-1}{4} - \frac{1}{20} = \frac{2x-3}{5}$$

$$(ix)\ \frac{x-2}{2} = 5 - \frac{x+10}{9}$$

$$(x)\ \frac{x-1}{3} + \frac{x-3}{4} = x - 4$$

Selected Answers: (iii) $x = 10$ **(vi)** $x = 9$ **(ix)** $x = 8$

3.3.2 Quadratic Equations

Equations which can be written in the form:

where $a, b, c \in \mathbb{R}$ are real constants with $a \neq 0$ are called *quadratic equations*. Examples:

Take the equation $x^2 - 5x + 6 = 0$. When $x = 2$:

When $x = 3$:

Solving a quadratic equation involves finding the values of x which satisfy the equation. An analogous problem is finding the *roots* of the quadratic function:

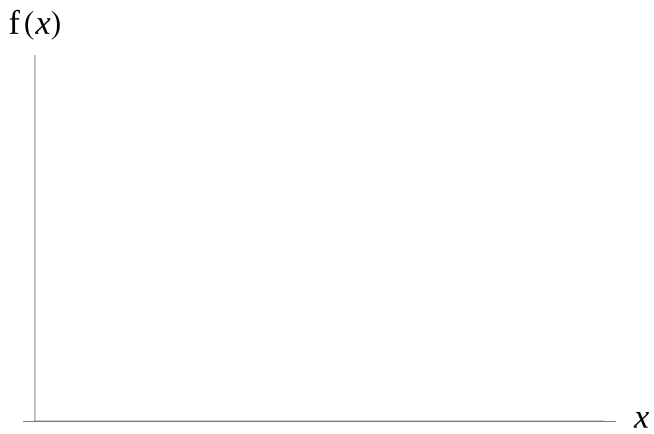

Figure 3.3: The *roots* of $f(x)$ occur where the graph cuts the x-axis.

Unfortunately this great Fundamental Principle of Equations doesn't apply very well as the x occurs twice so we have to change tack somewhat.

Finding the Roots

The *No Zero Divisors Theorem* is *the* crucial theorem in this section:

Hence what we can do is *factorise the quadratic* — i.e. write it as a product.

The simplest recipe for factorisation uses the following theorem:

Fact/Theorem

Let $f(x) = x^2 + bx + c$ be a quadratic function. Suppose $f(x)$ can be written in the form

Then

If this is the case, then α and β are the roots of $f(x)$.

Proof. Just multiply out •

Remark

This means that a quadratic has form

Hence suppose given a quadratic f:

If numbers α and β can be found such that

then

which has roots α and β according to the No Zero Divisors Theorem. If you are only interested in the roots of the quadratic then you shouldn't really need to factorise the quadratic but for good reasons I would prefer us to factorise the quadratic if we can.

Examples

Factorise the following quadratic functions and hence write down the roots.

1. $x^2 - 7x + 10$.

   Solution: We are looking for numbers α and β such that $\alpha + \beta = 7$ and $\alpha \cdot \beta = 10$:

   Hence by the No Zero Divisors Theorem the roots are 5 and 2.

2. $x^2 - 5x - 14$.

   Solution: We want $\alpha + \beta = 5$ and $\alpha \cdot \beta = -14$:

3. $x^2 + 12x + 32$.

   Solution: We want $\alpha + \beta = -12$ and $\alpha \cdot \beta = 32$:

4. Solve the equation $1 + \sqrt{x+1} = x$.

   Solution: It's an equation: what is difficult about it? How do we get rid of it?

   Now we have $x = 0$ and $x = 3$...

5. Solve for y: $(y-2)^2 - 7(y-2) + 10 = 0$.

   Solution: The routine thing to do here is to multiply out and see what happens... but there is a pattern here. What is it?

Remark

The first complication here is when the coefficient of x^2 is not 1. That is

where $a \neq 1$. Recall the aim of our endeavours is to find roots. The roots of $f(x)$ are solutions to the equation

If we take out a from the quadratic function

we have reduced to a problem we know we can solve. We either have $a = 0$ or $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$ — which we know how to solve.

Examples

Factorise the following quadratic functions and hence find the roots.

1. $f(x) = 3x^2 - 15x - 42$.

   Solution: Take out the three

   We've actually factorised this above:

   The roots are 5 and -2.

2. $f(x) = 2x^2 - x - 6$.

   Solution: Take out the 2:

   Ugh; so we want $\alpha + \beta = \frac{1}{2}$ and $\alpha \cdot \beta = -3$.

   So the roots are 2 and $-3/2$ — but that wasn't very satisfactory.

3. Solve the equation $\dfrac{6}{x} - \dfrac{5}{2x-1} = 1$.

   Solution: It's an equation; what is difficult about it:

Remark

An equivalent (and in my opinion far, far superior — this method handles cases with $a \neq 1$ much better) method for finding factors of a quadratic f is as follows. Find a re-writing of $f(x) = ax^2 + bx + c$ as

such that $b = k + m$ (so as not to change the quadratic) and $km = ac$. Then you will be able to factorise as we did before and the roots of $f(x)$ will apparent. In practise this method is known as *doing the cross*.

Examples

1. Solve the equation $3x^2 + 10x = 8$.

   Solution: Firstly this is not yet in the form we want which is quadratic equal to zero:

   Now I find factors of $3 \times (-8) = -24$ that I can combine to make 10 and hence rewrite $+10x$:

   Now I take out what is common out of the first two terms and then take something out of the second two terms that will leave a common term to both. Like this:

   Now I use the No Zero Divisors Theorem:

2. Solve the equation $12x^2 - 4x - 5 = 0$.

   Solution:

3. Factorise $2x^2 - 19x + 9$.

   Solution:

4. Factorise $5x^2 + 7x - 3$.

   Solution: O.K.; let us find a rewriting of 7 that is made of the factors of $5 \times (-3) = -15$...

Sometimes the quadratic will not have simple roots and then the two techniques for finding the roots will not suffice. There is a formula for these cases. Of course the formula may always be used to extract the two roots but the above methods are preferable.

Only Two Roots???

In each case we have seen that there are two roots — is this always going to be the case?

Proposition

Suppose $f(x) = ax^2 + bx + c$**.** *Then* $f(x)$ *may be re-written as:*

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b}{4a}\right).$$

Proof. Multiply out •

Remark

It can be shown that because of this, the graph of $f(x)$ is similar to the graph of $f(x) = x^2$ (it is got by translating and stretching the axes.):

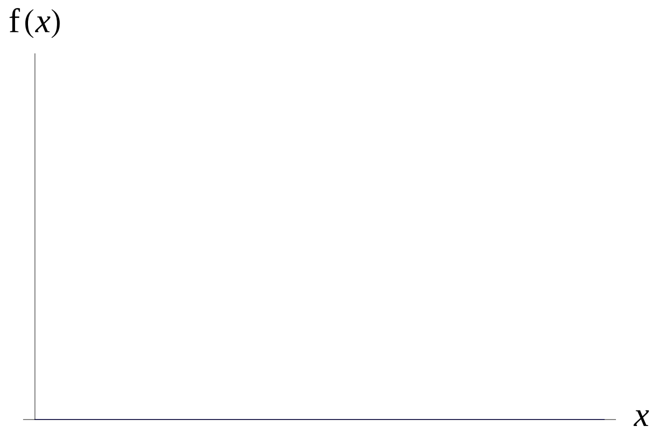

Figure 3.4: Quadratic functions look like x^2 — hence they can only cut the x-axis at most twice.

In fact we can go further and solve to show the above rewriting of $f(x)$ to show that:

Proposition

Suppose $f(x) = ax^2 + bx + c$**.** *Then the roots of* $f(x)$ *are given by:*

$$x_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{3.1}$$

Proof. Solve

$$a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b}{4a}\right) = 0.$$

Remark

Examining this formula, note that if $b^2 - 4ac < 0$ then there is no (real) number equal to $\sqrt{b^2 - 4ac}$ as a real number squared is always positive. The roots are said to be *unreal* and later it will be seen that they are *complex*.

In this case the graph of $f(x)$ does not cut the x-axis at any point:

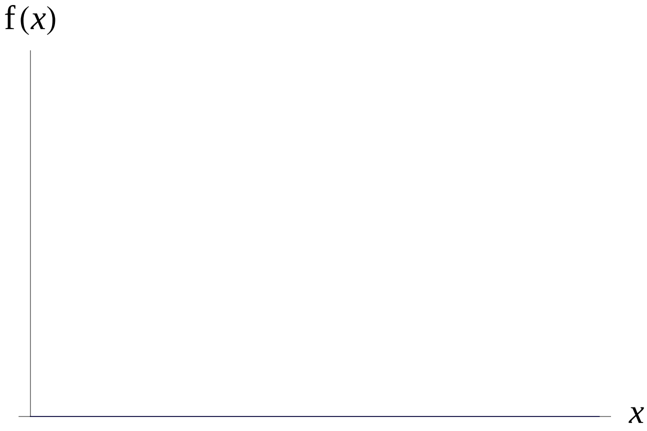

Figure 3.5: Quadratic functions with $b^2 - 4ac < 0$ have no real roots.

If $b^2 - 4ac = 0$ then the roots are real and equal,

In this case the graph has as a tangent the x-axis:

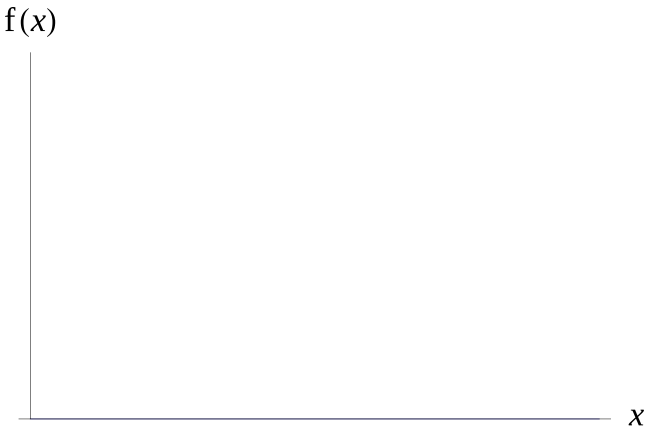

Figure 3.6: Quadratic functions with $b^2 - 4ac = 0$ have two real, repeated roots.

It is only in the case that $b^2 - 4ac > 0$ do we have two distinct, real roots.

Examples

1. Find the roots of $f(x) = 5x^2 + 7x - 3$ correct to two decimal places.

   Solution: We have already seen that we can't factorise this so we use the formula:

2. Show that $3x^2 + 4x + 9$ has no real roots.

   Solution: We simply examine $b^2 - 4ac = 16 - 4(3)(9) < 0$ which implies that the roots are unreal/complex.

Exercises

To check your answer, substitute in for x. Please use the α-β method or the cross method. You can use your own method in tests & exams.

1. Find the roots of the following functions:

$$(i)\,(2x-5)(x+3) \quad (ii)\,2x(3x-1) \quad (iii)\,x^2-8x+12$$
$$(iv)\,x^2-7x-18 \quad (v)\,x^2-9x-10 \quad (vi)\,x^2+5x-36$$
$$(vii)\,2x^2-x-6 \quad (viii)\,3x^2+x-10 \quad (ix)\,3x^2+19x-14$$
$$(x)\,5x^2-13x+6$$

2. Factor and solve for x

$$(i)\,3x^2-4x=0 \quad (ii)\,3x^2-7x=0$$
$$(iii)\,4x^2-1=0 \quad (iv)4x^2-49=0$$

3. Solve for x:

$$(2x-5)(x-2)=15.$$

4. Solve the equation:

$$(x+3)^2=(x+1)(2x+3).$$

5. Solve

$$\log(x-1)+\log(x+2)=2\log(x+2).$$

6. Solve the equation

$$x^2-13x+42=0.$$

   Hence solve the equation

$$(2t-3)^2=13(2t-3)+42=0.$$

7. Solve the equation

$$2x^2+9x-5=0.$$

   Hence solve the equation

$$2\left(x-\frac{1}{2}\right)^2+9\left(x-\frac{1}{2}\right)-5=0.$$

8. Solve the equation

$$6x^2-11x-10=0.$$

   Hence solve the equation

$$6(t-1)^2-11(t-1)-10=0.$$

9. Solve the equation
$$x^2 - 19x + 70 = 0.$$
Hence solve the equation
$$y^4 - 19y^2 + 70 = 0.$$

10. Solve each of the following equations
$$\begin{array}{cc} (i) & \frac{x+7}{3} + \frac{2}{x} = 4 \\ (ii) & \frac{1}{x+1} + \frac{x}{5} = 1 \\ (iii) & \frac{1}{x+1} - \frac{1}{x+2} = \frac{1}{2} \\ (iv) & \frac{2}{x-1} - \frac{2}{x} = \frac{1}{3} \\ (v) & \frac{3}{x-1} - \frac{2}{x+1} = 1 \\ (vi) & \frac{5}{2x+3} - \frac{2}{4x-3} = \frac{1}{3} \end{array}$$

11. Solve the equation $x^2 + 5x - 14 = 0$. Hence find the four values of t that satisfy the equation
$$\left(t - \frac{8}{t}\right)^2 + 5\left(t - \frac{8}{t}\right) - 14 = 0.$$

12. Solve the equation $x^2 - 12x + 27 = 0$. Hence find the four roots of the function
$$g(t) = \left(2t - \frac{5}{t}\right)^2 - 12\left(2t - \frac{5}{t}\right) + 27.$$

13. Find the roots of the following functions, correct to two decimal places:
(i) $x^2 + 2x - 5$ (ii) $3x^2 - x - 1$ (iii) $5x^2 - 4x - 2$

14. Solve for x: (i) $3x^2 + 7x = 2$ (ii) $3x^2 + 5x = 3$

15. Express the following equations in the form $ax^2 + bx + c = 0$ and hence use the quadratic formula to solve for x correct to two decimal places:
$$(i)\ \frac{7}{x} = 3 + 2x.$$
$$(ii)\ \frac{1}{x+1} + \frac{2}{x-3} = 4.$$

16. Find the roots of:
$$3x^2 - 2x - 2.$$
Hence find the roots, correct to one decimal place, of the function
$$g(z) = 3(2z - 1)^2 - 2(2z - 1) - 2.$$

17. Given that $x^2 + y^2 + z^2 = 29$, calculate the two possible values of y when $x = 2$ and $z = -3$.

18. If 2 is a root of $f(x) = 3x^2 - 4x + c = 0$, find the value of c.

19. Find the two values of y which satisfies the equation
$$y^3 + \frac{27}{y^3} = 28.$$

20. A real number is such that the sum of the number and its square is 10. Find the two real numbers which have this property.

21. Two consecutive numbers have product 42. Write down an equation and hence find two solutions (e.g. 5 and 6 AND 11 and 12).

22. Find the values of x for which:
(i) $\sqrt{x+3} = x - 3$ (ii) $x + \sqrt{x} = 2$.

23. * Show that for all values of $k \in \mathbb{R}$, $f(x) = kx^2 - 3kx + k$ has real roots.

24. * Show that the equation $(x-1)(x+7) = k(x+2)$ always has two distinct solutions, where $k \in \mathbb{R}$.

25. * By plugging in
$$x_+ = \frac{-b + \sqrt{b^2 - 4ac}}{2a},$$
into $f(x) = ax^2 + bx + c$, show that x_+ is a root of $f(x)$.

26. * By multiplying out, prove that:
$$f(x) = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b}{4a}\right). \tag{3.2}$$
where $f(x) = ax^2 + bx + c$.
Now solve for x, thus deriving the formula for the roots of a quadratic function.

3.3.3 Transposition of Formulae

These are solved exactly the same as above. Remember letters just stand for real numbers — manipulate them as such.

Examples

1. Solve for x: $2y = \frac{x}{4} - 3$.

   Solution:

2. Given $a = b + (c^2 - 1)d$ write c in terms of a, b and d.

   Solution:

3. Solve for d: $a = \frac{c}{2s}\left(\frac{h^2}{d-h}\right)$.

   Solution:

A Slight Complication

Sometimes the variable which you are trying to isolate appears twice:

Examples

1. If $x = \dfrac{t-3}{2t-1}$, write t in terms of x.

   Solution:

2. Solve for b: $\dfrac{ab}{3} = \dfrac{b}{2} + c$.

   Solution:

3. Solve for n: $p = \sqrt{\dfrac{m+n}{n}}$

   Solution

4. Solve for x: $y^3 = \sqrt{\dfrac{1+x}{1-x}}$.

   Solution: At each stage remove the problem by doing the same thing to both sides. What is in the way?

Exercises

In the following we have "Lhs = Rhs; variable". Solve for "variable".

1. $E = P + k$; k.
2. $F = ma$; m.
3. $y = mx + c$; m.
4. $E = mc^2$; c.
5. $E = V/R$; R.
6. $v^2 = u^2 + 2as$; u.
7. $t = (3+v)k + c$; v.
8. $s = ut + at^2/2$; a.
9. $z = p/(2s+q)$; s.
10. $s = a/(1-r)$; r.
11. $A = 4\pi r^2$; r.
12. $x = \sqrt{y+z}$; y.
13. $E = mgh + mv^2/2$; m.
14. $v = \pi h(R^2 + Rr + r^2)/3$; h.
15. $s = n(2a + (n-1)d)/2$; d.
16. $A = P(1+r)^3$; r.

17. $3\log_a y - \log_a(x+1) = \log_a 2$, y.

Selected answers:

6. $m = \dfrac{y-c}{x}$
9. $u = \sqrt{v^2 - 2as}$
12. $\dfrac{p-q^2}{2z}$
15. $x^2 - z$
18. $\dfrac{2(s-na)}{n(n-1)}$

3.3.4 Simultaneous Equations

Suppose that you want to find a pair of values (x, y) that have the property that

$$\begin{aligned} x + y &= 12 \\ 3x - 6y &= -2 \end{aligned}$$

We call this a *set of simultaneous equations*. In terms of an applied example, x and y might be some unknowns: the first equation represents some relationship/condition between x and y and the second equation refers to another. We might want to find a pair (x, y) that satisfies both relationships/conditions: that satisfies *both equations*.

We will learn in the next section that the set of points that satisfy an equation of the form

constitutes a line. Therefore, geometrically at least, finding the solution of a set of simultaneous equations is the same as finding the intersection of two lines

The Substitution Method

Given two *simultaneous equations*, for example:

$$\begin{aligned} 3x - 4y &= 4 \\ x - y &= 6 \end{aligned}$$

Step 1: Take the first equation and write y (or x if you so please) in terms of x:

Step 2: Now replace any instance of y (respectively x) in the second equation with y in terms of x — i.e. eliminate y from the second equation so it is an equation in x alone:

Step 3: Now solve this for x:

Step 4: Now substitute back into y in terms of x to find y:

Examples

1. Solve the simultaneous equations:

$$\begin{aligned} 3x - 9y &= -6 \\ x - y &= 2 \end{aligned}$$

   Solution: First find perhaps x in terms of y:

   Now write x in terms of y and solve for y:

   Hence find x:

   Therefore the answer is $(4, 2)$.

2. Solve the simultaneous equations:

$$\begin{aligned} x - 0.9y &= -0.6 \\ 2.1x - 4.1y &= 3 \end{aligned}$$

   Solution: First find perhaps x in terms of y:

   Now write x in terms of y and solve for y:

   Hence find x:

   Therefore the answer, correct to two decimal places, is $(2.34, -1.98)$.

3. Twelve workmen on a building site earn a total of €6,050 per week between them. Labourers earn €450 per week and Tradesmen earn €580 per week. How many of each is employed? **Note that this question is the same as the one from the Sample Test**.

   Solution: Here there are two unknowns. When we don't know what something is we should say

   Now there are two pieces of information here. First the total number of workmen is twelve. This induces an equation:

   If one labourer makes €450 per week, two labourers make €450×2 per week, three labourers make €450×3 per week how much money do x labourers make?

   Hence we have simultaneous equations. First find perhaps x in terms of y:

   Now write x in terms of y and solve for y:

   Hence find x:

   Therefore the answer is $(7, 5)$.

4. Solve the simultaneous equations:

$$x - y = -2$$
$$y - 4x + x^2 = 0$$

   Solution: First find perhaps y in terms of x:

Now write x in terms of y and solve for y:

Hence find y:

Therefore the answers are $(2,4)$ and $(1,3)$.

Exercises
Solve the following simultaneous equations:

(i)

$$\begin{aligned} 5x + 7y &= 0 \\ -3x + 4y &= 2 \end{aligned}$$

(ii)

$$\begin{aligned} 5a + 7b &= 0 \\ 6a - 8b &= -4 \end{aligned}$$

(iii)

$$\begin{aligned} 5x_1 + 7x_2 &= 0 \\ 7x_1 + 18x_2 &= 2 \end{aligned}$$

Selected Solutions: Check answers by substituting in your values.

3.4 Functions Again*

A function is like a black box that when fed a number x spits out another number $f(x)$. A number x is inputted, something happens in the black box, and $f(x)$ is the output. For every input there is a *single* output. Say $f(2) = 4$ or -4; such an f is not a function. Technically, a function is a *rule that associates each number in a set (domain) with a single number in another set (co-domain).* We usually consider functions where the domain and codomain are real numbers. The number $f(x)$ is the *value of* f *at* x. The *range* of f is the set of all possible values of $f(x)$ as x varies throughout the domain.

Example

$f(x) = x^2$ is a function.

$$f : \mathbb{R} \to \mathbb{R},\ x \mapsto x^2.$$

Here the domain is the set of real numbers, $\mathbb{R}$. The co-domain is also $\mathbb{R}$. Each number in $\mathbb{R}$ is associated with a single number in $\mathbb{R}$; $2 \mapsto 4$, $-1 \mapsto 1$, $\sqrt{2} \mapsto 2$, etc. The range of f is $[0, \infty)$ - the non-negative real numbers.

The most common method for visualising a function is its graph. If f is a function, then its graph is the set of pairs:

$$\text{graph}(f) = \{(x, f(x) \,|\, x \in \mathbb{R})\}. \tag{3.3}$$

The graph of $f(x) = x^2$ in $[-1, 1]$:

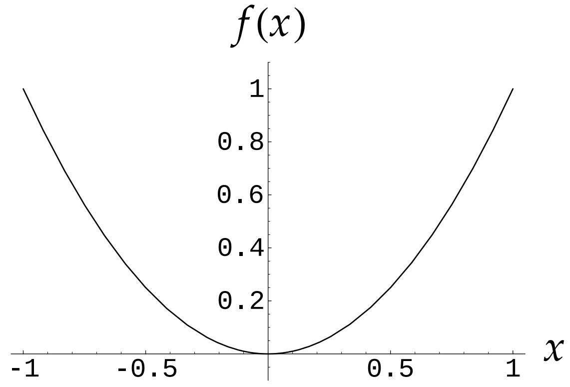

Figure 3.7: The graph of $f(x) = x^2$.

Many functions are continuous — that is their graph can be drawn without lifting the pen from the page.

3.4.1 Types of Functions

Common functions include:

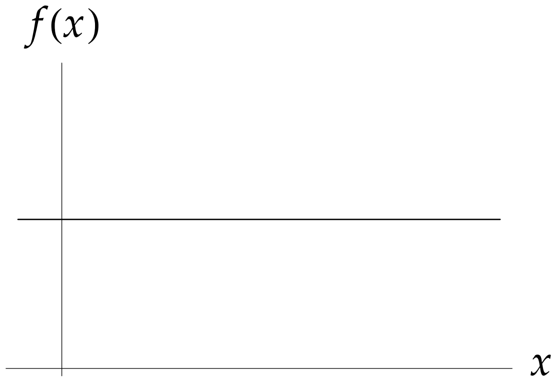

Figure 3.8: Constant function; e.g. $f(x) = 4$.

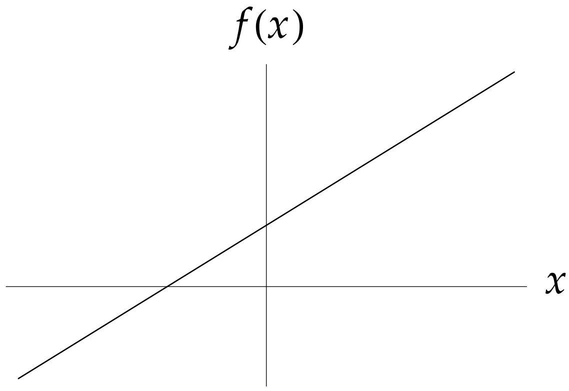

Figure 3.9: Linear function; e.g. $f(x) = 2x + 4$.

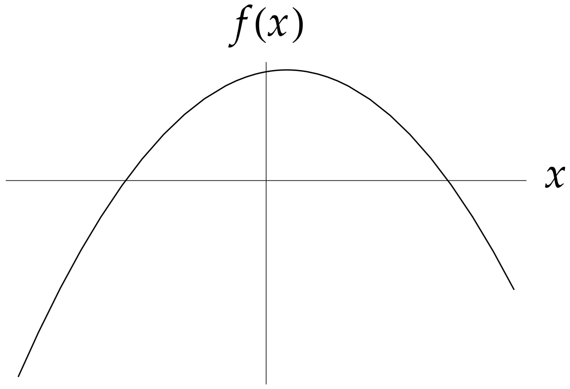

Figure 3.10: Quadratic function; e.g. $f(x) = -6x^2 + 2x + 10$.

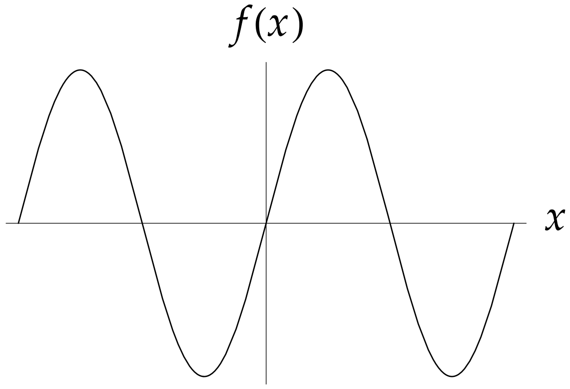

Figure 3.11: Trigonometric function; e.g. $f(x) = \sin x$.

3.4.2 Properties of Functions

Many questions that can be asked about such functions are answered by looking at the graph of the function.

Range

The range of f:

$$\text{Range}(f) = \{f(x) \mid x \in \mathbb{R}\}, \tag{3.4}$$

can be easily seen from the graph of f.

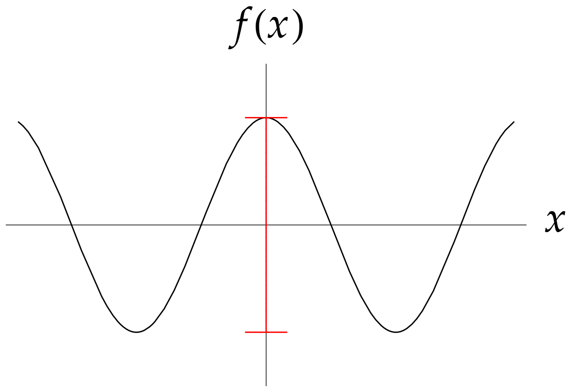

Figure 3.12: If the function is *continuous*, the interval between $y_{\max}$ and $y_{\min}$, $[y_{\min}, y_{\max}]$ comprises the range.

Period of a Function

If a function f is such that:

$$f(x + p) = f(x), \; \forall\, x \in \mathbb{R}, \tag{3.5}$$

f is said to have period p. This can be seen in a graph:

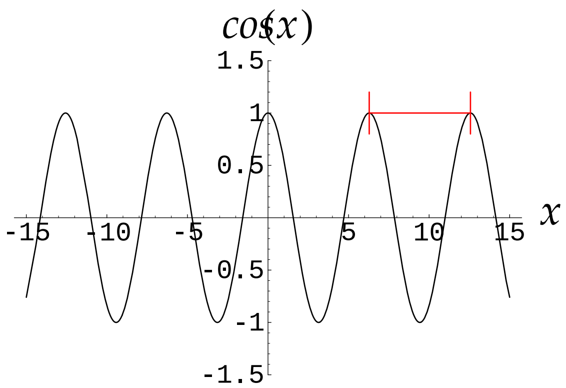

Figure 3.13: $f(x) = \cos x$ repeats itself every $2\pi \approx 6.28$; hence has period $p = 2\pi$

Roots or Zeros of a Function

Let f be a function. A *root* or *zero* of f is a solution of the equation $f(x) = 0$. Given a graph of $f(x)$, the set of roots of f is the set of points where f cuts the y-axis (these points are $(x, 0)$ - the x is a root).

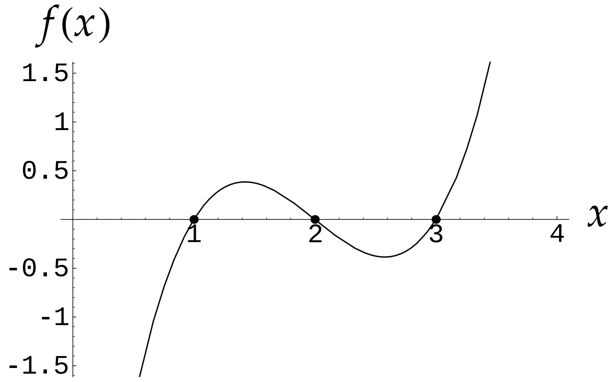

Figure 3.14: The function $f(x) = x^3 - 6x^2 + 11x - 6$ has roots $x = 1, 2, 3$. This is obvious if it is noted $f(x) = (x - 1)(x - 2)(x - 3)$ - if not the graph shows roughly where the zeros are.

Maxima and Minima

It will soon be seen that to locate the local maxima and minima of a function f; the equation:

$$f'(x) \overset{!}{=} 0, \tag{3.6}$$

is solved for x. A graph of f also shows these local maxima and minima:

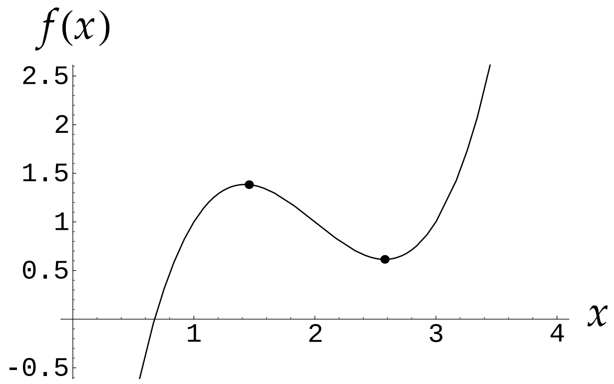

Figure 3.15: This cubic function has a global maxima and mimina at the two points as shown.

3.4.3 The Duality Principle*

We have seen that every function, an algebraic object, gives rise to a geometric object, its graph:

Now does every geometric object, graph, correspond to an algebraic object? Yes is the answer. Given any curve $S \subset$ Plane there is a function $f(x, y)$ such that

Therefore we have a duality principle:

> *A point (x_0, y_0) is on a curve if and only if it satisfies the equation of the curve.*

Example: The Equation of a Line

What do we need to define a line:

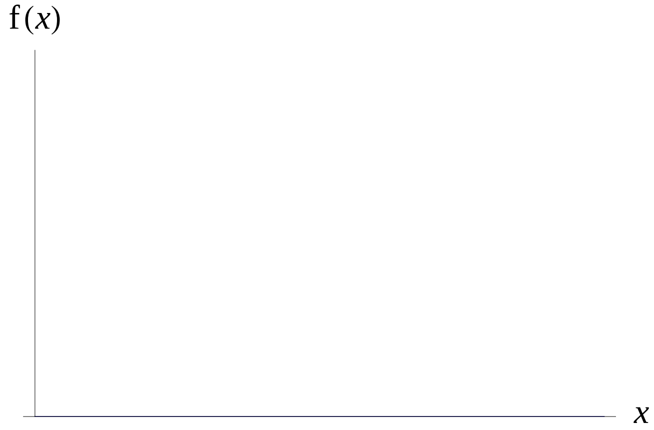

Figure 3.16: A point and a slope defines a line. What does it take for a point (x, y) to be on this line?

(x, y) is on the line if and only if the slope from (x_1, y_1) to (x, y) is equal to m. What is slope?

One of the most remarkable and powerful aspects of duality and hence mathematics is that many questions about geometry may be answered algebraically.

Example: The Intersection of Two circles

Consider the two circles as shown. Find the coordinates where they intersect.

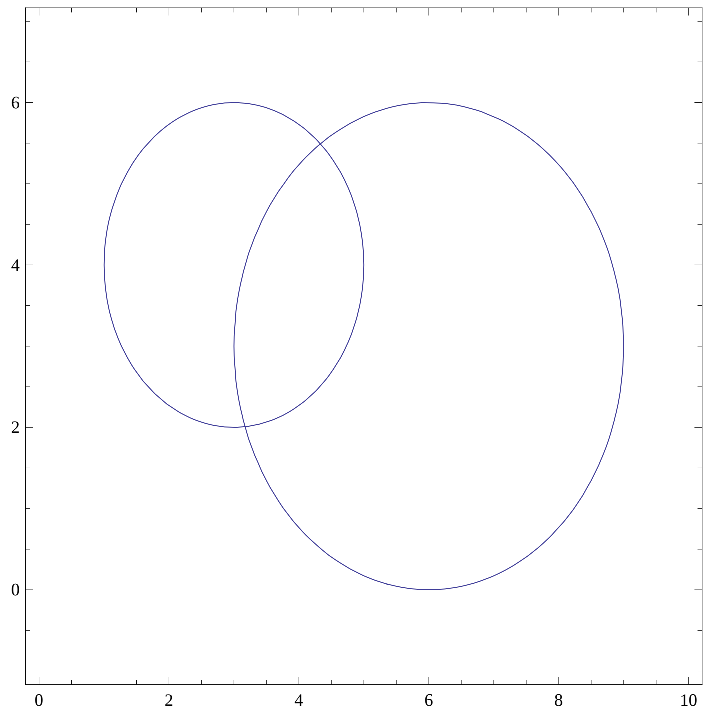

Figure 3.17: Find the coordinates of the intersections of the two circles. The left circle has centre $(3, 4)$ and radius 2. The right-circle has centre $(6, 3)$ and radius 3.

Solution: We show how to answer this algebraically. First of all what is a circle?

Now take a general point (x, y) on the plane. What does it take for the point (x, y) to be on the left circle? It can be either too far from the centre, too close to the centre or just right:

That is (x, y) is on the circle if and only if the distance from (x, y) to $(3, 4)$ is equal to 2. We can calculate the distance from a general point (x, y) to $(3, 4)$ using the distance formula from coordinate geometry[3]:

[3]this formula comes from Pythagoras Theorem

What we have here is a dictionary between the *geometric*

and the *algebraic*

If we take the general case of a circle with radius r and centre (h, k) we can see that all the points of such a circle satisfy the equation

$$(x - h)^2 + (y - k)^2 = r^2. \tag{3.7}$$

So we have a dictionary that sends the geometric object "circle with centre $(6, 3)$ and radius 3" to the algebraic object

Now what are intersections of curves?

So to find the intersections we find points (x, y) that satisfy

and

at the same time:

Conversely many questions about algebra can be answered geometrically!

Example: Difficult Algebra Problem

Consider the family of simultaneous system of equations

$$x^2 + y^2 + 2ax + 2by + c = 0$$
$$x^2 + y^2 + dx + ey + f = 0$$

where $a, b, c, d, e, f \in \mathbb{R}$ are constants. How many solutions can elements of this family have?

Solution: A careful rewriting shows that these equations may be rewritten as

$$(x - a)^2 + (y - b)^2 = (\sqrt{a^2 + b^2 - c})^2$$
$$(x - d)^2 + (y - e)^2 = (\sqrt{d^2 + e^2 - f})^2$$

So both of these equations represent circles so the question asks how many intersections can two circles have:

So the answer is none, one, two or an infinite number.

Exercises

1. True-False Quiz: Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why. If it is false, explain why or give an example that disproves the statement.
   (a) If f is a function, then $f(s+t) = f(s) + f(t)$.
   (b) If $f(s) = f(t)$ then $s = t$.
   (c) If f is a function, then $f(3x) = 3f(x)$.
2. An airplane flies from an airport and lands an hour later at another airport, 400 miles away. If t represents the time in minutes since the plane has left the terminal building, let $x(t)$ be the horizontal distance travelled and $y(t)$ be the altitude of the plane.
   (a) Sketch a possible graph of $x(t)$.
   (b) Sketch a possible graph of $y(t)$.
   (c) Sketch a possible graph of the ground speed.
   (d) Sketch a possible graph of the vertical speed.
3. If $f(x) = 3x^2 - x + 2$, find $f(2)$, $f(-2)$, $f(a)$, $f(-a)$, $f(a+1)$, $2f(a)$, $f(2a)$, $f(a^2)$, $[f(a)]^2$ and $f(a+h)$.
4. A spherical balloon with radius r inches has volumes $V(r) = \frac{4}{3}\pi r^3$. Find a function that represents the amount of air required to inflate a balloon from a radius of r inches to a radius of $r+1$ inches.
5. A rectangle has perimeter 20 m. Express the area of the rectangle as a function of the length of one of its sides.
6. A taxi company charges two euro for the first mile (or part of a mile) and 20 cent for each succeeding tenth of a mile (or part). Sketch the cost function C (in euros) of a ride as a function of the distance x travelled (in miles) for $0 < x < 2$.
7. (a) Find an equation for the family of lines with slope 2 and sketch several members of the family.
   (b) Find an equation for the family of lines such that $f(2) = 1$ and sketch several members of the family.
   (c) Which line belongs to both families.
8. The relationship between the Fahrenheit (F) and Celsius (C) temperature scales is given by the line $F = \frac{9}{5}C + 32$.
   (a) Sketch a graph of this function.
   (b) What is the slope of the graph and what does it represent.

3.4.4 Appendix: The Axioms of the Real Numbers

Closure

For any x, $y \in \mathbb{R}$;

$$x + y \in \mathbb{R},$$
$$x \times y \in \mathbb{R}.$$

Commutivity

For any x, $y \in \mathbb{R}$;

$$x + y = y + x,$$
$$x \times y = y \times x.$$

Associativity

For any x, y, $z \in \mathbb{R}$;

$$x + (y + z) = (x + y) + z,$$
$$x \times (y \times z) = (x \times y) \times z.$$

Identity

There is a special real number $0 \in \mathbb{R}$ such that:

$$0 + x = x,$$

for every $x \in \mathbb{R}$. Also there is a special number $1 \in \mathbb{R}$ $(1 \neq 0)$ such that:

$$1 \times x = x,$$

for every $x \in \mathbb{R}$.

Subtraction and Division

For every number x there corresponds a number $-x \in \mathbb{R}$ such that:

$$x + (-x) = 0.$$

Also if $x \neq 0$ there is a number $x^{-1} \in \mathbb{R}$ $(x^{-1} = 1/x)$ such that:

$$x \times x^{-1} = 1.$$

Distributive Law

For x, y, $z \in \mathbb{R}$,

$$x \times (y + z) = (x \times y) + (x \times z).$$

Chapter 4

Graphs

11:15 Restate my assumptions:

1. *Mathematics is the language of nature.*
2. *Everything around us can be represented and understood through numbers.*
3. *If you graph these numbers, patterns emerge. Therefore: There are patterns everywhere in nature*

Max Cohen

4.1 Introduction

A definition of science? How about

> *A science is any study that defines quantities and explores the relationships between them.*

This approach suggests that Mr. Cohen's third assumption should read:

> *There are patterns*[1] *everywhere in nature. Therefore: if you graph these numbers patterns emerge.*

How does science emerge? Science happens when someone asks a question. For example, why does an apple fall to Earth? After this the scientific method goes through a number of stages typically:

1. hypothesis: someone answers the question:
2. prediction: someone determines the consequences of the hypothesis
3. experiment: someone tests a prediction of the hypothesis
4. analysis: if the results of the experiment are particularly 'good' and the hypothesis not only answers the question but explains why it is so, the hypothesis may be accepted as a *theory*.

It is the experiment/analysis stage that makes astronomy a science and astrology... not a science.

[1] relationships/connections/etc

As an example consider the question:

> How does a light ray behave when it goes from one medium to another?

1. hypothesis: the principle of least time
2. prediction: the ratio of the sine of the angle of incidence to the sine of the angle of refraction is a constant
3. experiment: how might we do this?
4. analysis: a subject of this chapter

There are two main things that we will be doing in this chapter:

1. Suppose that we have two quantities/variables x and y which we *believe* have a relationship of the form:

   We can verify this relationship *experimentally* by graphing both x and y on an x-y plane and showing that indeed x and y share the relationship.
2. Suppose that we have two quantities/variables x and y which we *know* have a relationship of the form:

   for some constants a and b. By plotting x against y we will actually be able to measure a and b.

Before we start thinking about these things, however, we must do a study of coordinate geometry.

4.2 Coordinate Geometry: Lines

Just as points on a numberline can be identified with real numbers, points in a plane can be identified with pairs of numbers. We start by drawing two perpendicular coordinate *axes* that intersect at the origin O on each line (where the coordinate in $(0,0)$). Usually one line is horizontal with positive direction to the right and is called the x-axis; the other line is vertical with positive direction upward and is called the y-axes:

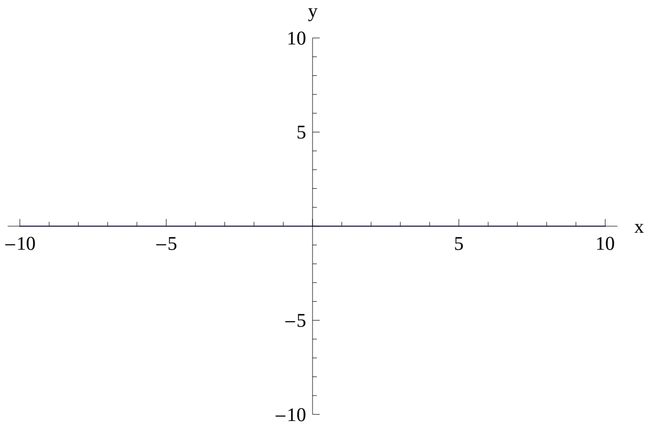

Figure 4.1: When graphing a function, usually the x-axis describes the input variable and the y-axes describes the output variable. To each point P we can associate a pair (a, b) and to each pair (c, d) we can associate a point Q.

This coordinate system is called the *Cartesian coordinate system* in honour of the French mathematician René Descartes. We have already shown that the equation of the line of slope m containing the point (x_1, y_1) is given by

To find the equation of L we use it's *slope*, which is a measure of the steepness of the line.

Formula: Slope

The *slope* of a line that passes through the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is:

Proof. We have

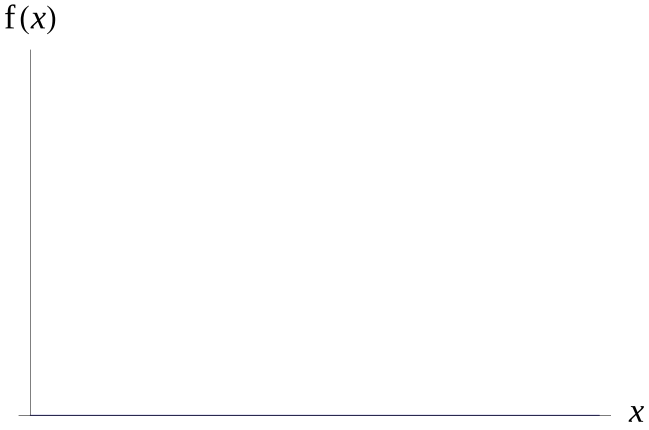

Figure 4.2: The *slope* of a line is the ratio of how much you go up, as you across: *slope* $= |\uparrow|/|\rightarrow|$

The slope of a line is constant — we can take the slope between *any* two points on the line and arrive at the same answer.

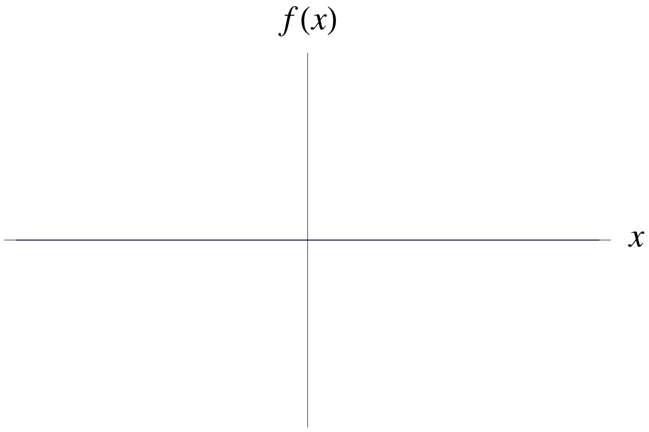

Figure 4.3: Notice that lines with positive slope slant upwards to the right, whereas lines with negative slope slant downwards to the right. Notice also that the steepest lines are those whose slope is big — be it in the positive or negative sense.

Formula: Equation of a Line I

The *equation of the line* passing through the point $P(x_1, y_1)$ and having slope m is

Examples

1. Find the equation of the line through $(1, -7)$ with slope $-1/2$.

   Solution:

2. Find the equation of the line through the points $(-1, 2)$ and $(3, -4)$.

   Solution:

Consider again the line defined by the point $(2, 3)$ and slope 4:

$$y - 3 = 4(x - 2).$$

This shows that any equation of the form:

for m, c constants is another way of writing the equation of the line.
What about c?

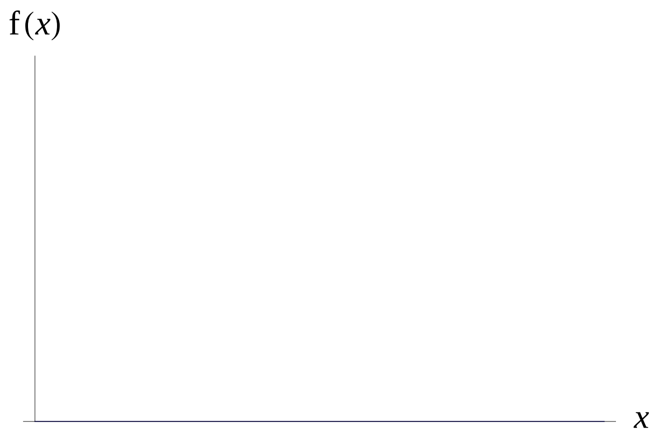

Figure 4.4: When the x-coordinate is zero, the graph of the line cuts the y-axis.

So c is where the line cuts the y-axis. I would recommend that inasmuch as possible write your lines in this form.

Formula: Equation of Line II

The equation of the line with slope m and y-intercept c is given by:

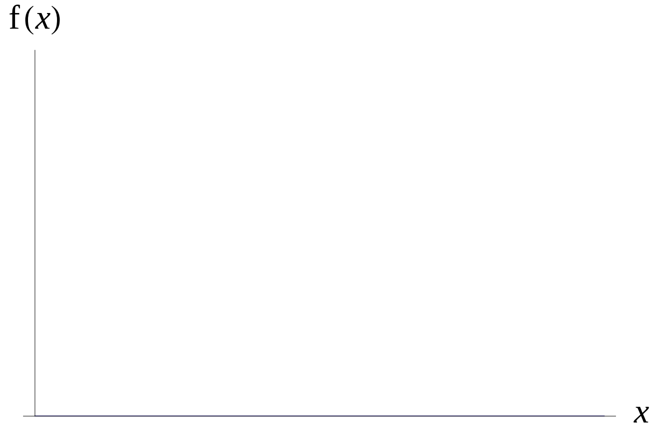

Figure 4.5: If a line is horizontal, then its slope is 0, so its equation is $y = c$, where c is the y-intercept. A vertical line does not have a slope, but we can write its equation as $x = a$, where a is the x-intercept, because the x-intercept of every point on the line is a.

Observe that every equation of the form:

$$ax + by + c = 0 \tag{4.1}$$

is that of a line:

We call an equation of the form (4.1) a *linear equation*. Many authors use the $ax + by + c = 0$ form for the equation of a line — but if you get into the habit of solving this for $y = mx + c$ you know two things about the line straight away: its slope and its y-intercept. Conversely, *every* equation of the form $y = mx + c$ or $ax + by + c = 0$ is a line.

Examples

1. Sketch the graph of the line $3x - 5y = 15$.

   Solution: Simply solve for y

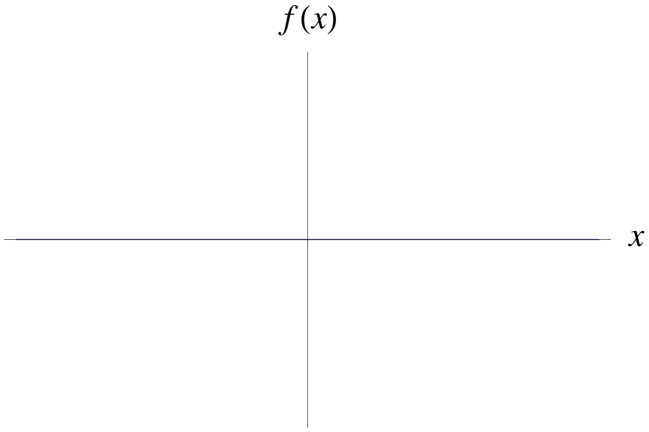

Figure 4.6: This line has a slope of $3/5$ and a y-intercept of -3.

2. Suppose we are given two lines $L_1 \equiv y = x - 1$ and $L_2 \equiv y = -x + 8$. How do we find their intersection? Find their intersection.

   Solution: What is an *intersection*:

   What does it take to be on L_1?

   What does it take to be on L_2

   What does it take to be in the intersection, which we could write $L_1 \cap L_2$ (*said* L_1 *intersection* L_2.)?

   Hence we are looking for a solution (x, y) to the *simultaneous equations:*

   Now the geometry should inform us that there is a *unique* solution[2]. One way to look at this is to say well all the points on L_1 have coordinates $(x, x - 1)$ and the lines on L_2 have coordinates $(x, -x + 8)$.

[2] unless the lines are parallel — or equal.

Is there a point where these general points coincide? Well we will need their y coordinates to coincide:

So at the point where $x = 3$ we know that $(x, x-1) = (x, -x+8)$. Hence the intersection is $(3, 2)$. This method, while nicely geometric, is not the standard method. Also this method only seems to work for lines: what about equations of the form:

Well to be honest these two equations *do* represent lines but we might not always know this. This is where the substitution method or your old row method (boo, hiss, etc.) come in.

Formula: Distance

Suppose that $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points in the plane. Then the distance between them $|PQ|$ is given by:

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}. \tag{4.2}$$

Proof. Consider

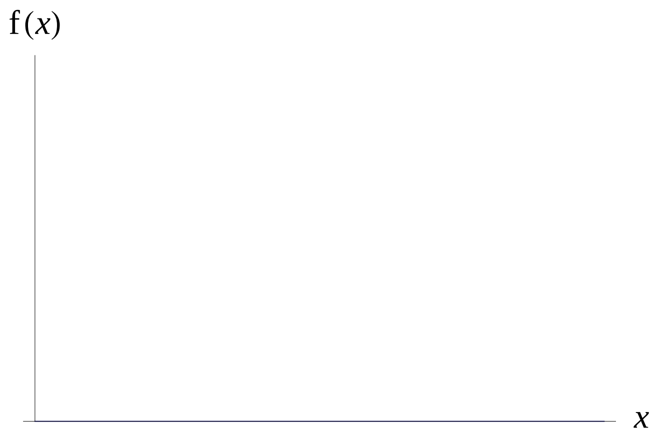

Figure 4.7: This distance formula comes straight from Pythagoras Theorem.

Parallel and Perpendicular Lines

Slopes can be used to show that lines are parallel or perpendicular:

Theorem

1. Two distinct lines are parallel if and only if they have the same slope.
2. Two lines[3] with slopes m_1 and m_2 are perpendicular if and only if

$$m_1 \times m_2 = -1. \tag{4.3}$$

Proof. 1. Suppose that L_1 and L_2 are two distinct lines with a slope of m:

To find their intersection we solve $mx + c_1 = mx + c_2 \Leftrightarrow c_1 = c_2$ — but this is impossible as the lines are distinct. Hence the lines have no intersection •

2. Usually done using a $\tan(A - B)$ formula which we haven't derived — but I have a much nicer proof! Let L_1 and L_2 be the lines and suppose first that the lines are perpendicular. Lines are *perpendicular* if the angle between them is 90°. Now translate everything to the origin:

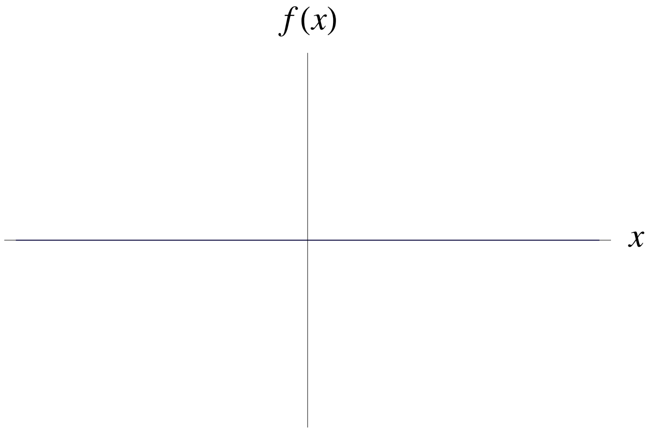

Figure 4.8: Now if L_1 is perpendicular to L_2 then $\triangle opq$ is right-angled.

Now $|op|^2 = 1 + m_1^2$ and $|oq|^2 = 1 + m^2$ by Pythagoras Theorem. Also the length marked m_2 is actually $-m_2$... Thus we have, by more Pythagoras

$$\begin{aligned} (m_1 - m_2)^2 &= 1 + m_1^2 + 1 + m_2^2 \\ \Rightarrow m_1^2 - 2m_1m_2 + m_2^2 &= 2 + m_1^2 + m_2^2 \\ \Rightarrow -2m_1m_2 &= 2 \\ \Rightarrow m_1m_2 &= -1. \end{aligned}$$

Using the converse to Pythagoras Theorem, we also have that $m_1 \times m_2 = -1$ implies that $L_1 \perp L_2$ •

[3] this fails if the lines are of the form $y = c$ or $x = a$: the slope of the line $x = a$ is undefined

Examples

1. A line contains the points $(1, 3)$ and $(5, 16)$. Find it's equation.

   Solution: As we don't know the y-intercept we should use the $y - y_1$... equation. We have a point — we must find the slope:

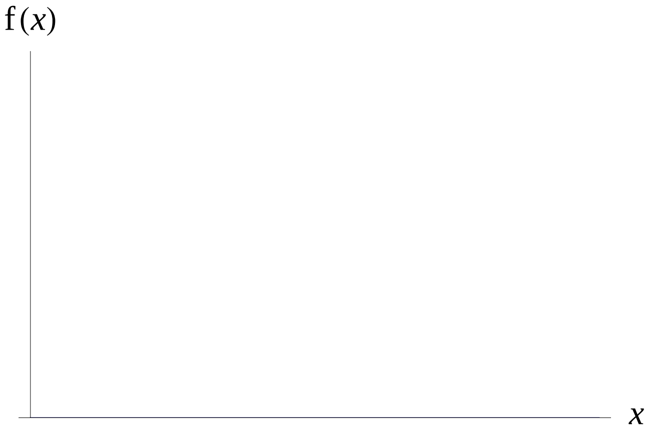

Figure 4.9: Using the equation $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ we get $m = \dfrac{13}{4}$. However don't think that slope is a formula — it is an idea. By looking at the picture we can also say that $m = \dfrac{13}{4}$.

   Hence we write:

2. Find the slope of the line $x + 3y = 2$.

   Solution: Simply write in the form $y = mx + c$:

3. Find the x and y-intercepts of the line $3x - 5y = 11$.

   Solution:

   For the x-intercept:

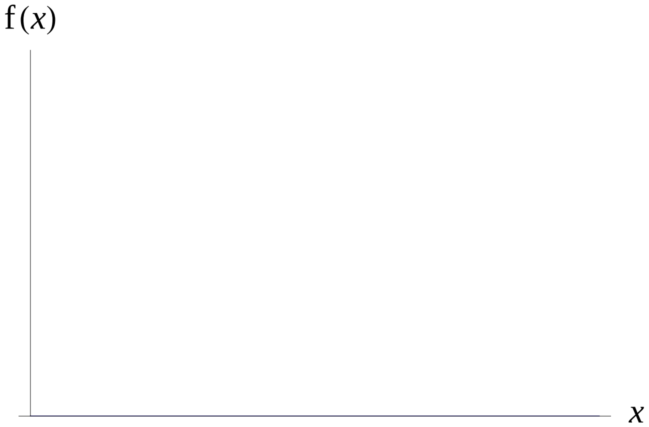

Figure 4.10: A curve intercepts the x-axis when the y-coordinate is equal to zero and intercepts the y-axis when the x-coordinate is zero.

For the y-intercept:

4. Does the point $(1, 4)$ lie above or below the line $3x - y = 3$?

   Solution: First we write $y = mx + c$:

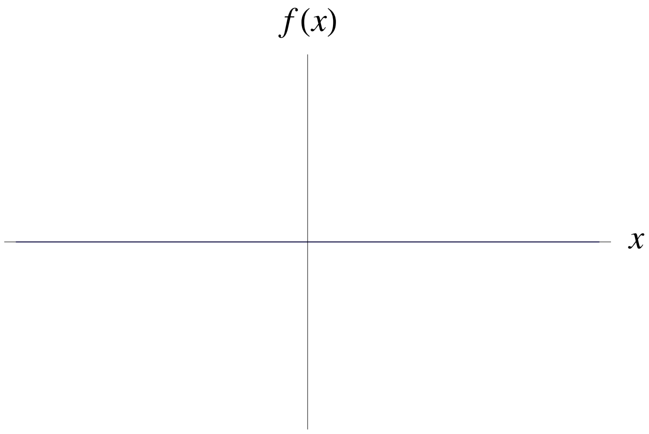

Figure 4.11: $y = 0$ at $x = 1$ so $(1, 4)$ lies above the line.

5. Find the distance between the points $(-1, 2)$ and $(3, 4)$.

   Solution: We can use the distance equation:

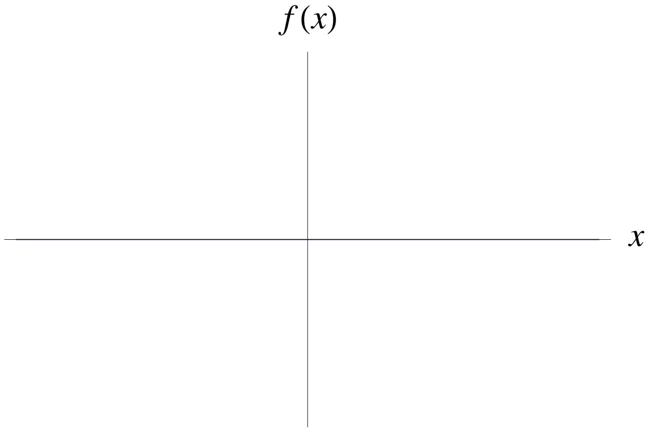

Figure 4.12: Using the equation $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ we get $d = 2\sqrt{5}$. However don't think that distance is a 'just' formula — here we can find it using Pythagoras Theorem.

6. Prove that $(4, 14)$ does not lie on the line $x - 4y = 11$.

   Solution: The equation of a curve is formulated such that

   > *A point is on a curve if and only if it satisfies the equation of the curve.*

   We have $x - 4y = 4 - 42 \neq 11$ therefore $(4, 14)$ does not lie on the line $x - 4y = 11$.

7. Find the equation of the line L through the point $(5, 2)$ that is parallel to the line $4x+6y+5 = 0$.

   Solution: If L is parallel to $4x + 6y + 5 = 0$ then it has the same slope. Hence find the slope of $4x + 6y + 5 = 0$ by solving for y:

   That is we are looking for the equation of the line of slope $-\frac{2}{3}$ containing the point $(5, 2)$:

8. Show that the lines $2x + 3y = 1$ and $6x - 4y - 1 = 0$ are perpendicular.

   Solution: Find the slopes of both

Exercises

1. Find the slope of the line through $P(-3,3)$ and $Q(-1,-6)$.
2. Use slopes to show that the triangle formed by the points $A(6,-7)$, $B(11,-3)$, $C(2,-2)$ is a right-angled triangle.
3. Use slopes to show that the points $A(-1,3)$, $B(3,11)$ and $C(5,15)$ lie on the same line.
4. Sketch the graph of
   (i) $x = 3$ (ii) $y = -2$.
5. Find the graph of the equation that satisfies the given conditions:

$$(i)\ \text{Through } (2,-3),\text{ slope } 6.$$

$$(ii)\ \text{Through } (-3,-5),\text{ slope } -7/2.$$

$$(iii)\ \text{Slope } 3,\ y\text{-intercept } -2.$$

Selected Answers: (iii) $y = 3x - 2$.

4.3 Direct and Indirect Proportion

A great many physical laws are of the form:

$$A = kB. \tag{4.4}$$

That is a physical quantity A is fundamentally related to another B in that A is a scalar multiple of B. k is called the *constant of proportionality.* In this example if B is doubled, A is doubled. If B is trebled, A is trebled. If A is halved, B is halved, etc. In this case A is said to be *directly proportional* to B, $A \propto B$.

These laws are commonly discovered when a physicist observes quantities that seem to behave in the manner of (4.4). That is two physical quantities are observed or believed to be directly proportional. At this point the physicist will run an experiment to test his or her hunch. A shall be measured as B is varied and the physicist shall collect the data and record pairs of values (B, A). When the experiment is completed, the physicist shall then graph the data collected. Note two aspects of the graph:

1. The graph of A against B is a ***straight line through the origin***
2. The ***slope*** of the graph is the ***constant of proportionality***

A different type of proportionality is when A is *inversely proportional* to B. This asserts that the quantity A is directly proportional to $1/B$:

$$A \text{ inversely proportional to } B \;\Leftrightarrow\; A \propto \frac{1}{B}. \tag{4.5}$$

In this case

$$A = k\frac{1}{B},$$
$$\Leftrightarrow AB = k.$$

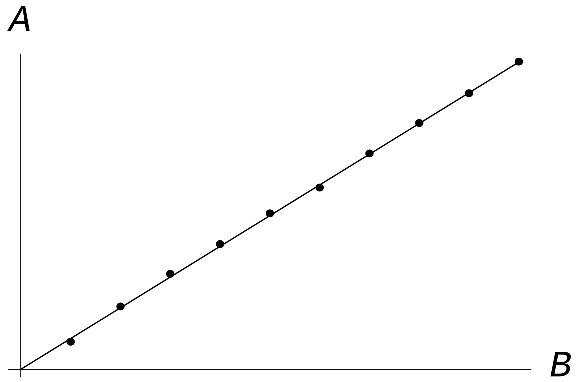

Figure 4.13: A is directly proportional to B; $A = kB$, slope $= k$.

4.4 Graphing

If A vs B is being graphed, the following guidelines should be followed[4]:

1. The independent variable is put on the x-axis, the dependent variable is put on the y-axis. i.e. if testing $A = kB$; $A \sim y$ is the dependent variable, $B \sim x$ is the independent variable.
2. Ascertain the maximal and minimal values of A and B. Use these to choose an appropriate scaling[5].
3. Title the graph: *Graph of A vs B.*
4. Annotate axes by:
$$\frac{\text{physical quantity}}{\text{unit}}$$
5. Put ticks on every large box and annotate same.
6. Zeroes about the origin.
7. Place the data points on the graph with *arrowheads.*
8. *Produce* the curve as required. This means for a $A = kB$ graph produce a ***straight line through the origin***!

4.4.1 Example

In an experiment to verify Snell's Law and hence measure the refractive index of glass, the following data was recorded:

Angle of Incidence, $i/°$	**Angle of Refraction**, $r/°$
10	7.5
15	11
20	15
25	18.5
30	22
35	22.5
40	29

[4] we'll do an example

[5] you'll have to generate your own feel for scaling the axes

Use this data to produce a graph that verifies Snell's Law and hence calculate the refractive index of glass.

Linear Models

When we say that an output y is a *linear function* of an input x, we mean that the graph of the function is a line, so we can use the slope-intercept form of the equation of a line to write a formula for the function as:

where m is the slope of the line and c is the y-intercept. A characteristic feature of linear functions is that they grow at a constant rate. For instance

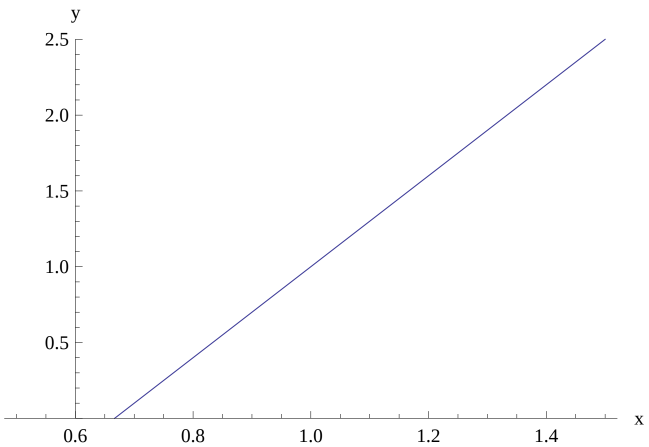

Figure 4.14: A graph of the linear function $y = 3x - 2$. Note that whenever x increases by 0.1, the value of y increase by 0.3. So y increases 3 times as fast as x. Thus, the slope of graph of $y = 3x - 2$, namely 3, can be interpreted as the rate of change of y with respect to x.

Example

As dry air moves upwards, it expands and cools. If the ground temperature is 20°C and the temperature at a height of 1 km is 10°C, express the temperature T (in ° C) as a function of the height h (in kilometres), assuming that a linear model is appropriate. Draw the graph of the function. What does the slope represent? What is the temperature at a height of 2.5 km?

Solution: First we suppose that temperature is a linear function of height:

Now that means we just have to find the equation of the line going through the points $(0, 20)$ and $(1, 10)$:

Figure 4.15: We could find the slope between these two points but why not just use a $y = mx + c$ model? The slope is -1, the y-intercept is 20 so we have $T = -10h + 20$.

The slope represents the rate of change of temperature with respect to height: a change in height of 1 km results in a change in temperature of $-10°$ C. We *extrapolate* that the temperature at a height of 2.5 km is given by:

If there is no physical law or principle to help us formulate a model, we construct an *empirical model*, which is based entirely on collected data. We seek a curve that "fits" the data in the sense that it captures the basic trend of the date points.

Example

The below table lists the average carbon dioxide level in the atmosphere, measured in parts per million at Mauna Loa Observatory from 1980 to 2000. Use the data to find a linear model for the carbon dioxide level. Use the linear model to estimate the average CO_2 level for 1987 and to predict the level for 2010. According to this model, when will the CO_2 level exceed 400 parts per million?

Year	CO_2 level (in ppm)
1980	338.7
1982	341.1
1984	344.4
1986	347.2
1988	351.5
1990	354.2
1992	356.4
1994	358.9
1996	362.6
1998	366.6
2000	369.4

Solution: O.K. the first thing we must do is graph this data. Let $y =$ be the CO_2 level and x the year. Now *roughly* draw a line of best fit to this data. This line is of the form:

Now to find out the 'best' values of m and c we find the equation of the line of best fit. To do this take two points on the line of best fit; say

Now find the slope between them

Now use the equation of the line and write it in the form $y = mx + c$:

Now we have $y = f(x)$ and we can *interpolate* the CO_2 level in 1987:

Extrapolate the CO_2 level in 2010:

Finally the CO_2 level hits 400 when $y = 400$:

4.5 Reduction of a Non-linear Relationship to a Linear Form*

4.5.1 Non-Linear Relationships

We saw earlier that many mathematical relationships take the form of a linear relationship:

In particular, if we know that some process is a linear one, but we don't know the *parameters* m and c, then we can make measurements of y and x, and plot a straight-line graph of y vs x:

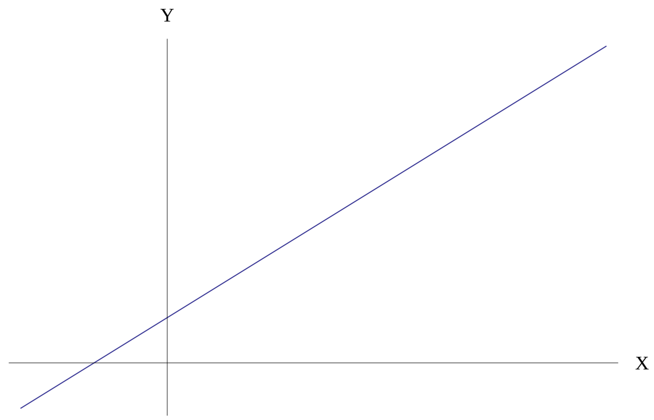

Figure 4.16: If y and x are related via $y = mx + c$, then the graph is a straight-line and we know that m is the slope and c is the y-intercept.

Example: Radioactive Decay

However not all mathematical relationships take such a tractable form. For example, it can be shown that the decay of a radioactive material is modeled by:

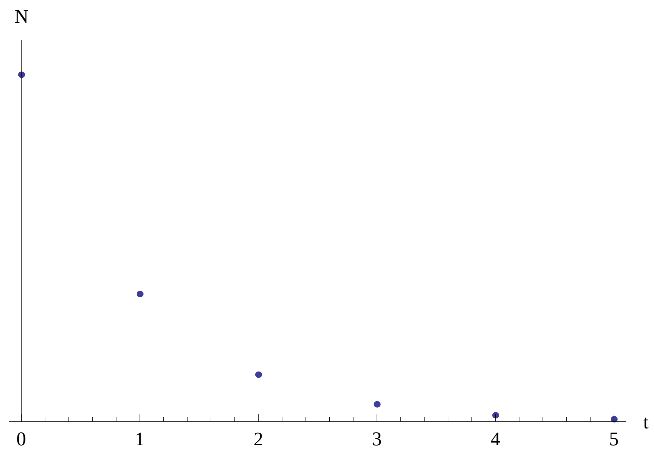

Figure 4.17: In Radioactive Decay, the number of particles reduced exponentially: $N(t) = N_0 e^{-\lambda t}$. It is very difficult to extract the parameters N_0 and λ from such a plot.

In this case $N(t)$ is the number of particles at a time t, N_0 is the initial number of particles and k is a constant. If we were to take measurements of $N(t_i)$ at various times t_i, and plotted them this is what we would get.

However — and this will be our tool in this section — we can convert the multiplication and powers in the decay equation by taking logs.

Review of Logs

We define the *natural logarithm to base e* as the inverse function of $f(x) = e^x$. Another way of looking at logarithms is to consider them as a way of converting multiplication to addition; division to subtraction; and powers to multiplication — by way of the *laws of logs*:

Examples

Write each of the following in linear form:

1. $Y = a \cdot b^X$

   Solution:

2. $Y = aX^b$

   Solution:

3. $y = ae^{bX}$.

   Solution:

So for example going back to the radioactive decay example, what we can do is, as before, measure $N(t)$ and t at various t, but this time we will plot $\log(N(t))$ vs t as

The graph of $\log(N(t))$ vs t will be a straight line. Furthermore, the parameters $-k$ and $\log N_0$ — which we want to calculate — will be the given by the slope and y-intercept.

What if it's not all Multiplication and Powers?

The distance traveled by an object in t seconds, with initial speed u and constant acceleration a is given by:

This is not in linear form; can we put it in linear form?

Note also that we can write the laws

$$Y = aX^2 + b$$
$$Y = aX^2 + bX$$
$$Y = \frac{a}{X} + b$$
$$Y = \frac{a}{X} + bX$$

in linear form also:

Exercises

For the following, a and b are the constants. Please write in linear form $Y = aX + b$; identifying Y, a, X, b:

$$P = ae^{bt}.$$
$$R = aV^b.$$
$$y = ae^{bx}.$$
$$y = ax^b.$$
$$v = ae^{tb}.$$
$$T = aL^b.$$
$$p = at^b.$$
$$G = ax^2 + bx$$
$$F = \frac{1}{3}at^2 + bt$$

4.6 Manipulation of Data and Plotting of Graphs*

Introduction

Consider the spread of a disease in a population. Suppose a medical team recorded the number of people infected over a series of days and compiled the data:

Day, t	8	12	18	23	28
Infectants, I	36	59	96	129	164

Suppose the relationship is thought to be of the form:

$$I(t) = at^b, \tag{4.6}$$

for some constants $a, b \in \mathbb{R}$. This would be an example of *polynomial growth*. Today we will look at a method of testing whether this data satisfies this law.
To fit a non-linear curve to data we follow do the following:

1. Write the non-linear relationship in linear form $Y = mX + c$ identifying Y, m, X and c.
2. Tabulate the 'new' Y and X values.
3. Plot Y vs X and draw a line of best fit.
4. Find the equation of the line of best fit and hence write down the values of m and c.

The Method

1. Write the equation in linear form. This means that starting with $y = f(x)$ we end up with:

   where $Y = g_1(x, y)$ and $X = g_2(x, y)$ (more usually each depend on y or x); and m and c are independent of x and y.
2. Take your data for y and x and manipulate the data to come up with a table:

y	x	Y	X
$\vdots$	$\vdots$	$\vdots$	$\vdots$

3. Plot the points Y vs X on graph paper.
4. If there is a reasonable straight-line fit we say that the law is verified. Draw the line if this is the case.

Examples

Show that the following data satisfies $I = at^b$.

t	I
8	36
12	59
18	96
23	129
28	164

Hence find the best values of a and b.

Solution

1. Taking the log of both sides (which base??):

   That is, we have $Y = \log I$ and $X = \log t$.

2. We now manipulate the data to fit the linear model:

t	I	$X = \log t$	$Y = \log I$
8	36		
12	59		
18	96		
23	129		
28	164		

3. Now we plot X vs Y — morryah $\log t$ vs $\log I$.

4. Hence the law is verified.

Now we use the graph to find m and c:

Exercises

1. v is believed to be related to t according to the law $v = ae^{bt}$.

t	27.3	37.5	42.5	47.8	56.3
v	700	190	100	50	17

Show by plotting a graph of $\log v$ against t that the law relating these quantities is as stated. [HINT: The linear form is $\log v = bt + \log a$.]

2. n an experiment on moments, a bar was loaded with a load W , at a distance x from the fulcrum. The results of the experiment were:

x	10	20	30	40	50	60
W	55	27.5	18.33	13.75	11	9.17

Verify that a law of the form $W = ax^b$ is true where a and b are constants by plotting $\log W$ against $\log x$.

3. It is believed that x and y are related by a law of the form $y = ar^{kx}$ where a and k are constants. Values of x and y were measured and the results are as shown:

x	0.25	0.9	2.1	2.8	3.7	4.8
y	6.0	10.0	25.0	42.5	85.0	198.0

Show by plotting $\log y$ against x that the law is true.