MATH6040 — Technological Maths 201

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Contents

0.1 Introduction

Lecturer

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This page will comprise the webpage for this module and as such shall be the venue for course announcements including definitive dates for the tests. This page shall also house such resources as links (such as to exam papers), as well supplementary material. Please note that not all items here are relevant to MATH6040; only those in the category 'MATH6040'. Feel free to use the comment function therein as a point of contact.

Module Objective

This module introduces students to the basic techniques of matrices and vectors. It also builds on the differential and integral calculus which was examined in Technological Mathematics 2.

Module Content

Matrix Algebra

Definitions and notation. Addition, subtraction, multiplication of matrices. Determinants. Matrix inversion. Application to the solution of simultaneous linear equations. Cramer's rule. The singular case, inconsistent equations. Gaussian elimination - partial pivoting.

Vector Algebra

Magnitude and direction. Component form in two and three dimensions. Addition and subtraction of vectors: triangle and parallelogram laws. Scalar product, vector product, scalar triple product. Application to geometry, trigonometry, resolution of forces, moments, work done.

Further Differentiation

Related rates of change. Differentiation of implicit functions and parametric functions. Partial differentiation with application to small changes, error analysis.

Further Integration

Techniques of integration including integration by parts and inverse trigonometric substitution. Applications of definite integrals: work done by variable force, expanding gas; centroid of a plane area; volume, mass and centre of gravity of solid of revolution; second moment of area, moment of inertia.

Assessment

Total Marks 100: End of Year Written Examination 70 marks; Continuous Assessment 30 marks.

Continuous Assessment

The Continuous Assessment will be comprised of a two-hour written tests worth 15%, in weeks 5 & 8.
Absence from a test will not be considered accept in truly extraordinary cases. Plenty of notice will be given of the test date. For example, routine medical and dental appointments will not be considered an adequate excuse for missing the test.

Lectures

It will be vital to attend all lectures as many of the examples, proofs, etc. will be completed by us in class.

Tutorials

The aim of the tutorials will be to help you achieve your best performance in the tests and exam.

Exercises

There are many ways to learn maths. Two methods which arent going to work are

1. reading your notes and hoping it will all sink in
2. learning off a few key examples, solutions, etc.

By far and away the best way to learn maths is by doing exercises, and there are two main reasons for this. The best way to learn a mathematical fact/ theorem/ etc. is by using it in an exercise. Also the doing of maths is a skill as much as anything and requires practise.
There are exercises in the notes for your consumption. The webpage may contain a link to a set of additional exercises. Past exam papers are fair game. Also during lectures there will be some things that will be *left as an exercise.* How much time you can or should devote to doing exercises is a matter of personal taste but be certain that effort is rewarded in maths.

Reading

Your primary study material shall be the material presented in the lectures; i.e. the lecture notes. Exercises done in tutorials may comprise further worked examples. While the lectures will present everything you need to know about MATH6040, they will not detail all there is to know. Further references are to be found in the library. Good references include:

- John Bird 2006, *Higher Engineering Mathematics*, Fifth Edition Ed., Newnes Oxford
- K.A. Stroud & Dexter J.Booth 2007, *Engineering Mathematics*, Sixth Edition Ed., Macmillan

The webpage may contain supplementary material, and contains links and pieces about topics that are at or beyond the scope of the course. Finally the internet provides yet another resource. Even Wikipedia isn't too bad for this area of mathematics! You are encouraged to exploit these resources; they will also be useful for further maths modules.

Exam

The exam format will roughly follow last year's. Acceding to the maxim that learning off a few key examples, solutions, etc. is bad and doing exercises is good, solutions to past papers shall not be made available (by me at least). Only by trying to do the exam papers yourself can you guarantee proficiency. If you are still stuck at this stage feel free to ask the question come tutorial time.

Chapter 1

Matrix Algebra

It is my experience that proofs involving matrices can be shortened by 50% if one throws the matrices out.

Emil Artin

Motivation: Network Flows

Suppose we have a network of one-way flows as shown in the diagram:

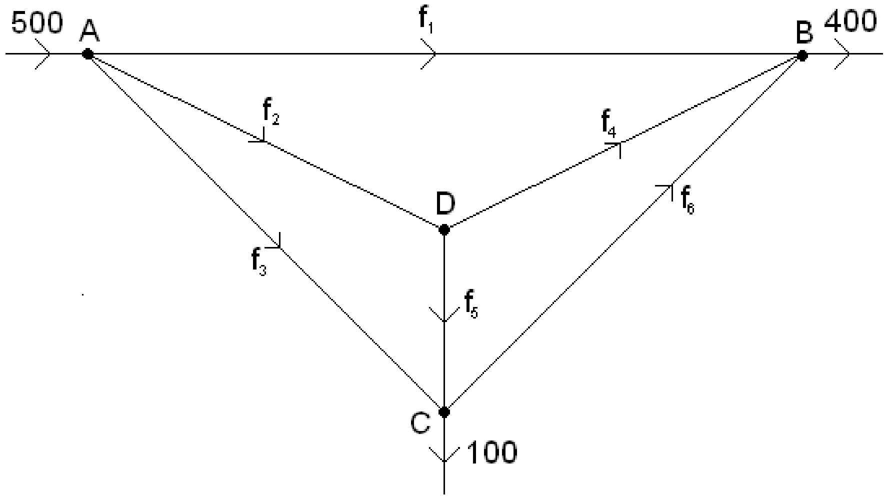


Figure 1.1: The flow into junction A is 500 , and 400 and 100 emerge from B and C. Suppose the flows are $f_1, \ldots, f_6$ as shown. From the simple rule that the flow into a junction must equal to flow out, how much can we tell about the internal workings of the network?

Equating the flow in with the flow out at each junction we get:

$$\begin{array}{lccc} \text{Junction } A & 500 & = & f_1 + f_2 + f_3 \\ \text{Junction } B & f_1 + f_4 + f_6 & = & 400 \\ \text{Junction } C & f_3 + f_5 & = & f_6 + 100 \\ \text{Junction } D & f_2 & = & f_4 + f_5 \end{array} .$$

This gives four equations in six variables $f_1, \ldots, f_6$:

$$\begin{aligned} f_1 + f_2 + f_3 &= 500 \\ f_1 + f_4 + f_6 &= 400 \\ f_3 + f_5 - f_6 &= 100 \\ f_2 - f_4 - f_5 &= 0 \end{aligned}$$

1.1 Matrices

Last semester we introduced the idea of a *function*. A function is a mathematical object that for an input produces a single output. The functions we examined last semester were functions of *a single variable* which means that we put in one number and got out one number:

Here the set of inputs (*domain*) and the set of outputs (*target*) are both the set of real numbers, $\mathbb{R}$; and we write $f : \mathbb{R} \to \mathbb{R}$. There is nothing stopping us from defining functions on sets other than the real numbers. For example, let $S = \{\text{pubs in Annascaul}\} = \{S, P, B, H, F, R\}$ and define the following function

A simple extension of the set of real numbers — the number line; is the set of points on the plane:

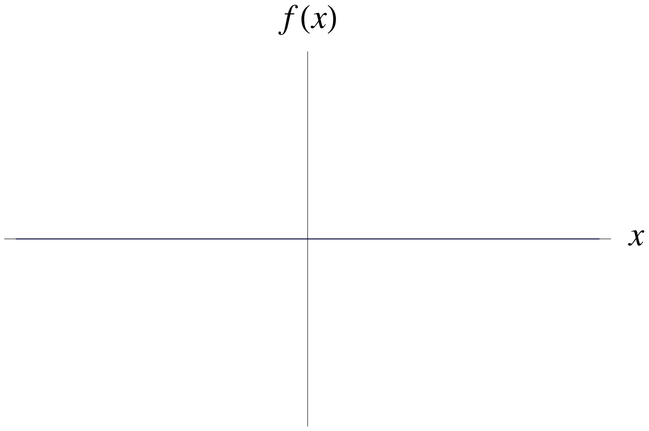


Figure 1.2: Take the collection of pairs of numbers (x, y). We call this set the *plane* and denote it by $\mathbb{R}^2$

We can extend again to the set of points in space:

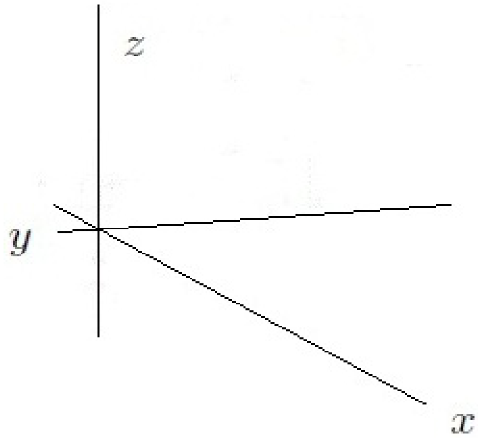


Figure 1.3: Take the collection of triples of numbers (x, y, z). We call this set *space* and denote it by $\mathbb{R}^3$

In this section we study a certain class of functions between the line, the plane and space:

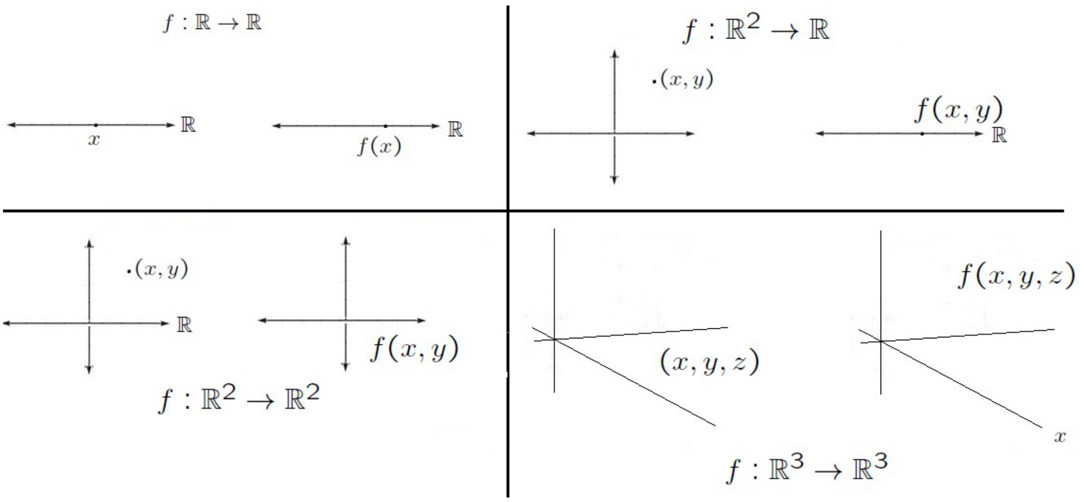


Figure 1.4: Functions between lines, planes and space.

We will study the *linear* maps between these sets. A map $f : A \to B$ is said to be *linear* if

It turns out that if we write elements of the line as (x), elements of the plane as $\begin{pmatrix} x \\ y \end{pmatrix}$ and elements of space as $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, then these linear maps can be written as *matrices* and the action of the matrix A on a point in $\mathbf{x}$ is given by *matrix multiplication.*

Examples

$$A = \begin{pmatrix} 1 & 0 \\ 2.6 & -8 \end{pmatrix}$$
$$B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$C = \begin{pmatrix} 1 & 0 & 3 \\ -16 & 0 & \sqrt{26} \end{pmatrix}$$

Remarks

1. A matrix with n rows and m columns is said to have *dimension* $n \times m$ or be an $n \times m$ matrix. For example, A is a 2×2 matrix; B is a 2×1 matrix, and C is a 2×3 matrix. We will see that an $n \times m$ matrix A is a function $A : \mathbb{R}^m \to \mathbb{R}^n$.

2. A *square* matrix is an $n \times n$ matrix and is a function $A : \mathbb{R}^n \to \mathbb{R}^n$.

3. The (i, j)-entry of a matrix is the number in the ith row and jth column.

1.1.1 Addition of Matrices

Two matrices of equal dimension may be added together to produce another matrix of the same dimension. This sum is a matrix whose elements are obtained by adding corresponding elements.

The *zero matrix* is denoted $\mathbf{0}$, and has only 0 as its entries. It satisfies

Just like zero for the real numbers.

1.1.2 Scalar Multiplication of a Matrix

Any matrix may be multiplied by a scalar (some $k \in \mathbb{R}$) by multiplying each element by the number.

By definition $-A = (-1)A$, so that $A - B$ means $A + (-B)$. Properties of matrix addition and scalar multiplication include:

$$A + B = B + A; \quad (A + B) + C = A + (B + C); \quad k(A + B) = kA + kB;$$

$$(k + l)A = kA + lA; \quad (kl)A = k(lA); \quad A - A = 0; \quad 0A = \mathbf{0}.$$

Note that these mirror the properties of ordinary addition and multiplication.

If A is an $m \times n$ matrix then the *transpose* of A, denoted A^T, is the $n \times m$ matrix whose got by exchanging the rows and columns of A. Properties of the transpose operation include:

$$(A^T)^T = A; \quad (kA)^T = kA^T; \quad (A+B)^T = A^T + B^T.$$

If $A : \mathbb{R}^n \to \mathbb{R}^m$ then $A^T : \mathbb{R}^m \to \mathbb{R}^n$.

1.1.3 Equality of Matrices

Two matrices are *equal as matrices* if they have same dimension and each corresponding element is equal.

Example

Suppose

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{ and}$$
$$B = \begin{pmatrix} 1 & 0 \\ 2.6 & -8 \end{pmatrix}$$

and one is told $A = B$. Thence $a = 1, b = 0, c = 2.6$ and $d = -8$.

1.1.4 Definition

A matrix A is *conformable* with a matrix and B if the dimension of A is $n \times k$ and the dimension of B is $k \times m$ for some $k \in \mathbb{N}$.

Remarks

1. Only a notion of multiplication between conformable matrices is considered. In this case the product of an $n \times k$ matrix and a $k \times m$ matrix is a $n \times m$ matrix.
2. This means that a matrix A may be multiplied by a matrix B to form the product AB if and only if the number of columns in A is equal to the number of rows in B.
3. Note also that if A is conformable with B it does not follow that B is conformable with A. For example, a 2×3 matrix be multiplied by a 3×4 matrix to produce a 2×4 matrix but a 3×4 matrix may not be multiplied by a 2×3 matrix
4. Two square matrices of equal dimension may be multiplied together to produce another square matrix of the same dimension. However note that the order of multiplication is important. It will be seen in general that for square matrices A and B;

   $$AB \neq BA \tag{1.1}$$

   That is the axiom of commutivity for real numbers $xy = yx$, $\forall x, y \in \mathbb{R}$; fails in general for an algebra of matrices.

Note that all of these make sense if we consider these matrices as functions. Note however that BA means that A acts first.

1.1.5 Definition

Let $A := [A]_{ij} = a_{ij}$ of dimension $n \times r$; and $B := [B]_{ij} = b_{ij}$ of dimension $r \times m$. Then the matrix product $AB = C = [C]_{ij}$ has matrix entries

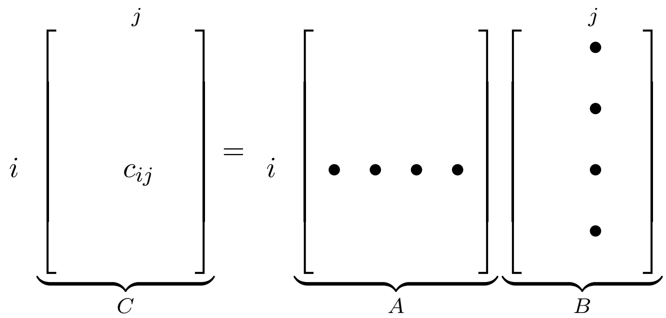
Remarks

This is the technical definition for any two conformable matrices A and B. The meaning of this equation will be discussed for the general case of two conformable matrices; and for the cases of $n \times m$ matrices with $n, m \leq 2$.

(i) The General Case;
Let A be a $n \times r$ matrix and B be a $r \times m$ matrix. What are the entries of $C = AB$? Well take the general entry that is in the i-th row and j-th column of C. This is the number c_{ij}. This is by (**??**):

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{ir}b_{rj}$$

So to find the (ij)-th element sum the numbers along the i-th row of A multiplied by the numbers along the j-th column of B:



(ii) A 1×2 matrix by a 2×1 matrix.

Note a 1×2 matrix by a 2×1 matrix is a 1×1 matrix. This is equivalent to a real number; in this case $ac + bd$.

(iii) A 1×2 matrix by a 2×2 matrix.

Note a 1×2 matrix by a 2×2 matrix is a 1×2 matrix.

(iv) A 2×2 matrix by a 2×1 matrix.

Note a 2×2 matrix by a 2×1 matrix is a 2×1 matrix.

(v) A 2×2 matrix by a 2×2 matrix.

I find the best way to remember — certainly if we are multiplying with 3×3 matrices — is as follows:

$$C = AB = \begin{pmatrix} \text{1st row by 1st column} & \text{1st row by 2nd column} & \cdots & \text{1st row by last column} \\ \text{2nd row by 1st column} & \text{2nd row by 2nd column} & \cdots & \text{2nd row by last column} \\ & & & \\ \text{last row by 1st column} & \text{last row by 2nd column} & \cdots & \text{last row by last column} \end{pmatrix} \tag{1.2}$$

Examples

1. If

$$A = \begin{bmatrix} 1 & 8 \\ 3 & -2 \\ 0 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 & 9 \\ -2 & 7 \end{bmatrix},$$

find AB.

Solution:

2. Given the matrices

$$A = \begin{bmatrix} 5 & -3 \\ -2 & -4 \\ 2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 & 3 \\ -3 & 0 & 2 \end{bmatrix},$$

determine the following sums/products if defined

(a) $2A + B$

(b) $2A + B^T$

(c) BA

Solution:

(a) This is not defined because $A : \mathbb{R}^2 \to \mathbb{R}^3$ and $B : \mathbb{R}^3 \to \mathbb{R}^2$.

(b) We have

(c) We calculate

Summer 2012: Question 1 (h)

For the matrices

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 4 & 2 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -1 & 5 \\ 0 & -2 & 1 \end{pmatrix}$$

show that $(A + B)^T = A^T + B^T$.

[5 marks]

Solution: First of all we calculate $A + B$:

Now we find the transpose:

Now find the transposes of A and B:

and add them together:

Summer 2012: Question 2 (a) (i)

For the matrices

$$A = \begin{pmatrix} 2 & -1 & 0 \\ 3 & 2 & -1 \end{pmatrix}, \ C = \begin{pmatrix} 5 & -3 \\ 2 & 1 \\ 3 & 4 \end{pmatrix}.$$

Determine each of the following

1. $A + C$
2. CA
3. AC

Solution:

1. This is not defined because $A : \mathbb{R}^3 \to \mathbb{R}^2$ and $C : \mathbb{R}^2 \to \mathbb{R}^3$.
2. We calculate

3. We calculate

Autumn 2012: Question 1 (h)

For the matrices

$$B = \begin{pmatrix} 2 & 1 & -3 \\ 3 & -2 & 4 \end{pmatrix}, \; C = \begin{pmatrix} 1 & 0 & 3 \\ -2 & 0 & 2 \\ 4 & 5 & -3 \end{pmatrix}.$$

show that $(BC)^T = C^T B^T$.

[5 marks]

Solution: First we calculate BC:

Now we take the transpose

Now find C^T and B^T:

and multiply them together:

Autumn 2012: Question 2 (a) (i)

For the matrices

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 3 & -2 & -1 \end{pmatrix}, \; B = \begin{pmatrix} 1 & 2 & 1 \\ -3 & 2 & 0 \end{pmatrix}, \; C = \begin{pmatrix} 5 & 3 \\ 2 & -1 \\ 3 & 4 \end{pmatrix}.$$

Determine each of the following, if defined

1. $A + B$
2. $A + C$
3. $(A + B)^T$
4. AC
5. CA

Solution:

1. We calculate
2. This is not defined because $A : \mathbb{R}^3 \to \mathbb{R}^2$ and $C : \mathbb{R}^2 \to \mathbb{R}^3$.
3. We write
4. We calculate
5. We calculate

Exercises

1. Let $A = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 3 \\ -1 & 0 \\ 1 & 4 \end{bmatrix}$. Compute the following (where possible):

$$(i)\ 3A - 2B \ \ (ii)\ 5C \ \ (iii)\ 4A^T - 3C \ \ (iv)\ B + D \ \ (v)\ (A+C)^T \ \ (vi)\ A - D.$$

2. Find A if
   (a) $5A - \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = 3A - \begin{bmatrix} 5 & 2 \\ 6 & 1 \end{bmatrix}$.
   (b) $3A + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 5A - 2\begin{bmatrix} 3 \\ 0 \end{bmatrix}$.
   (c) $\left(3A^T + 2\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right) = \begin{bmatrix} 8 & 0 \\ 3 & 1 \end{bmatrix}$.
   (d) $\left(2A^T - 5\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}\right)^T = 4A - 9\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$.

3. Compute the following matrix products (if possible):
   (a) $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & 4 \end{bmatrix}\begin{bmatrix} 2 & 3 & 1 \\ 1 & 9 & 7 \\ -1 & 0 & 2 \end{bmatrix}$.
   (b) $\begin{bmatrix} 1 & 3 & -3 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ -2 & 1 \\ 0 & 6 \end{bmatrix}$.
   (c) $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$.
   (d) $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}\begin{bmatrix} a' & 0 & 0 \\ 0 & b' & 0 \\ 0 & 0 & c' \end{bmatrix}$.
   (e) $\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} -1 & 6 \\ 1 & 0 \end{bmatrix}$.
   (f) $\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 1 & 2 \end{bmatrix}$.

4. Let A, B and C be matrices.
   (a) If A^2 can be formed, what can be said about the size of A.
   (b) If AB and BA can both be formed, describe the sizes of A and B.
   (c) If ABC can be formed, A is 3×3 and C is 5×5, what size is B.

1.2 Matrix Inverses

What does a 2×2 matrix *do*? Well it sends points (x, y) on the plane to other points on the plane:

An inverse of a matrix sends the points back to where they came from so that hitting a point with A and then by 'A^{-1}' will send it back from whence it came.

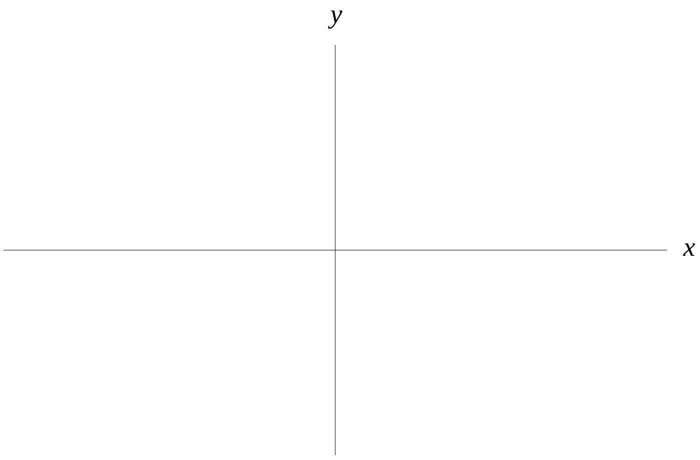


Figure 1.5: The action of a matrix and its inverse.

This means that if you hit a point with A and then A^{-1} the point goes nowhere. The matrix that send a point to itself is known as the *identity* matrix, I and is given by:

This means that if A is a matrix and A^{-1} it's inverse that

We do also require that $A^{-1}A = I$. Not all matrices are invertible however. Consider the 2×2 matrix given by

In contrast to arithmetic in the real numbers, $\mathbb{R}$, every non-zero number x has a *multiplicative inverse* x^{-1} given by the number $1/x$ with the property:

where '1' is the *multiplicative identity* with the special property that for all $x \in \mathbb{R}$:

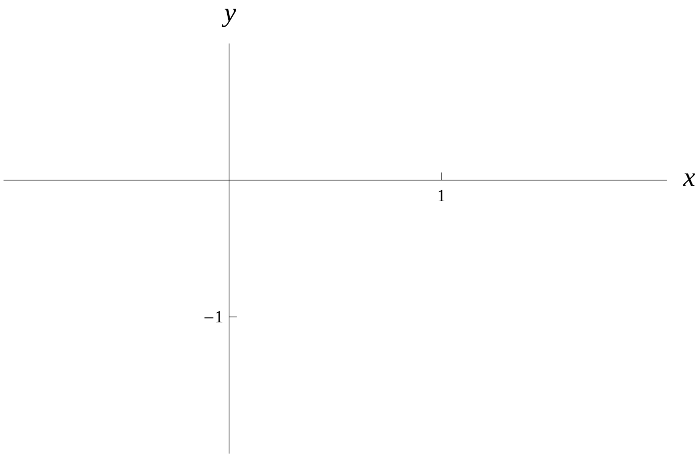


Figure 1.6: How can you send back from the origin?

The identity matrix I that is like the 1 for matrix multiplication:

How can we tell if a matrix has an inverse? Can I matrix have more than one inverse? If we know that it has an inverse how do we find it?

1.2.1 Inverses of 2×2 Matrices

Let A be a 2×2 matrix

where $a, b, c, d \in \mathbb{R}$ are real numbers. If you solve the equations

for x, y, z, w then you have the inverse of A. It is tricky but not impossible to show that

1.2.2 Formula

Given a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad - bc \neq 0$, the (multiplicative) *inverse* of A, A^{-1} is the matrix

$$\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \tag{1.3}$$

This formula is in the tables.

There is no such neat formula for 3×3 [1] but we have a method of finding the inverse of a matrix that works for 3×3 and larger — as well as for 2×2 matrices.

[1]well there is but it's a bit 'big': http://ardoris.wordpress.com/2008/07/18/general-formula-for-the-inverse-of-a-3x3-matrix/

Example

Find the inverse matrix of

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 3 \end{bmatrix}.$$

Solution: Using the formula

Exercises: Find the inverses of the matrices

$$\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}, \qquad \begin{bmatrix} 4 & 6 \\ 3 & 2 \end{bmatrix}, \qquad \begin{bmatrix} 3 & 4 \\ -2 & 1 \end{bmatrix} \qquad \begin{pmatrix} 2 & -c \\ c & 3 \end{pmatrix}$$

Answers:

$$\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}, \qquad \frac{1}{10}\begin{bmatrix} -2 & 6 \\ 3 & -4 \end{bmatrix}, \qquad \frac{1}{11}\begin{bmatrix} 1 & -4 \\ 2 & 3 \end{bmatrix}, \qquad \frac{1}{c^2+6}\begin{pmatrix} 3 & c \\ -c & 2 \end{pmatrix},$$

1.2.3 Row Operations

The method which we will use to find the inverse of a matrix is by a method called Gauss-Jordan Elimination. When we are finished, we will have a method that is straightforward to implement and finds matrix inverses with an amount of elementary calculations. The method will not show any of the theoretical scaffolding behind the method but here we will give a flavour of how the method works.

We start with a matrix A whose inverse we would like to compute. What we are going to do is multiply the matrix A by a sequence of matrices $E_1, E_2, \dots, E_n$ until we have that

Now can we say anything about the matrix $E_n \cdots E_2 E_1$?

Now how do we know which matrices E_i to multiply by... and do we have to multiply them altogether at the end? These matrices are called *elementary matrices* and there are three types.

1. *Switches Rows* Given a matrix A, there is a matrix $E_{i\leftrightarrow j}$ that switches the ith and jth rows of A.

   Example:

2. *Multiplies a row by a constant* Given a matrix A, there is a matrix $E_{i\to ki}$ that multiplies the ith row of A by a constant k.

Example:

3. *Adds a row to another row* Given a matrix A, there is a matrix $E_{i\to i+j}$ that adds the jth row to the ith row.

   Example:

These last two matrices may be combined

to produce a matrix $E_{i\to i+kj}$ that adds a constant multiple of the jth row to the ith row. So we have these three (four) types of matrices and they have three (four) associated *elementary row operations*; r_i means the ith row.

1. *Switch Rows* Given a matrix A, the row operation $r_i \leftrightarrow r_j$ switches the ith and jth rows of A.

   Example:

2. *Multiply a row by a constant* Given a matrix A, the row operation $r_i \to kr_i$ multiplies the ith row of A by a constant k.

   Example:

3. *Add a row to another row* Given a matrix A, the row operation $r_i \to r_i + r_j$ adds the jth row to the ith row.

   Example:

4. *Add a constant multiple of a row to another row* Given a matrix A, the row operation $r_i \to r_i + kr_jj$ adds k times the jth row to the ith row.

   Example:

Theorem

Suppose that E is an elementary matrix that has the action of the elementary row operation r. *Then*

$$E = rI \tag{1.4}$$

where I is the identity matrix.

Proof. Beyond the scope of the course •

This means that to multiply by an elementary matrix E you can instead just do the elementary row operation that E induces... it would be a lot easier to just forget about elementary matrices and just use elementary row operations. Luckily we can do this. If we write the matrix A we want to invert beside the identity matrix like this:

what we can do is keep doing row operations on A until it is has been transformed into the identity — *while doing exactly the same row operations on the identity matrix*:

Then the matrix B here is A^{-1}! How? Well if we do row operations $r_1, \dots, r_n$ to A, then we also do them to the identity I:

We know that this is the same as multiplying by elementary matrices $E_1, \dots, E_n$:

Now as discussed above if $E_n \cdots E_2 E_1 A = I$ then $E_n \cdots E_1 = A^{-1}$... and what does $E_n \cdots E_1 I$ equal to?

So we have a method of calculating the inverse of A. Write

Apply elementary row operations to A *and* I until we have

Then the matrix left on the right is A^{-1}. Now is there a neat way to get from $A \longrightarrow I$ using row operations? The answer is yes: the Gauss-Jordan algorithm.

1.2.4 Gaussian-Jordan Elimination

Recall the elementary row operations (EROs); r_i means the ith row.

- swapping any two rows:

- multiplying any row by a constant:

- adding any row to any other:

- combining the last two: adding a multiple of a row to another row:

Recall that you want to get, say, a 3×3 matrix in the form of the identity:

The best way to do this is to work like this

Taking a matrix and applying EROs until it is the identity is called the *Gauss-Jordan Algorithm*. We can write down the algorithm as follows. This version assumes that the matrix is 3×3 and invertible but extends in the natural way for an $n \times n$ matrix. You can show that the Gauss-Jordan Algorithm cannot be done if A is *not* invertible.

1. (a) If the $(1,1)$ entry is a zero use the ERO $r_1 \leftrightarrow r_j$ to swap the first row with a row which has a non-zero entry in the first column.

   (b) If the $(1,1)$ entry is $a \neq 0$ then use the ERO $r_1 \rightarrow \dfrac{1}{a} r_1$ to multiply the first row by $\dfrac{1}{a}$/divide the first row by a to turn it into a 1. This 1 in the $(1,1)$ entry is the *first pivot*.

   (c) Now *use the first pivot* to turn the entries below the pivot into zeroes using the ERO $r_i \rightarrow r_i + kr_1$.

2. (a) If the $(2,2)$ entry is a zero use the ERO $r_2 \leftrightarrow r_3$ to swap the second row with the third row.

   (b) If the $(2,2)$ entry is $b \neq 0$ then use the ERO $r_2 \to \dfrac{1}{b} r_2$ to multiply the second row by $\dfrac{1}{b}$/divide the first row by b to turn it into a 1. This 1 in the $(2,2)$ entry is the *second pivot*.

   (c) Now *use the second pivot* to turn the entry below the pivot into zeroes using the ERO $r_3 \to r_3 + kr_2$.

3. The $(3,3)$ entry is $c \neq 0$. Use the ERO $r_3 \to \dfrac{1}{c} r_3$ to multiply the second row by $\dfrac{1}{c}$/divide the first row by c to turn it into a 1. This 1 in the $(3,3)$ entry is the *third pivot*.

4. Use the third pivot to turn the $(1,3)$ and $(2,3)$ entries into a zero using EROs of the form $r_2 \to r_2 + kr_3$ and $r_1 \to r_1 + kr_3$.

5. Use the second pivot to turn the $(1,2)$ entry into a zero using an ERO of the form $r_1 \to r_1 + kr_2$.

When you have implemented Gauss-Jordan elimination the matrix is said to be in *reduced row form*.

Examples

1. Use Gauss-Jordan elimination to find A^{-1} where

$$A = \begin{bmatrix} 1 & 0 & 8 \\ 2 & 5 & 3 \\ 1 & 2 & 3 \end{bmatrix}.$$

Solution: First of all we write it in the $[A \,|\, I]$ form:

We first note that the $(1,1)$ entry is fine so we look at the 2 and 1 below it. *Use the pivot to 'kill' entries below it.* We do this with the EROS $r_2 \to r_2 - 2r_1$ and $r_3 \to r_3 - r_1$. **Make sure to do the same to the matrix on the right**:

Now we turn out attention to the $(2,2)$ entry. This is to be the next pivot. *To make a pivot multiply/divide.* We do this with the ERO $r_2 \to \frac{1}{5} r_2 = r_2 \div 5$. Then we will use this pivot to 'kill' the 2 in the $(3,2)$ entry. We will do this with $r_3 \to r_3 - 2r_1$. **Make sure to do the same to the matrix on the right**:

Luckily we now have a pivot in the $(3,3)$ position so this entry is done; nearly there. We use the pivot here to 'kill' the two entries above it using $r_2 \to r_2 + 3r_3$ and $r_1 \to r_1 - 8r_1$. **Make sure to do the same to the matrix on the right**:

Finally the $(1,2)$ entry is already a 0 so we are in reduced row form so we are done. Now by our theory the matrix on the right is A^{-1} so we write:

If you want to check that you haven't made a mistake you could check that $A^{-1}A = AA^{-1} = I$. We will calculate $A^{-1}A$ to verify our work:

2. Determine A^{-1} where $A = \begin{bmatrix} 1 & 1 & -1 \\ -3 & 2 & -1 \\ 3 & -3 & 2 \end{bmatrix}$.

Solution: First of all we write it in the $[A \,|\, I]$ form:

We first note that the $(1,1)$ entry is fine so we look at the -3 and 3 below it. *Use the pivot to 'kill' entries below it.* We do this with the EROS $r_2 \to r_2 + 3r_1$ and $r_3 \to r_3 - 3r_1$. **Make sure to do the same to the matrix on the right**:

Now we turn out attention to the $(2,2)$ entry. This is to be the next pivot. *To make a pivot multiply/divide.* We do this with the ERO $r_2 \to \frac{1}{5}r_2 = r_2 \div 5$. Then we will use this pivot to 'kill' the -6 in the $(3,2)$ entry. We will do this with $r_3 \to r_3 + 6r_1$. **Make sure to do the same to the matrix on the right**:

Now we turn out attention to the $(3,3)$ entry. This is to be the next pivot. *To make a pivot multiply/divide.* We do this with the ERO $r_3 \to 5r_3$. We use the pivot here to 'kill' the two entries above it using $r_2 \to r_2 + \frac{4}{5}r_3$ and $r_1 \to r_1 + r_3$. **Make sure to do the same to the matrix on the right**:

Finally we must 'kill' the 1 in the $(1,2)$. We do this using the pivot at $(2,2)$ by using $r_1 \to r_1 - r_2$:

Now by our theory the matrix on the right is A^{-1} so we write:

Exercises

1. Find the inverse of the matrix
$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 3 \\ 1 & 3 & 1 \end{bmatrix}.$$

Ans: $\begin{bmatrix} 7 & 2 & -6 \\ -3 & -1 & 3 \\ 2 & 1 & -2 \end{bmatrix}$

2. Find the inverse of the matrix
$$B = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}.$$

Ans: $\begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix}$

1.2.5 Matrix Equations

How do you solve the equation

In reality you don't need division at all. What could you do instead?

This number, $\frac{1}{5}$ is known as the *multiplicative inverse* of the real number 5 in pure maths circles. Instead of division, you have multiplication by the inverse:

Suppose A and B are two known matrices. How do you solve the matrix equation and isolate the matrix X?

Rather than say *divide by* A we instead multiply both sides by the (multiplicative) inverse of A; i.e. the inverse A^{-1}:

Now the crucial, crucial difference between a matrix equation and (real) number equation is that the *order of multiplication* matters. For matrices $AB \neq BA$ necessarily. Hence we must multiply both sides on the right — or multiply both sides on the left... and you can't do a hodge-podge.

Autumn 2012: Question 2 (a) (iii)

Given the matrices

$$A = \begin{pmatrix} 1 & -3 \\ 4 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 16 & 32 \\ 36 & 84 \end{pmatrix}$$

determine the matrix X such that $XA = C$.

Solution: We need to eliminate the A from the right-hand side. How do we do this?

Now we can find A^{-1} using the formula in the tables or via Gauss-Jordan elimination. I will do this example using Gauss-Jordan elimination[2]. First we write in the (augmented) $[A\,|\,I]$ form:

[2]which of course gives the same solution

The first pivot is there so we use it to 'kill' the four underneath. How?

Now we turn the 18 into a pivot by multiplying by $\frac{1}{18}$/ dividing by 18. Then we can use this pivot to 'kill' the -3 and put the matrix in reduced row form:

Hence we know that

Our algebra has shown that $X = CA^{-1}$ so we multiply

Summer 2012: Question 2 (a) (iii)

Given the matrices

$$B = \begin{pmatrix} 1 & 3 \\ 5 & 7 \end{pmatrix}, \quad D = \begin{pmatrix} 14 & 30 \\ 38 & 86 \end{pmatrix}$$

determine the matrix X such that $BX = D$.

Solution: We need to eliminate the B from the right-hand side. How do we do this?

Now we can find B^{-1} using the formula in the tables:

Our algebra has shown that $X = B^{-1}D$ so we multiply

Exercises

1. Find the matrix B such that A is another matrix such that

$$A = \begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix} \quad \text{and} \quad AB = \begin{pmatrix} 0 & -3 \\ 2 & 4 \end{pmatrix}$$

   Ans: $\begin{pmatrix} -14 & -37 \\ 10 & 26 \end{pmatrix}$

2. **Winter '12 Q. 2 (a) (i)** Given

$$A = \begin{pmatrix} 1 & -2 \\ 3 & 5 \end{pmatrix}, \qquad C = \begin{pmatrix} 12 & 22 \\ 16 & 24 \end{pmatrix}.$$

   Determine X is X satisfies $XA = C$.

   Ans: $-\frac{1}{11}\begin{pmatrix} 6 & -46 \\ -8 & -56 \end{pmatrix}$

3. Let A be an invertible matrix. Show that if $AX = AY$ then $X = Y$. Show that if $PA = QA$ then $P = Q$.

4. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Verify that $AB = CA$, A is invertible, but that $B \neq C$.

1.3 Linear Systems

Yes but what has all this got to do with network flows? Well let us set of some kind of a network flow with losses:

Recalling that we want flow-in equal to flow-out (at each junction). We end up with simultaneous equations of the form:

It is still not clear what the hell this has got to do with equations. However a bit of rewriting gives us a glimmer of hope. O.K. We using the *coefficients* to make a matrix M:

Example

Use the inverse of methods to solve

$$\begin{aligned} -x + y + z - 3, \\ -2x - 3y - z = 2, \\ 2x - 3y - z = 1. \end{aligned}$$

Verify y *using Cramer's Rule.*
Solution :

1.4 Determinants

There is a quantity related to square matrices called the *determinant*. You will have encountered determinants before and may be able to calculate the determinants of 2×2 and 3×3 matrices. For MATH7021 we only need to calculate determinants of 2×2 matrices. What *are* they geometrically though?

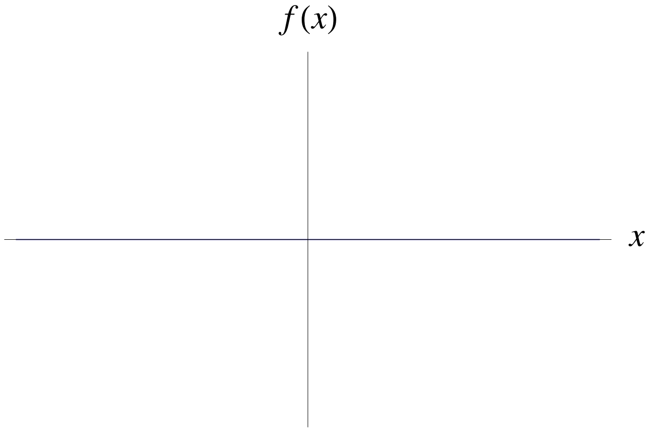


Figure 1.7: If a 2×2 matrix M_1 is applied to a region of area A then the area of the image will be $\det M_1 \times A$. This means that a matrix M_2 with $\det M_2 = 0$ cannot be invertible as an infinite number of points will be sent to a single point or line — dimension arguments show that we cannot send these points back faithfully.

Definition & Formula

Let A be a square matrix with column vectors $c_1, c_2, \ldots, c_n$. The *determinant* of A, $\det A$ is the volume of the parallelepiped generated by the column vectors. The determinant of a 2×2 matrix is given by:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc. \tag{1.5}$$

$|A|$ is another notation for the determinant of A. We have the following which we may have learnt about in a previous module.

Theorem

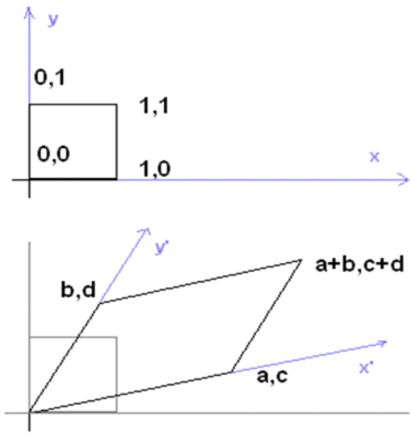
A square matrix A is invertible if and only if $\det A$ is non-zero.

Proof. If $\det A = 0$ then clearly A can not be invertible according to Figure 1.7. There is a formula for the inverse of a square matrix: $A^{-1} = \dfrac{1}{\det A} C$ where C is a matrix defined in terms of A. If $\det A \neq 0$ then this formula is valid and A is invertible •

1.5 Cramer's Rule

Suppose that a linear system has the form

then the system can be rewritten in the form



$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a+b \\ c+d \end{bmatrix}$$

Figure 1.8: We can somehow get a handle on this by taking as unit area/volume the *parallelepiped* generated by the basis vectors. It is not too difficult to show that the area of the parallelogram is $ad - bc$ and that this generalises neatly to any dimension $n \geq 4$.

where A is a square matrix, $\mathbf{b}$ is the vector of constants and $\mathbf{x}$ is the vector of variables:

Now introduce the idea of the inverse of a matrix. We might have encountered these before but again as an aid to understanding it might be fruitful to consider the geometry of the situation. If A is invertible[3], then we can 'hit' both sides with A^{-1} to obtain the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

[3]if there exists a matrix A^{-1} such that $AA^{-1} = A^{-1} = I$ where I is the identity matrix

That is if we have a square matrix of coefficients A and of if A is invertible then the solution is unique.

Formula

Suppose that we have a square linear system

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \equiv A\mathbf{x} = \mathbf{b} \equiv [A \,|\, b] \tag{1.6}$$

with $\det A \neq 0$. Then the solutions are given as

$$x_i = \frac{\det A_i}{\det A} \tag{1.7}$$

where A_i is the matrix formed by replacing column i by the constant vector $\mathbf{b}$:

When we are solving for say x, y and maybe z we simply use

$$x = \frac{D_x}{D}, \ y = \frac{D_y}{D}, \ z = \frac{D_z}{D} \tag{1.8}$$

Example

Use Cramer's Rule to solve the linear system

$$\begin{aligned} 4x + 5y &= 8 \\ x - y &= 11 \end{aligned}$$

Solution: First we find $x = \dfrac{D_x}{D}$:

Now we find $y = \dfrac{D_y}{D}$:

Remark

This seems too good to be true — such a simple formula for solving simultaneous equations. Much quicker than Gaussian elimination methods and even *ad hoc* methods... Don't be fooled. Cramer's Rule only applies when the linear system is square and has a unique solution. The only way to know this in advance is to calculate the determinant of the coefficient matrix. For 2×2 this is simple. For 3×3 a little harder but not impossible. However as the size of the system increases Cramer's Rule takes comparatively longer and longer to implement in comparison to Gaussian elimination methods because determinants of larger matrices take an increasingly long time to compute.

Why the hell do we even use Cramer's Rule so? Well suppose that you have a real physical linear system (e.g. some kind of network flow) that you know must somehow have a unique solution. Sometimes you will not be interested in all of the variables but only one: this is where Cramer's Rule's strength lies. With Gaussian elimination methods you either have none of the solutions or one of the solutions. With Cramer's Rule you can find only one variable if you want. In MATH7021 we will be using it in an exam situation to check an answer or approximation.

Also I have omitted the proof. It relies on a number of properties of determinants.

Example

Use only determinants to determine if the following homogenous system of linear equations has either the trivial solution or non-trivial solutions.

$$\begin{aligned} 2x - 4y - 5z &= 0 \\ 3x + y - 4z &= 0 \\ x - 6y - z &= 0. \end{aligned}$$

Solution: