MATH6040 — Technological Maths 201

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Contents

0.1 Introduction

Lecturer

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This page will comprise the webpage for this module and as such shall be the venue for course announcements including definitive dates for the tests. This page shall also house such resources as links (such as to exam papers), as well supplementary material. Please note that not all items here are relevant to MATH6040; only those in the category 'MATH6040'. Feel free to use the comment function therein as a point of contact.

Module Objective

This module introduces students to the basic techniques of matrices and vectors. It also builds on the differential and integral calculus which was examined in Technological Mathematics 2.

Module Content

Matrix Algebra

Definitions and notation. Addition, subtraction, multiplication of matrices. Determinants. Matrix inversion. Application to the solution of simultaneous linear equations. Cramer's rule. The singular case, inconsistent equations. Gaussian elimination - partial pivoting.

Vector Algebra

Magnitude and direction. Component form in two and three dimensions. Addition and subtraction of vectors: triangle and parallelogram laws. Scalar product, vector product, scalar triple product. Application to geometry, trigonometry, resolution of forces, moments, work done.

Further Differentiation

Related rates of change. Differentiation of implicit functions and parametric functions. Partial differentiation with application to small changes, error analysis.

Further Integration

Techniques of integration including integration by parts and inverse trigonometric substitution. Applications of definite integrals: work done by variable force, expanding gas; centroid of a plane area; volume, mass and centre of gravity of solid of revolution; second moment of area, moment of inertia.

Assessment

Total Marks 100: End of Year Written Examination 70 marks; Continuous Assessment 30 marks.

Continuous Assessment

The Continuous Assessment will be comprised of a two-hour written tests worth 15%, in weeks 5 & 8.
Absence from a test will not be considered accept in truly extraordinary cases. Plenty of notice will be given of the test date. For example, routine medical and dental appointments will not be considered an adequate excuse for missing the test.

Lectures

It will be vital to attend all lectures as many of the examples, proofs, etc. will be completed by us in class.

Tutorials

The aim of the tutorials will be to help you achieve your best performance in the tests and exam.

Exercises

There are many ways to learn maths. Two methods which aren't going to work are

1. reading your notes and hoping it will all sink in
2. learning off a few key examples, solutions, etc.

By far and away the best way to learn maths is by doing exercises, and there are two main reasons for this. The best way to learn a mathematical fact/ theorem/ etc. is by using it in an exercise. Also the doing of maths is a skill as much as anything and requires practise.
There are exercises in the notes for your consumption. The webpage may contain a link to a set of additional exercises. Past exam papers are fair game. Also during lectures there will be some things that will be *left as an exercise.* How much time you can or should devote to doing exercises is a matter of personal taste but be certain that effort is rewarded in maths.

Reading

Your primary study material shall be the material presented in the lectures; i.e. the lecture notes. Exercises done in tutorials may comprise further worked examples. While the lectures will present everything you need to know about MATH6040, they will not detail all there is to know. Further references are to be found in the library. Good references include:

- John Bird 2006, *Higher Engineering Mathematics*, Fifth Edition Ed., Newnes Oxford
- K.A. Stroud & Dexter J.Booth 2007, *Engineering Mathematics*, Sixth Edition Ed., Macmillan

The webpage may contain supplementary material, and contains links and pieces about topics that are at or beyond the scope of the course. Finally the internet provides yet another resource. Even Wikipedia isn't too bad for this area of mathematics! You are encouraged to exploit these resources; they will also be useful for further maths modules.

Exam

The exam format will roughly follow last year's. Acceding to the maxim that learning off a few key examples, solutions, etc. is bad and doing exercises is good, solutions to past papers shall not be made available (by me at least). Only by trying to do the exam papers yourself can you guarantee proficiency. If you are still stuck at this stage feel free to ask the question come tutorial time.

Chapter 1

Matrix Algebra

It is my experience that proofs involving matrices can be shortened by 50% if one throws the matrices out.

Emil Artin

Motivation: Network Flows

Suppose we have a network of one-way flows as shown in the diagram:

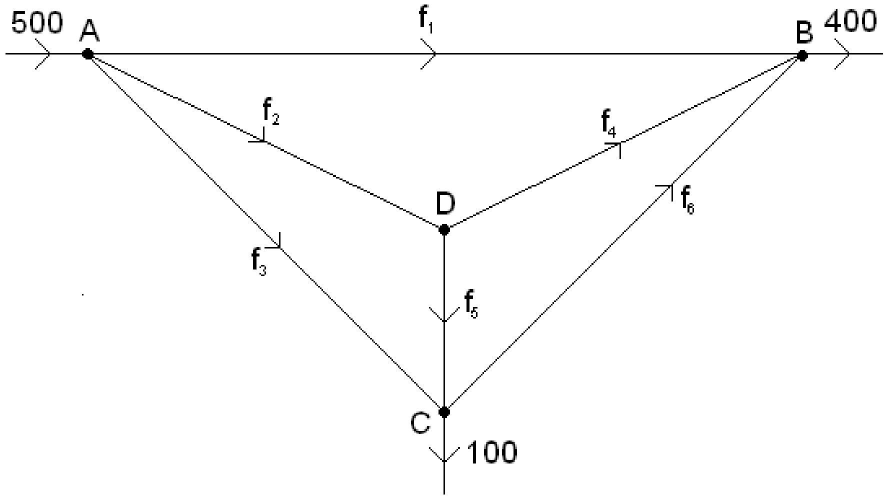


Figure 1.1: The flow into junction A is 500 , and 400 and 100 emerge from B and C. Suppose the flows are $f_1, \ldots, f_6$ as shown. From the simple rule that the flow into a junction must equal to flow out, how much can we tell about the internal workings of the network?

Equating the flow in with the flow out at each junction we get:

$$\begin{array}{lccc} \text{Junction } A & 500 & = & f_1 + f_2 + f_3 \\ \text{Junction } B & f_1 + f_4 + f_6 & = & 400 \\ \text{Junction } C & f_3 + f_5 & = & f_6 + 100 \\ \text{Junction } D & f_2 & = & f_4 + f_5 \end{array}.$$

This gives four equations in six variables $f_1, \ldots, f_6$:

$$\begin{aligned} f_1 + f_2 + f_3 &= 500 \\ f_1 + f_4 + f_6 &= 400 \\ f_3 + f_5 - f_6 &= 100 \\ f_2 - f_4 - f_5 &= 0 \end{aligned}$$

1.1 Matrices

Last semester we introduced the idea of a *function*. A function is a mathematical object that for an input produces a single output. The functions we examined last semester were functions of *a single variable* which means that we put in one number and got out one number:

Here the set of inputs (*domain*) and the set of outputs (*target*) are both the set of real numbers, $\mathbb{R}$; and we write $f : \mathbb{R} \to \mathbb{R}$. There is nothing stopping us from defining functions on sets other than the real numbers. For example, let $S = \{\text{pubs in Annascaul}\} = \{S, P, B, H, F, R\}$ and define the following function

A simple extension of the set of real numbers — the number line; is the set of points on the plane:

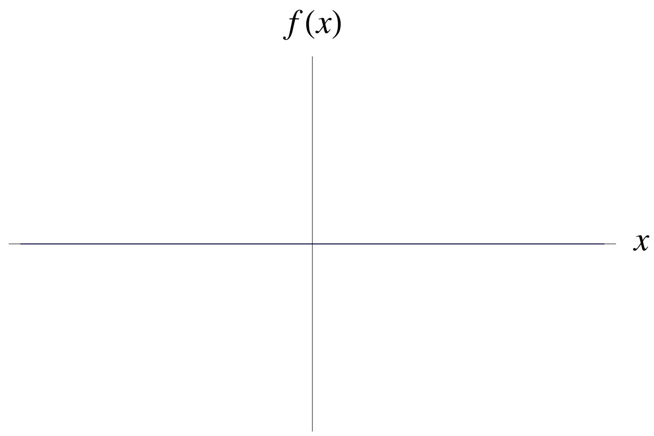


Figure 1.2: Take the collection of pairs of numbers (x, y). We call this set the *plane* and denote it by $\mathbb{R}^2$

We can extend again to the set of points in space:

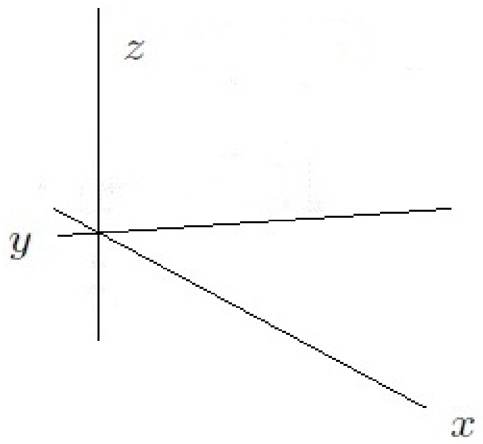


Figure 1.3: Take the collection of triples of numbers (x, y, z). We call this set *space* and denote it by $\mathbb{R}^3$

In this section we study a certain class of functions between the line, the plane and space:

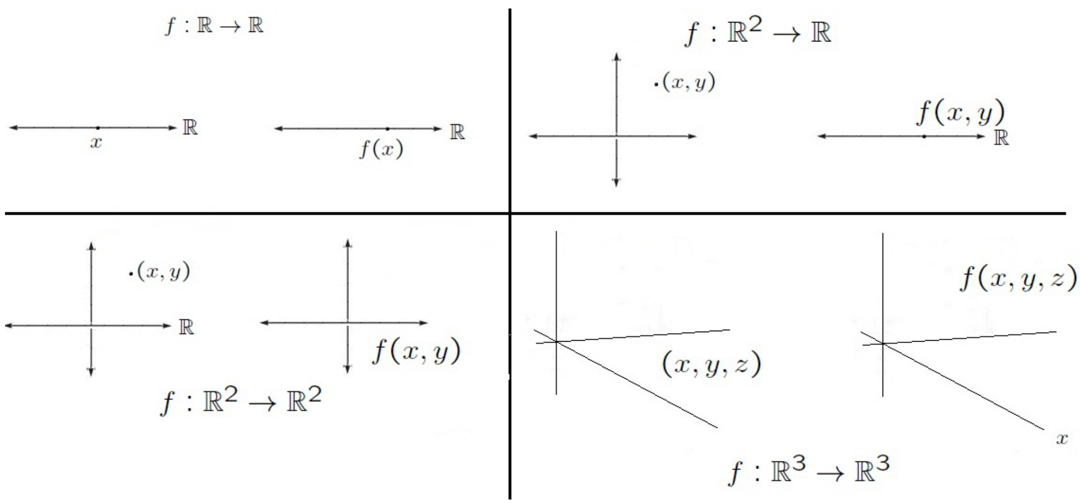


Figure 1.4: Functions between lines, planes and space.

We will study the *linear* maps between these sets. A map $f : A \to B$ is said to be *linear* if

It turns out that if we write elements of the line as (x), elements of the plane as $\begin{pmatrix} x \\ y \end{pmatrix}$ and elements of space as $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, then these linear maps can be written as *matrices* and the action of the matrix A on a point in $\mathbf{x}$ is given by *matrix multiplication*.

Examples

$$A = \begin{pmatrix} 1 & 0 \\ 2.6 & -8 \end{pmatrix}$$
$$B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$C = \begin{pmatrix} 1 & 0 & 3 \\ -16 & 0 & \sqrt{26} \end{pmatrix}$$

Remarks

1. A matrix with n rows and m columns is said to have *dimension* $n \times m$ or be an $n \times m$ matrix. For example, A is a 2×2 matrix; B is a 2×1 matrix, and C is a 2×3 matrix. We will see that an $n \times m$ matrix A is a function $A : \mathbb{R}^m \to \mathbb{R}^n$.
2. A *square* matrix is an $n \times n$ matrix and is a function $A : \mathbb{R}^n \to \mathbb{R}^n$.
3. The (i, j)-entry of a matrix is the number in the ith row and jth column.

1.1.1 Addition of Matrices

Two matrices of equal dimension may be added together to produce another matrix of the same dimension. This sum is a matrix whose elements are obtained by adding corresponding elements.

The *zero matrix* is denoted $\mathbf{0}$, and has only 0 as its entries. It satisfies

Just like zero for the real numbers.

1.1.2 Scalar Multiplication of a Matrix

Any matrix may be multiplied by a scalar (some $k \in \mathbb{R}$) by multiplying each element by the number.

By definition $-A = (-1)A$, so that $A - B$ means $A + (-B)$. Properties of matrix addition and scalar multiplication include:

$$A + B = B + A; \quad (A + B) + C = A + (B + C); \quad k(A + B) = kA + kB;$$

$$(k + l)A = kA + lA; \quad (kl)A = k(lA); \quad A - A = 0; \quad 0A = \mathbf{0}.$$

Note that these mirror the properties of ordinary addition and multiplication.

If A is an $m \times n$ matrix then the *transpose* of A, denoted A^T, is the $n \times m$ matrix whose got by exchanging the rows and columns of A. Properties of the transpose operation include:

$$(A^T)^T = A; \quad (kA)^T = kA^T; \quad (A+B)^T = A^T + B^T.$$

If $A : \mathbb{R}^n \to \mathbb{R}^m$ then $A^T : \mathbb{R}^m \to \mathbb{R}^n$.

1.1.3 Equality of Matrices

Two matrices are *equal as matrices* if they have same dimension and each corresponding element is equal.

Example

Suppose

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{ and}$$
$$B = \begin{pmatrix} 1 & 0 \\ 2.6 & -8 \end{pmatrix}$$

and one is told $A = B$. Thence $a = 1, b = 0, c = 2.6$ and $d = -8$.

1.1.4 Definition

A matrix A is *conformable* with a matrix and B if the dimension of A is $n \times k$ and the dimension of B is $k \times m$ for some $k \in \mathbb{N}$.

Remarks

1. Only a notion of multiplication between conformable matrices is considered. In this case the product of an $n \times k$ matrix and a $k \times m$ matrix is a $n \times m$ matrix.
2. This means that a matrix A may be multiplied by a matrix B to form the product AB if and only if the number of columns in A is equal to the number of rows in B.
3. Note also that if A is conformable with B it does not follow that B is conformable with A. For example, a 2×3 matrix be multiplied by a 3×4 matrix to produce a 2×4 matrix but a 3×4 matrix may not be multiplied by a 2×3 matrix
4. Two square matrices of equal dimension may be multiplied together to produce another square matrix of the same dimension. However note that the order of multiplication is important. It will be seen in general that for square matrices A and B;

   $$AB \neq BA \tag{1.1}$$

   That is the axiom of commutivity for real numbers $xy = yx$, $\forall\, x, y \in \mathbb{R}$; fails in general for an algebra of matrices.

Note that all of these make sense if we consider these matrices as functions. Note however that BA means that A acts first.

1.1.5 Definition

Let $A := [A]_{ij} = a_{ij}$ of dimension $n \times r$; and $B := [B]_{ij} = b_{ij}$ of dimension $r \times m$. Then the matrix product $AB = C = [C]_{ij}$ has matrix entries

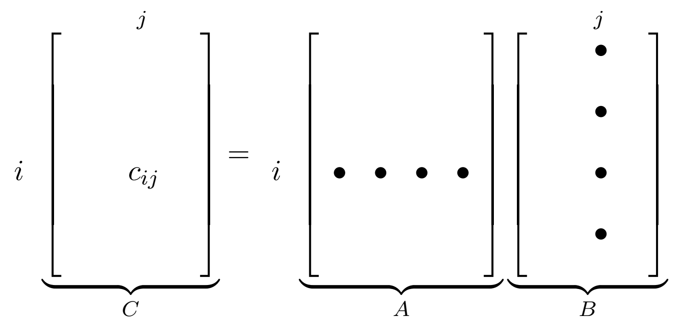
Remarks

This is the technical definition for any two conformable matrices A and B. The meaning of this equation will be discussed for the general case of two conformable matrices; and for the cases of $n \times m$ matrices with $n, m \leq 2$.

(i) The General Case;
Let A be a $n \times r$ matrix and B be a $r \times m$ matrix. What are the entries of $C = AB$? Well take the general entry that is in the i-th row and j-th column of C. This is the number c_{ij}. This is by (**??**):

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{ir}b_{rj}$$

So to find the (ij)-th element sum the numbers along the i-th row of A multiplied by the numbers along the j-th column of B:



(ii) A 1×2 matrix by a 2×1 matrix.

Note a 1×2 matrix by a 2×1 matrix is a 1×1 matrix. This is equivalent to a real number; in this case $ac + bd$.

(iii) A 1×2 matrix by a 2×2 matrix.

Note a 1×2 matrix by a 2×2 matrix is a 1×2 matrix.

(iv) A 2×2 matrix by a 2×1 matrix.

Note a 2×2 matrix by a 2×1 matrix is a 2×1 matrix.

(v) A 2×2 matrix by a 2×2 matrix.

I find the best way to remember — certainly if we are multiplying with 3×3 matrices — is as follows:

$$C = AB = \begin{pmatrix} \text{1st row by 1st column} & \text{1st row by 2nd column} & \cdots & \text{1st row by last column} \\ \text{2nd row by 1st column} & \text{2nd row by 2nd column} & \cdots & \text{2nd row by last column} \\ & & & \\ \text{last row by 1st column} & \text{last row by 2nd column} & \cdots & \text{last row by last column} \end{pmatrix} \tag{1.2}$$

Examples

1. If

$$A = \begin{bmatrix} 1 & 8 \\ 3 & -2 \\ 0 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 & 9 \\ -2 & 7 \end{bmatrix},$$

find AB.

Solution:

2. Given the matrices

$$A = \begin{bmatrix} 5 & -3 \\ -2 & -4 \\ 2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 & 3 \\ -3 & 0 & 2 \end{bmatrix},$$

determine the following sums/products if defined

(a) $2A + B$

(b) $2A + B^T$

(c) BA

Solution:

(a) This is not defined because $A : \mathbb{R}^2 \to \mathbb{R}^3$ and $B : \mathbb{R}^3 \to \mathbb{R}^2$.

(b) We have

(c) We calculate

Summer 2012: Question 1 (h)

For the matrices

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 4 & 2 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & -1 & 5 \\ 0 & -2 & 1 \end{pmatrix}$$

show that $(A + B)^T = A^T + B^T$.

[5 marks]

Solution: First of all we calculate $A + B$:

Now we find the transpose:

Now find the transposes of A and B:

and add them together:

Summer 2012: Question 2 (a) (i)

For the matrices

$$A = \begin{pmatrix} 2 & -1 & 0 \\ 3 & 2 & -1 \end{pmatrix}, \; C = \begin{pmatrix} 5 & -3 \\ 2 & 1 \\ 3 & 4 \end{pmatrix}.$$

Determine each of the following

1. $A + C$
2. CA
3. AC

Solution:

1. This is not defined because $A : \mathbb{R}^3 \to \mathbb{R}^2$ and $C : \mathbb{R}^2 \to \mathbb{R}^3$.
2. We calculate
3. We calculate

Autumn 2012: Question 1 (h)

For the matrices

$$B = \begin{pmatrix} 2 & 1 & -3 \\ 3 & -2 & 4 \end{pmatrix}, \; C = \begin{pmatrix} 1 & 0 & 3 \\ -2 & 0 & 2 \\ 4 & 5 & -3 \end{pmatrix}.$$

show that $(BC)^T = C^T B^T$.

[5 marks]

Solution: First we calculate BC:

Now we take the transpose

Now find C^T and B^T:

and multiply them together:

Autumn 2012: Question 2 (a) (i)

For the matrices

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 3 & -2 & -1 \end{pmatrix}, \; B = \begin{pmatrix} 1 & 2 & 1 \\ -3 & 2 & 0 \end{pmatrix}, \; C = \begin{pmatrix} 5 & 3 \\ 2 & -1 \\ 3 & 4 \end{pmatrix}.$$

Determine each of the following, if defined

1. $A+B$
2. $A+C$
3. $(A+B)^T$
4. AC
5. CA

Solution:

1. We calculate
2. This is not defined because $A:\mathbb{R}^3\to\mathbb{R}^2$ and $C:\mathbb{R}^2\to\mathbb{R}^3$.
3. We write
4. We calculate
5. We calculate

Exercises

1. Let $A = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 3 \\ -1 & 0 \\ 1 & 4 \end{bmatrix}$. Compute the following (where possible):

$$(i)\, 3A - 2B \;\; (ii)\, 5C \;\; (iii)\, 4A^T - 3C \;\; (iv)\, B + D \;\; (v)\, (A+C)^T \;\; (vi)\, A - D.$$

2. Find A if

   (a) $5A - \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = 3A - \begin{bmatrix} 5 & 2 \\ 6 & 1 \end{bmatrix}$.

   (b) $3A + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 5A - 2\begin{bmatrix} 3 \\ 0 \end{bmatrix}$.

   (c) $\left(3A^T + 2\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right) = \begin{bmatrix} 8 & 0 \\ 3 & 1 \end{bmatrix}$.

   (d) $\left(2A^T - 5\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}\right)^T = 4A - 9\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$.

3. Compute the following matrix products (if possible):

   (a) $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & 4 \end{bmatrix}\begin{bmatrix} 2 & 3 & 1 \\ 1 & 9 & 7 \\ -1 & 0 & 2 \end{bmatrix}$.

   (b) $\begin{bmatrix} 1 & 3 & -3 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ -2 & 1 \\ 0 & 6 \end{bmatrix}$.

   (c) $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix}$.

   (d) $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}\begin{bmatrix} a' & 0 & 0 \\ 0 & b' & 0 \\ 0 & 0 & c' \end{bmatrix}$.

   (e) $\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} -1 & 6 \\ 1 & 0 \end{bmatrix}$.

   (f) $\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 1 & 2 \end{bmatrix}$.

4. Let A, B and C be matrices.

   (a) If A^2 can be formed, what can be said about the size of A.

   (b) If AB and BA can both be formed, describe the sizes of A and B.

   (c) If ABC can be formed, A is 3×3 and C is 5×5, what size is B.

1.2 Matrix Inverses

What does a 2×2 matrix *do*? Well it sends points (x, y) on the plane to other points on the plane:

An inverse of a matrix sends the points back to where they came from so that hitting a point with A and then by 'A^{-1}' will send it back from whence it came.

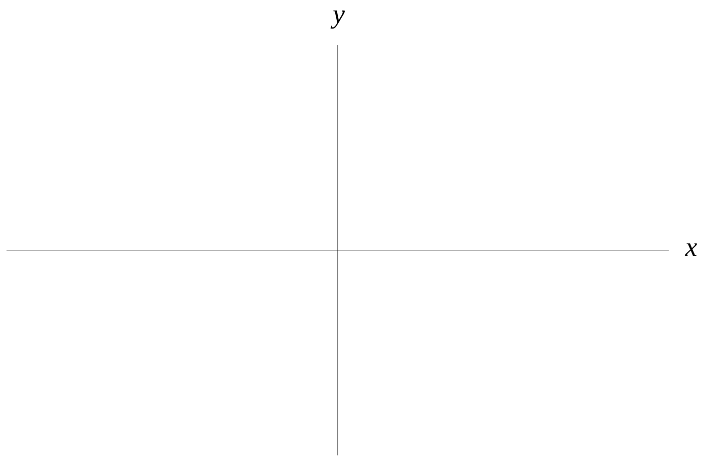


Figure 1.5: The action of a matrix and its inverse.

This means that if you hit a point with A and then A^{-1} the point goes nowhere. The matrix that send a point to itself is known as the *identity* matrix, I and is given by:

This means that if A is a matrix and A^{-1} it's inverse that

We do also require that $A^{-1}A = I$. Not all matrices are invertible however. Consider the 2×2 matrix given by

In contrast to arithmetic in the real numbers, $\mathbb{R}$, every non-zero number x has a *multiplicative inverse* x^{-1} given by the number $1/x$ with the property:

where '1' is the *multiplicative identity* with the special property that for all $x \in \mathbb{R}$:

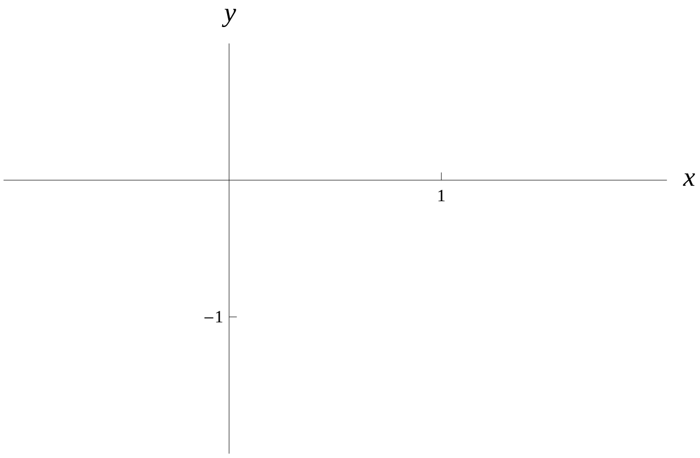


Figure 1.6: How can you send back from the origin?

The identity matrix I that is like the 1 for matrix multiplication:

How can we tell if a matrix has an inverse? Can I matrix have more than one inverse? If we know that it has an inverse how do we find it?

1.2.1 Inverses of 2×2 Matrices

Let A be a 2×2 matrix

where $a, b, c, d \in \mathbb{R}$ are real numbers. If you solve the equations

for x, y, z, w then you have the inverse of A. It is tricky but not impossible to show that

1.2.2 Formula

Given a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad - bc \neq 0$, the (multiplicative) *inverse* of A, A^{-1} is the matrix

$$\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \tag{1.3}$$

This formula is in the tables.

There is no such neat formula for 3×3 [1] but we have a method of finding the inverse of a matrix that works for 3×3 and larger — as well as for 2×2 matrices.

[1] well there is but it's a bit 'big': http://ardoris.wordpress.com/2008/07/18/general-formula-for-the-inverse-of-a-3x3-matrix/

Example

Find the inverse matrix of

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 3 \end{bmatrix}.$$

Solution: Using the formula

Exercises: Find the inverses of the matrices

$$\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}, \qquad \begin{bmatrix} 4 & 6 \\ 3 & 2 \end{bmatrix}, \qquad \begin{bmatrix} 3 & 4 \\ -2 & 1 \end{bmatrix} \qquad \begin{pmatrix} 2 & -c \\ c & 3 \end{pmatrix}$$

Answers:

$$\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}, \qquad \frac{1}{10}\begin{bmatrix} -2 & 6 \\ 3 & -4 \end{bmatrix}, \qquad \frac{1}{11}\begin{bmatrix} 1 & -4 \\ 2 & 3 \end{bmatrix}, \qquad \frac{1}{c^2+6}\begin{pmatrix} 3 & c \\ -c & 2 \end{pmatrix},$$

1.2.3 Row Operations

The method which we will use to find the inverse of a matrix is by a method called Gauss-Jordan Elimination. When we are finished, we will have a method that is straightforward to implement and finds matrix inverses with an amount of elementary calculations. The method will not show any of the theoretical scaffolding behind the method but here we will give a flavour of how the method works.

We start with a matrix A whose inverse we would like to compute. What we are going to do is multiply the matrix A by a sequence of matrices $E_1, E_2, \dots, E_n$ until we have that

Now can we say anything about the matrix $E_n \cdots E_2 E_1$?

Now how do we know which matrices E_i to multiply by... and do we have to multiply them altogether at the end? These matrices are called *elementary matrices* and there are three types.

1. *Switches Rows* Given a matrix A, there is a matrix $E_{i \leftrightarrow j}$ that switches the ith and jth rows of A.

   Example:

2. *Multiplies a row by a constant* Given a matrix A, there is a matrix $E_{i \to ki}$ that multiplies the ith row of A by a constant k.

Example:

3. *Adds a row to another row* Given a matrix A, there is a matrix $E_{i\to i+j}$ that adds the jth row to the ith row.

   Example:

These last two matrices may be combined

to produce a matrix $E_{i\to i+kj}$ that adds a constant multiple of the jth row to the ith row. So we have these three (four) types of matrices and they have three (four) associated *elementary row operations*; r_i means the ith row.

1. *Switch Rows* Given a matrix A, the row operation $r_i \leftrightarrow r_j$ switches the ith and jth rows of A.

   Example:

2. *Multiply a row by a constant* Given a matrix A, the row operation $r_i \to kr_i$ multiplies the ith row of A by a constant k.

   Example:

3. *Add a row to another row* Given a matrix A, the row operation $r_i \to r_i + r_j$ adds the jth row to the ith row.

   Example:

4. *Add a constant multiple of a row to another row* Given a matrix A, the row operation $r_i \to r_i + kr_jj$ adds k times the jth row to the ith row.

   Example:

Theorem

Suppose that E is an elementary matrix that has the action of the elementary row operation r. *Then*

$$E = rI \tag{1.4}$$

where I is the identity matrix.

Proof. Beyond the scope of the course •

This means that to multiply by an elementary matrix E you can instead just do the elementary row operation that E induces... it would be a lot easier to just forget about elementary matrices and just use elementary row operations. Luckily we can do this. If we write the matrix A we want to invert beside the identity matrix like this:

what we can do is keep doing row operations on A until it is has been transformed into the identity — *while doing exactly the same row operations on the identity matrix*:

Then the matrix B here is A^{-1}! How? Well if we do row operations $r_1, \dots, r_n$ to A, then we also do them to the identity I:

We know that this is the same as multiplying by elementary matrices $E_1, \dots, E_n$:

Now as discussed above if $E_n \cdots E_2 E_1 A = I$ then $E_n \cdots E_1 = A^{-1}$... and what does $E_n \cdots E_1 I$ equal to?

So we have a method of calculating the inverse of A. Write

Apply elementary row operations to A *and* I until we have

Then the matrix left on the right is A^{-1}. Now is there a neat way to get from $A \longrightarrow I$ using row operations? The answer is yes: the Gauss-Jordan algorithm.

1.2.4 Gaussian-Jordan Elimination

Recall the elementary row operations (EROs); r_i means the ith row.

- swapping any two rows:

- multiplying any row by a constant:

- adding any row to any other:

- combining the last two: adding a multiple of a row to another row:

Recall that you want to get, say, a 3×3 matrix in the form of the identity:

The best way to do this is to work like this

Taking a matrix and applying EROs until it is the identity is called the *Gauss-Jordan Algorithm*. We can write down the algorithm as follows. This version assumes that the matrix is 3×3 and invertible but extends in the natural way for an $n \times n$ matrix. You can show that the Gauss-Jordan Algorithm cannot be done if A is *not* invertible.

1. (a) If the $(1,1)$ entry is a zero use the ERO $r_1 \leftrightarrow r_j$ to swap the first row with a row which has a non-zero entry in the first column.

   (b) If the $(1,1)$ entry is $a \neq 0$ then use the ERO $r_1 \to \dfrac{1}{a} r_1$ to multiply the first row by $\dfrac{1}{a}$/divide the first row by a to turn it into a 1. This 1 in the $(1,1)$ entry is the *first pivot*.

   (c) Now *use the first pivot* to turn the entries below the pivot into zeroes using the ERO $r_i \to r_i + kr_1$.

2. (a) If the $(2,2)$ entry is a zero use the ERO $r_2 \leftrightarrow r_3$ to swap the second row with the third row.

   (b) If the $(2,2)$ entry is $b \neq 0$ then use the ERO $r_2 \to \dfrac{1}{b} r_2$ to multiply the second row by $\dfrac{1}{b}$/divide the first row by b to turn it into a 1. This 1 in the $(2,2)$ entry is the *second pivot*.

   (c) Now *use the second pivot* to turn the entry below the pivot into zeroes using the ERO $r_3 \to r_3 + k r_2$.

3. The $(3,3)$ entry is $c \neq 0$. Use the ERO $r_3 \to \dfrac{1}{c} r_3$ to multiply the second row by $\dfrac{1}{c}$/divide the first row by c to turn it into a 1. This 1 in the $(3,3)$ entry is the *third pivot*.

4. Use the third pivot to turn the $(1,3)$ and $(2,3)$ entries into a zero using EROs of the form $r_2 \to r_2 + k r_3$ and $r_1 \to r_1 + k r_3$.

5. Use the second pivot to turn the $(1,2)$ entry into a zero using an ERO of the form $r_1 \to r_1 + k r_2$.

When you have implemented Gauss-Jordan elimination the matrix is said to be in *reduced row form.*

Examples

1. Use Gauss-Jordan elimination to find A^{-1} where

$$A = \begin{bmatrix} 1 & 0 & 8 \\ 2 & 5 & 3 \\ 1 & 2 & 3 \end{bmatrix}.$$

Solution: First of all we write it in the $[A \,|\, I]$ form:

We first note that the $(1,1)$ entry is fine so we look at the 2 and 1 below it. *Use the pivot to 'kill' entries below it.* We do this with the EROS $r_2 \to r_2 - 2r_1$ and $r_3 \to r_3 - r_1$. **Make sure to do the same to the matrix on the right**:

Now we turn out attention to the $(2,2)$ entry. This is to be the next pivot. *To make a pivot multiply/divide.* We do this with the ERO $r_2 \to \frac{1}{5} r_2 = r_2 \div 5$. Then we will use this pivot to 'kill' the 2 in the $(3,2)$ entry. We will do this with $r_3 \to r_3 - 2r_1$. **Make sure to do the same to the matrix on the right**:

Luckily we now have a pivot in the $(3,3)$ position so this entry is done; nearly there. We use the pivot here to 'kill' the two entries above it using $r_2 \to r_2 + 3r_3$ and $r_1 \to r_1 - 8r_1$. **Make sure to do the same to the matrix on the right**:

Finally the $(1,2)$ entry is already a 0 so we are in reduced row form so we are done. Now by our theory the matrix on the right is A^{-1} so we write:

If you want to check that you haven't made a mistake you could check that $A^{-1}A = AA^{-1} = I$. We will calculate $A^{-1}A$ to verify our work:

2. Determine A^{-1} where $A = \begin{bmatrix} 1 & 1 & -1 \\ -3 & 2 & -1 \\ 3 & -3 & 2 \end{bmatrix}$.

   Solution: First of all we write it in the $[A \,|\, I]$ form:

   We first note that the $(1,1)$ entry is fine so we look at the -3 and 3 below it. *Use the pivot to 'kill' entries below it.* We do this with the EROS $r_2 \to r_2 + 3r_1$ and $r_3 \to r_3 - 3r_1$. **Make sure to do the same to the matrix on the right**:

Now we turn out attention to the $(2, 2)$ entry. This is to be the next pivot. *To make a pivot multiply/divide.* We do this with the ERO $r_2 \to \frac{1}{5}r_2 = r_2 \div 5$. Then we will use this pivot to 'kill' the -6 in the $(3, 2)$ entry. We will do this with $r_3 \to r_3 + 6r_1$. **Make sure to do the same to the matrix on the right**:

Now we turn out attention to the $(3, 3)$ entry. This is to be the next pivot. *To make a pivot multiply/divide.* We do this with the ERO $r_3 \to 5r_3$. We use the pivot here to 'kill' the two entries above it using $r_2 \to r_2 + \frac{4}{5}r_3$ and $r_1 \to r_1 + r_3$. **Make sure to do the same to the matrix on the right**:

Finally we must 'kill' the 1 in the $(1, 2)$. We do this using the pivot at $(2, 2)$ by using $r_1 \to r_1 - r_2$:

Now by our theory the matrix on the right is A^{-1} so we write:

Exercises

1. Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 3 \\ 1 & 3 & 1 \end{bmatrix}.$$

Ans: $\begin{bmatrix} 7 & 2 & -6 \\ -3 & -1 & 3 \\ 2 & 1 & -2 \end{bmatrix}$

2. Find the inverse of the matrix

$$B = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}.$$

Ans: $\begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix}$

1.2.5 Matrix Equations

How do you solve the equation

In reality you don't need division at all. What could you do instead?

This number, $\dfrac{1}{5}$ is known as the *multiplicative inverse* of the real number 5 in pure maths circles. Instead of division, you have multiplication by the inverse:

Suppose A and B are two known matrices. How do you solve the matrix equation and isolate the matrix X?

Rather than say *divide by* A we instead multiply both sides by the (multiplicative) inverse of A; i.e. the inverse A^{-1}:

Now the crucial, crucial difference between a matrix equation and (real) number equation is that the *order of multiplication* matters. For matrices $AB \neq BA$ necessarily. Hence we must multiply both sides on the right — or multiply both sides on the left... and you can't do a hodge-podge.

Autumn 2012: Question 2 (a) (iii)

Given the matrices

$$A = \begin{pmatrix} 1 & -3 \\ 4 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 16 & 32 \\ 36 & 84 \end{pmatrix}$$

determine the matrix X such that $XA = C$.

Solution: We need to eliminate the A from the right-hand side. How do we do this?

Now we can find A^{-1} using the formula in the tables or via Gauss-Jordan elimination. I will do this example using Gauss-Jordan elimination[2]. First we write in the (augmented) $[A\,|\,I]$ form:

[2] which of course gives the same solution

The first pivot is there so we use it to 'kill' the four underneath. How?

Now we turn the 18 into a pivot by multiplying by $\frac{1}{18}$/ dividing by 18. Then we can use this pivot to 'kill' the -3 and put the matrix in reduced row form:

Hence we know that

Our algebra has shown that $X = CA^{-1}$ so we multiply

Summer 2012: Question 2 (a) (iii)

Given the matrices

$$B = \begin{pmatrix} 1 & 3 \\ 5 & 7 \end{pmatrix}, \quad D = \begin{pmatrix} 14 & 30 \\ 38 & 86 \end{pmatrix}$$

determine the matrix X such that $BX = D$.

Solution: We need to eliminate the B from the right-hand side. How do we do this?

Now we can find B^{-1} using the formula in the tables:

Our algebra has shown that $X = B^{-1}D$ so we multiply

Exercises

1. Find the matrix B such that A is another matrix such that

$$A = \begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix} \quad \text{and} \quad AB = \begin{pmatrix} 0 & -3 \\ 2 & 4 \end{pmatrix}$$

   Ans: $\begin{pmatrix} -14 & -37 \\ 10 & 26 \end{pmatrix}$

2. **Winter '12 Q. 2 (a) (i)** Given

$$A = \begin{pmatrix} 1 & -2 \\ 3 & 5 \end{pmatrix}, \qquad C = \begin{pmatrix} 12 & 22 \\ 16 & 24 \end{pmatrix}.$$

   Determine X is X satisfies $XA = C$.

   Ans: $-\frac{1}{11}\begin{pmatrix} 6 & -46 \\ -8 & -56 \end{pmatrix}$

3. Let A be an invertible matrix. Show that if $AX = AY$ then $X = Y$. Show that if $PA = QA$ then $P = Q$.

4. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Verify that $AB = CA$, A is invertible, but that $B \neq C$.

1.3 Linear Systems

Yes but what has all this got to do with network flows? Well let us set of some kind of a network flow with losses:

Recalling that we want flow-in equal to flow-out (at each junction). We end up with simultaneous equations of the form:

It is still not clear what the hell this has got to do with matrices. However, we can rewrite the set of simultaneous equations as a matrix equation:

Sometimes we call the matrix A the *matrix of coefficients*, $\mathbf{x}$ the *unknowns* and $\mathbf{b}$ the *constants*. Look what happens when we multiply this out:

As expected. Recall that we want to find the column matrix $\mathbf{x}$ (in the next section we will call this a *vector*). How do we solve this equation:

Now that is the algebraic picture and pretty much all we need to do to solve these simultaneous equations. A *linear system* is just a fancy name for a set of simultaneous equations of the form

We can describe this situation geometrically also. For example consider the 2×2 system:

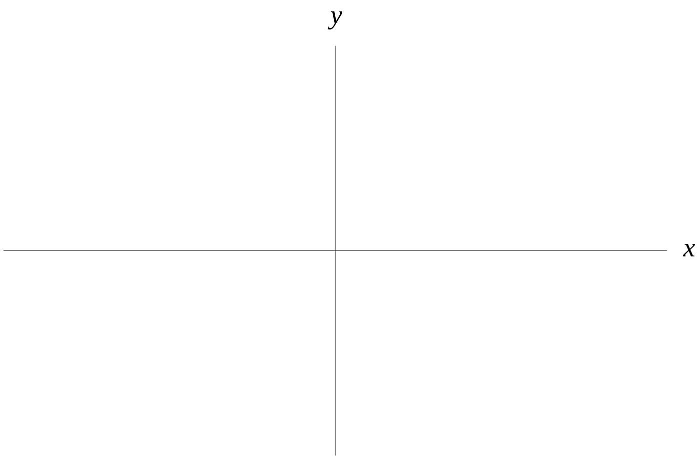


Figure 1.7: This linear system is seen to be equivalent to the problem, *which points* (x, y) *are sent to the point* $(6, 7)$ *by the matrix of coefficients* A. We use the inverse to go from $(6, 7)$ back to the solution (x, y).

Winter 2012: Question 1 (g)

Express the system of equations

$$\begin{aligned} x + 2y &= 3 \\ 3x + y &= -1 \end{aligned}$$

in matrix form. Use the inverse matrix method to solve it and check your answer.

[5 Marks]

Solution: We take the coefficients and constants and write

To solve this we multiply both sides on the left by A^{-1}:

Hence we must find the inverse of A. We use the equation in the tables:

Now we just multiply:

We can check our answer with a substitution:

Summer 2012: Question 1 (g)

Express the system of equations

$$\begin{aligned} x - 3y &= 1 \\ 2x + 4y &= 4 \end{aligned}$$

in matrix form. Use the inverse matrix method to solve it and check your answer.

[5 Marks]

We take the coefficients and constants and write

To solve this we multiply both sides on the left by A^{-1}:

Hence we must find the inverse of A. We use the equation in the tables:

Now we just multiply:

We can check our answer with a substitution:

Winter 2012: Question 2 (c)

Obtain the inverse of the matrix

$$\begin{pmatrix} 1 & 3 & 2 \\ 2 & -1 & -3 \\ 5 & 2 & 1 \end{pmatrix}.$$

Use this to solve the system of equations

$$\begin{aligned} x + 3y + 2z &= 3 \\ 2x - y - 3z &= -8 \\ 5x + 2y + z &= 9 \end{aligned}$$

Solution: To find the matrix inverse we apply the Gauss-Jordan algorithm once we have written the matrix in augmented form.

Straight away we have the first pivot and so can 'kill' the terms below:

Now we must make our second pivot by dividing by -7. When this is done we can use it to 'kill' the term underneath::

Now we can make our final pivot by dividing by four. Then we can use it to 'kill' the entries above:

Finally we kill the final entry using the second pivot:

Now we have that

We note that the system of equations may be written as

To find the unknowns multiply on the left by A^{-1}:

Summer 2012: Question 2 (c)

Obtain the inverse of the matrix

$$\begin{pmatrix} 1 & -2 & 1 \\ 3 & 2 & -1 \\ -1 & 3 & 5 \end{pmatrix}.$$

Use this to solve the system of equations

$$\begin{aligned} x - 2y + z &= -4 \\ 3x + 2y - z &= 8 \\ -x + 3y + 5z &= 0 \end{aligned}$$

Solution: To find the matrix inverse we apply the Gauss-Jordan algorithm once we have written the matrix in augmented form.

Straight away we have the first pivot and so can 'kill' the terms below:

Now we must make our second pivot by multiplying/dividing. When this is done we can use it to 'kill' the term underneath::

Now we can make our final pivot by multiplying/dividing. Then we can use it to 'kill' the entries above:

Finally we kill the final entry using the second pivot:

Now we have that

We note that the system of equations may be written as

To find the unknowns multiply on the left by A^{-1}:

Exercises

Find all the solutions of each of the following systems of linear equations using the inverse matrix method. Note that answers may be checked using substitution:

$$(i)\ \begin{aligned} x+2y&=1\\ 3x+4y&=-1\end{aligned} \qquad (ii)\ \begin{aligned} 3x+4y&=1\\ 4x+5y&=-3\end{aligned}$$

$$(iii)\ \begin{aligned} 2x+y+z&=-1\\ x+2y+z&=0\\ 3x-2z&=5\end{aligned} \qquad (iv)\ \begin{aligned} 3x-2y+z&=-2\\ x-y+3z&=5\\ -x+y+z&=-1\end{aligned}$$

$$(v)\ \begin{pmatrix}1&1&1\\3&5&1\\2&6&1\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}4\\3\\2\end{pmatrix} \qquad (vi)\ \begin{pmatrix}1&1&1\\3&5&4\\5&1&2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}3\\12\\10\end{pmatrix}$$

$$(vii)\ \begin{pmatrix}6&8&3\\1&1&1\\7&5&5\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}10\\0\\0\end{pmatrix} \qquad (viii)\ \begin{aligned} x+2y+3z&=2\\ 2x+3y+z&=0\\ 3x+y+4z&=6\end{aligned}$$

$$(ix)\ \begin{pmatrix}2&3&2\\3&1&-1\\4&-1&3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\5\\9\end{pmatrix} \qquad (x)\ \begin{aligned} 2x+4y+4z&=2\\ 4x+7y+9z&=3\\ 5x+9y+7z&=8\end{aligned}$$

Selected Answers: **(x)** $x=1$, $y=-1$, $z=1$ **(xi)** $x=2$, $y=-1$, $z=0$ **(xii)** $x=3$, $y=0$, $z=-1$

1.4 Determinants

There is a quantity related to square matrices called the *determinant*. For MATH6040 we only need to calculate determinants of 2×2 and 3×3 matrices. What *are* they geometrically though?

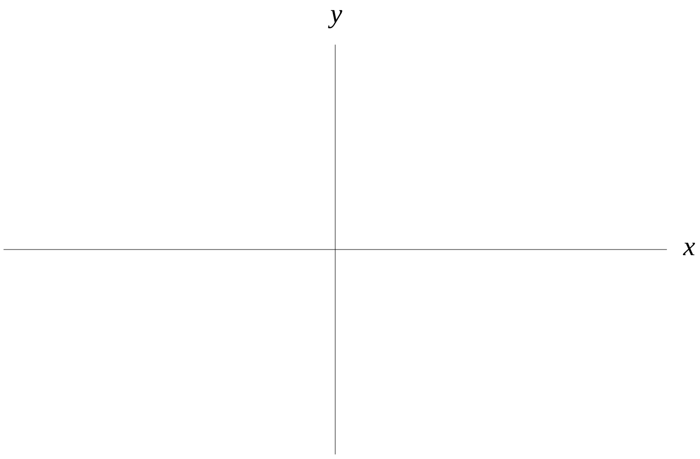
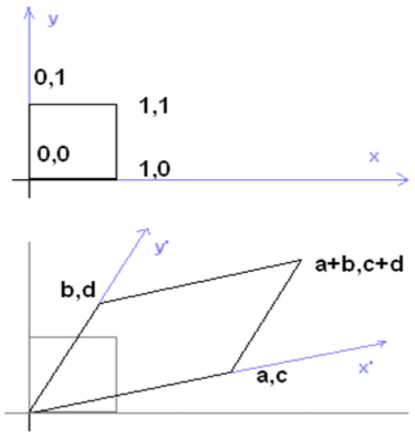


Figure 1.8: If a 2×2 matrix M_1 is applied to a region of area A then the area of the image will be $\det M_1 \times A$. This means that a matrix M_2 with $\det M_2 = 0$ cannot be invertible as an infinite number of points will be sent to a single point or line — dimension arguments show that we cannot send these points back faithfully.



$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a+b \\ c+d \end{bmatrix}$$

Figure 1.9: We can somehow get a handle on this by taking as unit area/volume the *parallelepiped* generated by the basis vectors. It is not too difficult to show that the area of the parallelogram is $ad - bc$ and that this generalises neatly to any dimension $n \geq 4$.

Definition & Formula

Let A be a square matrix with column vectors $c_1, c_2, \ldots, c_n$. The *determinant* of A, $\det A$ is the volume of the parallelepiped generated by the column vectors. The determinant of a 2×2 matrix is given by:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc. \tag{1.5}$$

$|A|$ is another notation for the determinant of A. We have the following which we may have learnt about in a previous module.

Theorem

A square matrix A is invertible if and only if $\det A$ is non-zero.

Proof. If $\det A = 0$ then clearly A can not be invertible according to Figure 1.8. There is a formula for the inverse of a square matrix: $A^{-1} = \dfrac{1}{\det A} C$ where C is a matrix defined in terms of A. If $\det A \neq 0$ then this formula is valid and A is invertible •

If the matrix of coefficients is not invertible then there are either *no* solutions or an infinite number of solutions.

Winter 2012: Question 2 (b)

Find the values of t for which the determinant of the matrix

$$\begin{bmatrix} 1 & 3+t \\ 3t & -6 \end{bmatrix}$$

is equal to zero.

[3 Marks]

Solution: We find the determinant

We now solve this equal to zero:

Now we want to find the roots of $t^2 + 3t + 2$:

Summer 2012: Question 2 (b)

Find the values of λ for which the determinant of the matrix

$$\begin{bmatrix} 6-\lambda & -3 \\ 2 & 1-\lambda \end{bmatrix}$$

is equal to zero.

[3 Marks]

Solution: We find the determinant

We now solve this equal to zero:

Now we want to find the roots of $\lambda^2 - 7\lambda + 12$:

Exercises Consider the matrices

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, B = \begin{pmatrix} 3 & 4 \\ 4 & 5 \end{pmatrix}.$$

Solve the equations $\det(A - \lambda I) = 0$ and $\det(B - kI) = 0$.

Answers: $\frac{1}{2}(5 + \sqrt{17})$, $4 \pm \sqrt{17}$.

Determinants of 3×3 Matrices

The full and proper way to calculate the determinant of any $n \times n$ square matrix is given by the *Laplace Expansion.* However as the justification for this is a little difficult and knowing why it works doesn't really help us answer questions later, I decided to give ye a little respite from theory and show you a shortcut that works for finding the determinant of a 3×3 matrix *only.* It is called *Sarrus Rule* and I consider it a memorisation scheme rather than real maths... but it will get you the answer. Like Gauss-Jordan I am going to call it an *algorithm*[3]. It can be shown to work all the using the Laplace Expansion.

Sarrus Algorithm

1. We first write down our matrix

2. Now we put the first and second column to the *right* of the matrix

3. Now we multiply along all of the diagonals

4. The determinant of the matrix is equal to the sum of the left-diagonals minus the sum of the right-diagonals.

If anyone doesn't like this method I will be happy to show them how to do a Laplace expansion in the tutorial.

[3] a mechanical procedure for solving a problem in a finite number of steps

1.5 Cramer's Rule

Suppose that a linear system has the form

where A is a square matrix, $\mathbf{b}$ is the vector of constants and $\mathbf{x}$ is the vector of variables:

Formula

Suppose that we have a square linear system

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \equiv A\mathbf{x} = \mathbf{b} \equiv [A \,|\, b] \tag{1.6}$$

with $\det A \neq 0$. Then the solutions are given as

$$x_i = \frac{\det A_i}{\det A} \tag{1.7}$$

where A_i is the matrix formed by replacing column i by the constant vector $\mathbf{b}$:

When we are solving for say x, y and maybe z we simply use

$$x = \frac{D_x}{D},\ y = \frac{D_y}{D},\ z = \frac{D_z}{D} \tag{1.8}$$

Example

Use Cramer's Rule to solve the linear system

$$\begin{aligned} 4x + 5y &= 8 \\ x - y &= 11 \end{aligned}$$

Solution: First we find $x = \dfrac{D_x}{D}$:

Now we find $y = \dfrac{D_y}{D}$:

Remark

This seems too good to be true — such a simple formula for solving simultaneous equations. Much quicker than Gaussian elimination methods and even *ad hoc* methods... Don't be fooled. Cramer's Rule only applies when the linear system is square and has a unique solution. The only way to know this in advance is to calculate the determinant of the coefficient matrix. For 2×2 this is simple. For 3×3 a little harder but not impossible. However as the size of the system increases Cramer's Rule takes comparatively longer and longer to implement in comparison to Gaussian elimination methods because determinants of larger matrices take an increasingly long time to compute.

Why the hell do we even use Cramer's Rule so? Well suppose that you have a real physical linear system (e.g. some kind of network flow) that you know must somehow have a unique solution. Sometimes you will not be interested in all of the variables but only one: this is where Cramer's Rule's strength lies. With Gaussian elimination methods you either have none of the solutions or one of the solutions. With Cramer's Rule you can find only one variable if you want. In MATH6040 we will be using it in an exam situation to check an answer or approximation.

Also I have omitted the proof. It relies on a number of properties of determinants.

Winter 2012: Question 2(c)

Consider system of equations

$$\begin{aligned} x + 3y + 2z &= 3 \\ 2x - y - 3z &= -8 \\ 5x + 2y + z &= 9 \end{aligned}$$

Use Cramer's Rule to verify the answer that $y = -3$.

Solution: We want to calculate $\dfrac{D_y}{D}$. First we find D using the Sarrus Algorithm:

1. We first write down our matrix

2. Now we put the first and second column to the *right* of the matrix

3. Now we multiply along all of the diagonals

4. The determinant of the matrix is equal to the sum of the left-diagonals minus the sum of the right-diagonals.

Now D_y is the determinant of the matrix of coefficients *except with the y-column replaced by the constants*; i.e. we want the determinant of

We apply the Sarrus Algorithm to this:

1. We first write down our matrix

2. Now we put the first and second column to the *right* of the matrix

3. Now we multiply along all of the diagonals

4. The determinant of the matrix is equal to the sum of the left-diagonals minus the sum of the right-diagonals.

Now we have

as required.

Summer 2012: Question 2 (c)

Consider the system of equations

$$\begin{aligned} x - 2y + z &= -4 \\ 3x + 2y - z &= 8 \\ -x + 3y + 5z &= 0 \end{aligned}$$

Use Cramer's Rule to verify the answer that $z = -1$.

Solution: We want to calculate $\dfrac{D_z}{D}$. First we find D using the Sarrus Algorithm:

1. We first write down our matrix

2. Now we put the first and second column to the *right* of the matrix

3. Now we multiply along all of the diagonals

4. The determinant of the matrix is equal to the sum of the left-diagonals minus the sum of the right-diagonals.

Now D_z is the determinant of the matrix of coefficients *except with the z-column replaced by the constants*; i.e. we want the determinant of

We apply the Sarrus Algorithm to this:

1. We first write down our matrix

2. Now we put the first and second column to the *right* of the matrix

3. Now we multiply along all of the diagonals

4. The determinant of the matrix is equal to the sum of the left-diagonals minus the sum of the right-diagonals.

Now we have

as required.
Exercise:
Use the inverse matrix method to solve

$$\begin{aligned} -x + y + z - 3, \\ -2x - 3y - z = 2, \\ 2x - 3y - z = 1. \end{aligned}$$

Verify y using Cramer's Rule.

This is the end of the section on matrices. There are more questions to be found in the past exam papers http://www.mycit.ie/examinations/exampapers/.

Chapter 2

Vector Algebra

> *The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve.*
>
> E. P. Wigner

The laws of physics which govern the known universe and thus engineering too, make a special distinction between *scalar* quantities and *vector* quantities. In the first instance we are interested in scalar quantities:

1. what is the length of that beam?
2. what is the mass of that gas?
3. how much time does it take for that metal to cool?
4. what temperature is that liquid?

All of these things can be described by a single number; e.g. 5 m, 125 g, 2 hours, 110 $^\circ$ C. However not everything can be modeled so easily. Consider the following question:

> *When a sled is pulled across the ice, there is a tension force and a friction force. What is the nett force?*

The issue here is that forces in different directions are very different quantities:

Quantities which have a strength/magnitude as well as a direction are known as *vector* quantities. In this chapter we will largely study vectors abstractly and get back to the interpretations and applications later.

2.1 Geometric & Coordinate Vectors

2.1.1 Plane Vectors

Throughout this chapter we shall try to do things both geometrically[1] & algebraically and in two dimensions (the plane $\mathbb{R}^2$) & three dimensions (space $\mathbb{R}^3$). Firstly we have vectors in the plane:

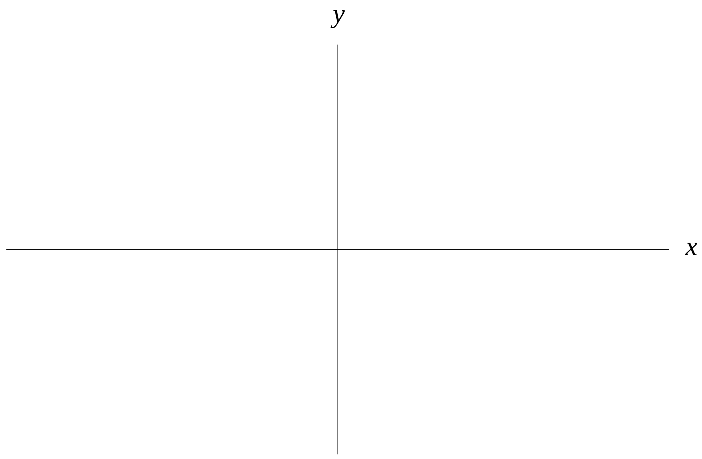


Figure 2.1: A plane vector **v** of any magnitude and any direction can be represented as shown. We will refer to this representation as the geometric picture and refer to the vector as a geometric vector.

The *magnitude* is the length of the vector while the *direction* is the angle made with the positive x-axis. Another way to represent vectors is as *coordinate vectors*:

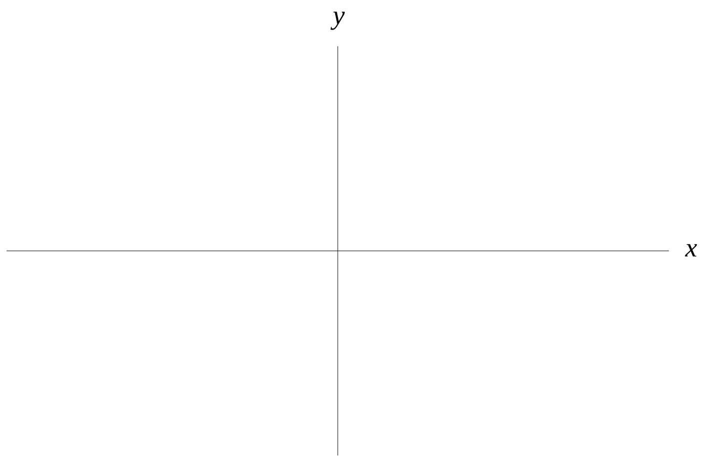


Figure 2.2: If you choose any point P on the plane, then this implicitly defines a vector **v**.

All of these plane coordinate vectors are of the form $\mathbf{v} = (a, b)$. We should be interested in how to convert from geometric to algebraic and vice versa.

[1] a bit further down the road this might be called the polar form of the vector

As ever a picture helps:

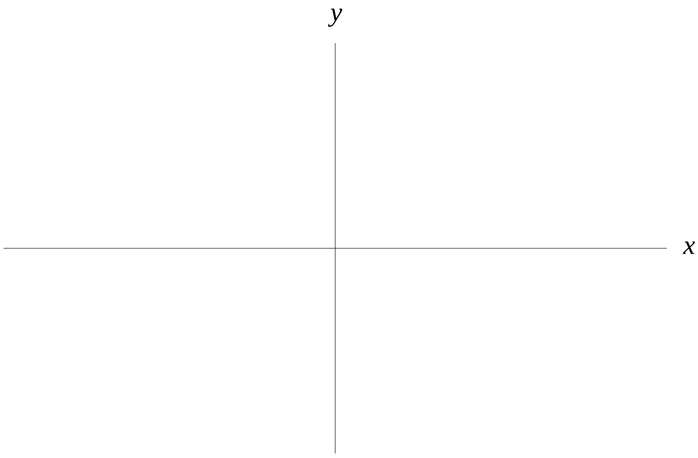


Figure 2.3: Given a plane vector $\mathbf{v}$ of magnitude $|\mathbf{v}|$ and direction θ, can we write it in the form $\mathbf{v} = (a, b)$ where a and b are both the coordinates of the 'point' and the horizontal/vertical lengths as shown?

We use trigonometry to do this. What is the cosine of θ?

What is the sine of θ?

Hence we have the following fact

A plane geometric vector $\mathbf{v}$ *of magnitude* $|\mathbf{v}|$ *and direction* θ *has coordinate form*

$$\mathbf{v} = (|\mathbf{v}|\cos\theta, |\mathbf{v}|\sin\theta). \tag{2.1}$$

Example

Write the vector with magnitude $\sqrt{3}$ and direction $\pi/6$ in coordinate form.

Solution: There is no harm in drawing another picture:
We have

So the answer is $(3/2, \sqrt{3}/2)$.

Going from the coordinate form to the geometric form isn't too difficult. There is a formula that says that if $\mathbf{v} = (a, b)$ then $|\mathbf{v}| = \sqrt{a^2 + b^2}$. This comes directly from Pythagoras Theorem.

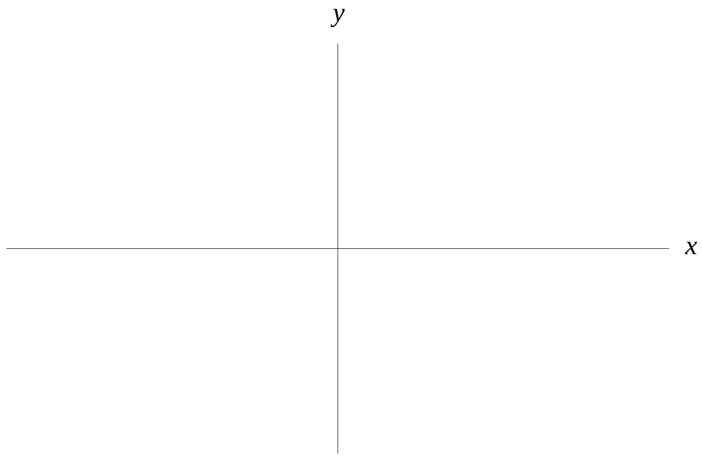


Figure 2.4: Rather than 'learn' a formula, drop a perpendicular and make up your 'formula' on the spot.

There is no such nice formula for the direction, I would prefer if for now we draw a picture.

Examples

Find the magnitude and direction of the vectors:

1. $\mathbf{u} = (1, 3)$
2. $\mathbf{v} = (-2, -2)$

Solutions:

1. Draw a picture:

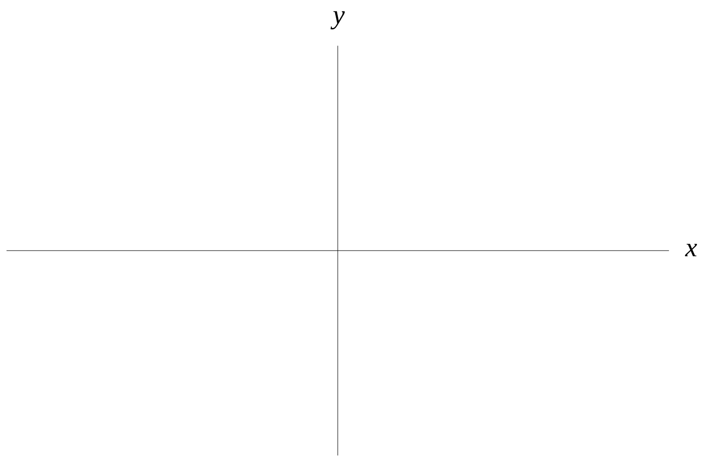


Figure 2.5: We can figure out the magnitude from Pythagoras. What is the angle θ?

So we have a magnitude of $\sqrt{10} \approx 3.162$ and a direction of $\tan^{-1}(3) \approx 1.249$ rad $\approx 71.565°$.

2. Again draw a picture:

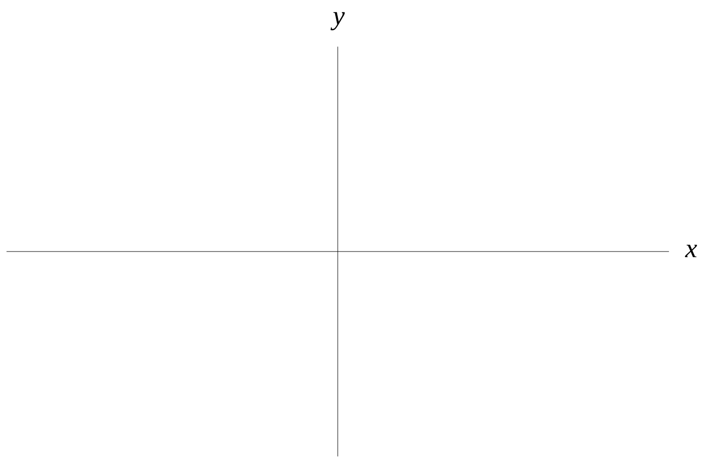


Figure 2.6: Here we must be careful in how we find θ. We can still use Pythagoras to find the magnitude.

So we have a magnitude of $\sqrt{8} = 2\sqrt{2} \approx 2.828$ and a direction of $\tan^{-1}(1) = \pi/4$ rad $= 45°$.

2.1.2 Vectors in Space

Everything here just generalises from the last section. The geometric picture is a lot harder in this picture although we will cover it for completeness.

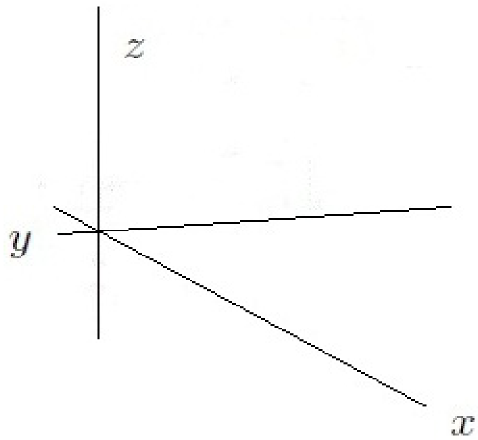


Figure 2.7: A vector **v** in space can be represented geometrically as shown. Note that we need *two* angles to represent direction: the *polar* angle θ and the *azimuthal* angle φ.

The *polar* angle is the angle between the *projection* of **v** onto the plane and the *positive* x-axis. The azimuthal angle φ is the angle between the vector and the *positive* z-axis. Another way to represent vectors in space is as *coordinate vectors*:

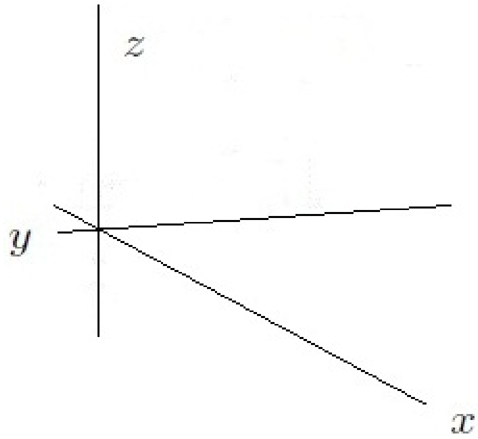


Figure 2.8: If you choose any point P in space, then this implicitly defines a vector $\mathbf{v}$.

Coordinate space vectors are of the form $\mathbf{v} = (a, b, c)$. We should be interested in how to convert from geometric to algebraic and vice versa. As ever a picture helps:

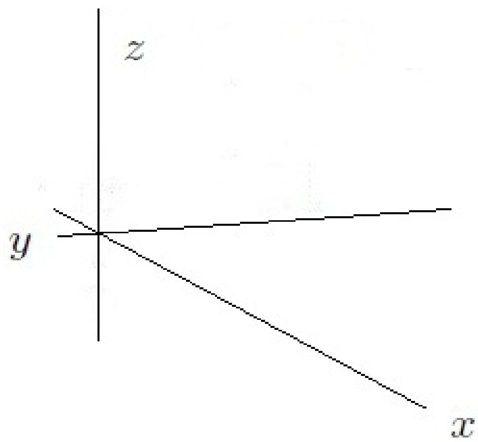


Figure 2.9: Given a vector $\mathbf{v}$ of magnitude $|\mathbf{v}|$ and directions θ and φ, can we write it in the form $\mathbf{v} = (a, b, c)$ where a, b and c are the coordinates of the 'point'.

We use trigonometry to do this. First of all we have that the length of the projection onto the plane is $|\mathbf{v}| \sin \varphi$. What is the cosine of θ?

What is the sine of φ?

This is similar to the case of the plane vector. We can similarly show that $c = |\mathbf{z}| \cos \varphi$.

Hence we have the following

A geometric space vector $\mathbf{v}$ *of magnitude* $|\mathbf{v}|$ *and directions* θ *and* φ *has coordinate form*

$$\mathbf{v} = (|\mathbf{v}|\sin\varphi\cos\theta, |\mathbf{v}|\sin\varphi\sin\theta, |\mathbf{v}|\cos\varphi). \tag{2.2}$$

One nice fact that we learn is that Pythagoras Theorem works in more than two dimensions.

Pythagoras Theorem in Space

Suppose that $\mathbf{v} = (a, b, c)$ is a vector in space. Then

$$|\mathbf{v}| = \sqrt{a^2 + b^2 + c^2} \tag{2.3}$$

Proof. Once again we draw a picture:

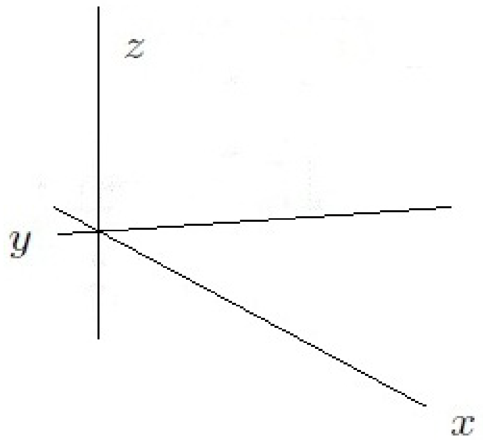


Figure 2.10: Project the vector onto the plane. Now look at the length of the projection.

By Pythagoras the length of the projection is given by

Now look at the triangle composed of the vector, the projected vector and the 'leg'. This is a right-angled triangle so satisfies Pythagoras so we have

Summer 2012: Question 3 (b) (i)

Find the magnitude of the vector $\mathbf{v} = (2, 3, 7)$.

Solution: We can just use Pythagoras

Exercise Find the magnitude of the vectors $\mathbf{a} = (-1, 3)$, $\mathbf{b} = (4, 0, 1)$ and $\mathbf{c} = (1, 5, 1)$.

2.2 Addition & Scalar Multiplication

2.2.1 Addition

Like forces can be added together, so can vectors. Adding geometric plane vectors is pretty straightforward although adding together space vectors is far easier in the coordinate picture. The idea when adding plane vectors is to remember that they are 'acting' at the same time. We describe two ways of adding together geometric vectors:

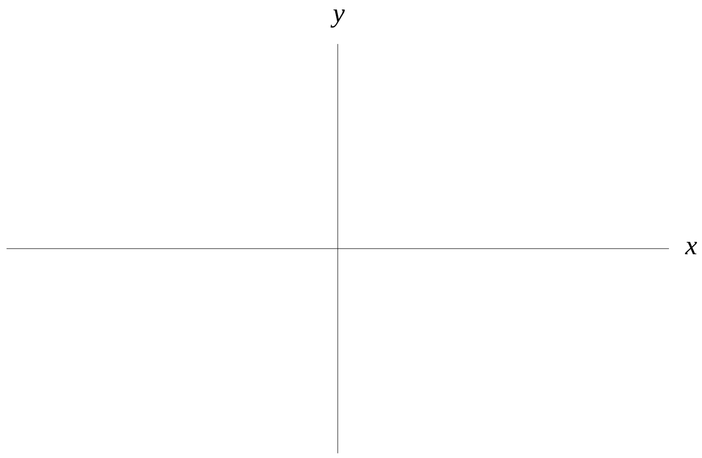


Figure 2.11: To add together two geometric vectors **a** and **b** using the *triangle law* we place them end-to-end as shown. The addition of the two vectors is the same as doing both: $\mathbf{c} = \mathbf{a} + \mathbf{b}$, the third side of the triangle.

There is another, completely equivalent method known as the *parallelogram law*:

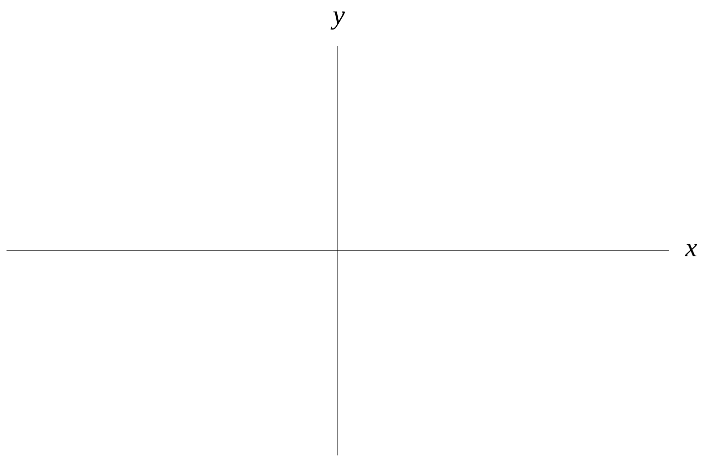


Figure 2.12: To add together two geometric vectors **a** and **b** using the *parallelogram law* we do triangle laws as shown.

Winter 2012: Question 3 (a) (i)

Let $\mathbf{a} = (2, 3)$ and $\mathbf{b} = (-3, 4)$. Sketch $\mathbf{a}$ and $\mathbf{b}$ as position vectors. Show how the triangular law is used to calculate $\mathbf{a} + \mathbf{b}$.

Solution: We draw a picture:

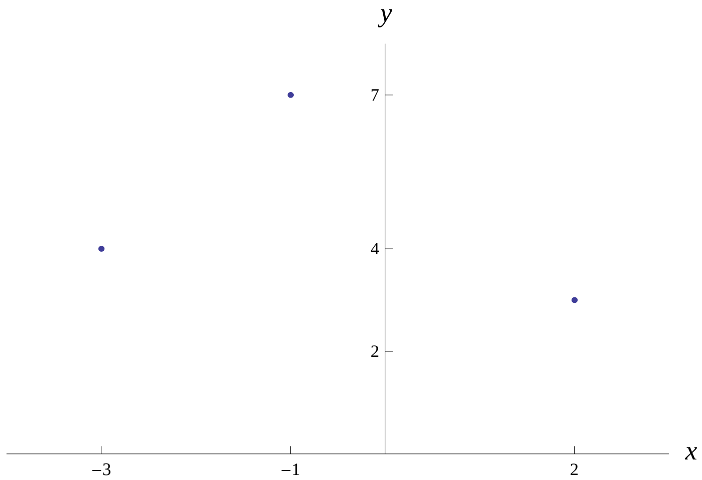


Figure 2.13: We clearly have $\mathbf{a} + \mathbf{b} = (-1, 7) = (2 + (-3), 3 + 4)$.

This example shows how easy it is to add vectors in the coordinate picture:

We have exactly the same set up for coordinate vectors in space:

2.2.2 Scalar Multiplication & the $\mathbf{i} - \mathbf{j} - \mathbf{k}$ Basis

The other thing with vectors is that we can multiply them by a scalar (real number):

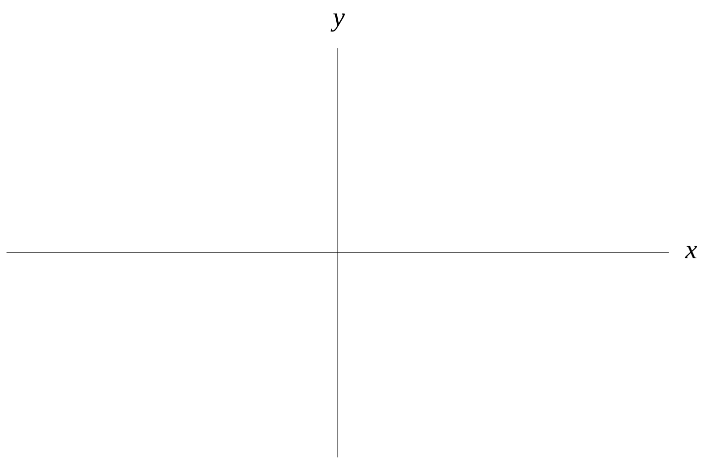


Figure 2.14: Multiplying by a scaler k with $|k| > 1$ stretches the vector; while if $|k| < 1$ the vector shrinks. If $k < 0$ then the direction of the vector is reversed.

This leads us to consider the following:

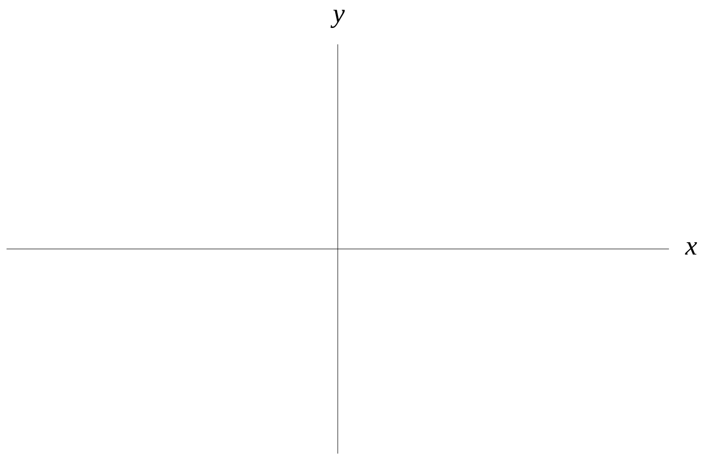


Figure 2.15: A plane vector is just the sum of a multiple of $(1, 0)$ and $(0, 1)$.

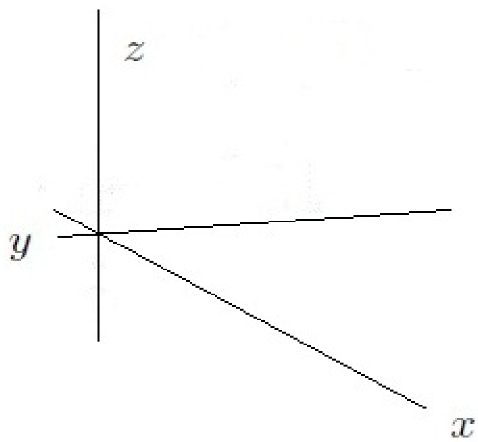


Figure 2.16: A vector in space is just a sum of multiples of $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$.

Hence we define the following:

and we write

Winter 2012: Question 1 (e)

Find the magnitude of the vector $3\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$.

Solution: We just use Pythagoras

2.2.3 Unit Vectors

We call $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ the *unit vectors*. This is because they have a magnitude/length of one. Suppose we want to find a unit vector of magnitude/length one in a direction other than x, y or z:

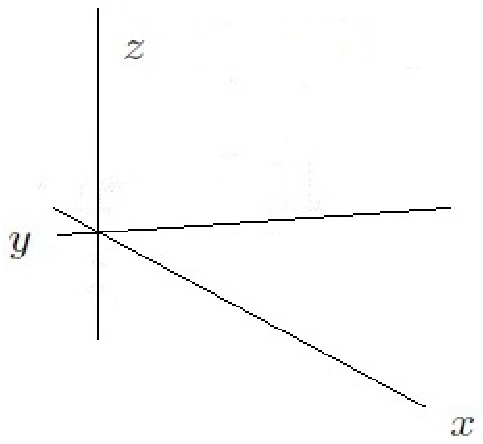


Figure 2.17: Suppose that we have a vector $\mathbf{v}$ and we want a unit vector that is in the same direction as $\mathbf{v}$. Suppose the magnitude/length of $\mathbf{v}$ is five. If we scalar multiply $\mathbf{v}$ by $\frac{1}{5}$ we get a vector of length one: a unit vector.

Unit Vector in a given direction

$\hat{\mathbf{v}} = \dfrac{\mathbf{v}}{|\mathbf{v}|}$ is the *unit vector in the direction of* $\mathbf{v}$.

Summer 2012: Question 3 (b)

For the vector $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} + 7\mathbf{k}$ find the magnitude of $\mathbf{v}$ and the unit vector in the direction of $\mathbf{v}$.

Solution: We find the magnitude using the Pythagoras formula:

Now we make this a unit vector by dividing (shrinking) it by it's magnitude $\sqrt{62}$:

Exercise: Find the magnitude of the vector $\mathbf{v} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$. Now find a unit vector in the same direction as $\mathbf{v}$.

2.3 Displacement Vectors

Given two vectors — equivalently two points — we might be interested in the vector **uv**: the vector *from* **u** *to* **v** as shown:

We can find **uv** quite easily:

Example

Given $\mathbf{u} = 5\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$ and $\mathbf{v} = 0.5\mathbf{i} - \mathbf{j} + 10\mathbf{k}$, find **uv** and **vu**.

Solution: We simply compute

2.4 Vector Products

Like scalars (numbers), we can add numbers. Like scalars, we can multiply vectors by scalars. Can we multiply vectors together? In this module we consider three *vector products*:

2.4.1 Dot/Scalar Products

In the roughest possible sense, the dot product is a measure of the correlation of two vectors. Vectors like this will have a large dot product:

Vectors like this, *perpendicular vectors*, will have a zero dot product:

Finally, vectors like this will have a large, *negative* dot product:

Geometric Definition

Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors. Consider the *projection of* $\mathbf{u}$ *onto* $\mathbf{v}$ as shown:

The dot product is defined as the product of the length of $\mathbf{v}$ by the length of the projection of $\mathbf{u}$ onto $\mathbf{v}$:

Luckily this has a nice, compact equational form. Suppose that the acute angle between $\mathbf{u}$ and $\mathbf{v}$ is θ. Then we can find the projection of $\mathbf{u}$ onto $\mathbf{v}$ in terms of $|\mathbf{u}|$ and θ:

So we have

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta \tag{2.4}$$

This formula allows us to see some nice geometric consequences.

Proposition

Suppose that u and v are two non-zero vectors such that the acute angle between them is θ. *Then*

1. $\mathbf{u} \cdot \mathbf{v} = 0$ implies that $\mathbf{u}$ and $\mathbf{v}$ are perpendicular/*orthogonal*; $\mathbf{u} \perp \mathbf{v}$.
2. $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|$ implies that $\mathbf{u}$ and $\mathbf{v}$ are parallel; $\mathbf{u} \parallel \mathbf{v}$.
3. $\mathbf{u} \cdot \mathbf{v} = -|\mathbf{u}||\mathbf{v}|$ implies that $\mathbf{u}$ and $\mathbf{v}$ are *anti*-parallel.

Proof. 1. In this case we have

As **u** and **v** are non-zero, their magnitudes are also. Hence we must have $\cos\theta = 0$:

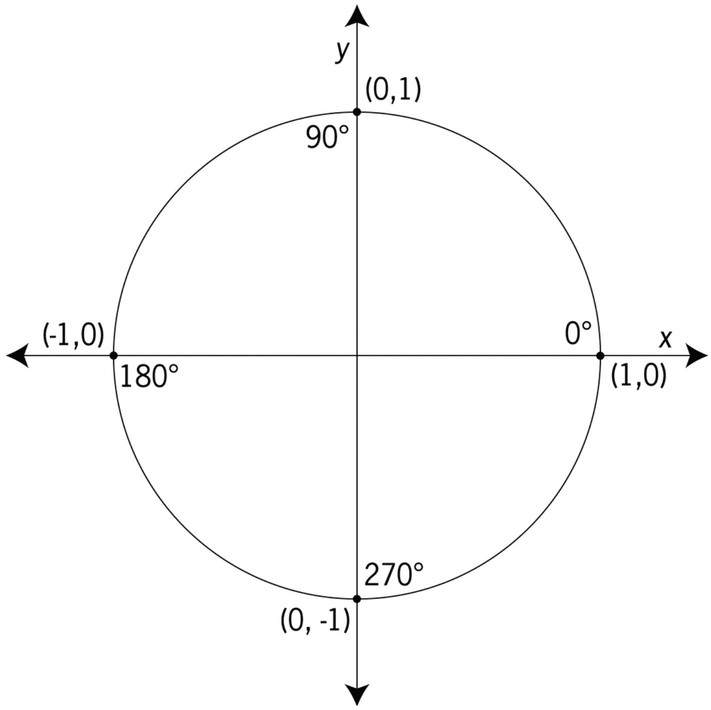


Figure 2.18: $\cos\theta = 0$ when $\theta = 90°$. That is the vectors are perpendicular.

2. In this case we have

   Hence we have $\cos\theta = 1$ and thus $\theta = 0$ — the vectors are parallel.

3. In this case we have

   Hence we have $\cos\theta = -1$ and thus $\theta = 180°$ — the vectors are *anti*-parallel.

Remark: In more abstract mathematics we use part 1. of this proposition to define perpendicularity/*orthogonality* in spaces more abstract than the plane and space.

Corollary

We have

$$\mathbf{i}\cdot\mathbf{j} = \mathbf{i}\cdot\mathbf{k} = \mathbf{j}\cdot\mathbf{k} = 0$$

and

$$\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1$$

Coordinate Definition

It takes a little bit of work, but we can show from the geometric definition that if we have three vectors **u**, **v** and **w** that the dot product is *linear*:

This allows to calculate the dot product of two coordinate form vectors **u** and **v** as follows:

Now all the cross-terms are equal to zero and we are simply left with the following. Suppose

$$\mathbf{u} = u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k} = (u_x, u_y, u_z) \text{ , and}$$

$$\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k} = (v_x, v_y, v_z).$$

$$\mathbf{u} \cdot \mathbf{v} = u_x v_x + u_y v_y + u_z v_z \tag{2.5}$$

Now the important part here is to move between the geometric and coordinate pictures:

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta = u_x v_x + u_y v_y + u_z v_z \tag{2.6}$$

This means that if we have two coordinate vectors, **u** and **v**, we can find the angle between them:

Summer 2012: Question 3 (b) (iii)

If $\mathbf{w} = \mathbf{i} + 4\mathbf{j} + t\mathbf{k}$ is perpendicular to $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} + 7\mathbf{k}$ find the value of t.

Solution: Recall that vectors are perpendicular just when their dot product is zero:

Hence if we want $\mathbf{w} \perp \mathbf{v}$ we have $\mathbf{w} \cdot \mathbf{v} = 0$:

Winter 2012: Question 3(b)

A triangle ABC has vertices $A(1,2,5)$, $B(-5,-3,-2)$ and $C(3,-1,1)$.

(i) Find the vectors **AB**, **BC** and **CA** and hence find their sum.

(ii) Determine the lengths of the sides of the triangle.

(iii) Determine the angle $\angle ABC$.

(iv) Find the area of the triangle ABC.

[11 Marks]

Solution: Before we start a rough sketch of the situation to help us:

(i) We simply use

We have

Now we add these

Remark: This result should be obviously from both a geometric and algebraic viewpoint. If you look at the vector

$$\mathbf{AB}+\mathbf{BC}+\mathbf{CA},$$

this is the same as going from $A \to B \to C \to A$ which is the same as going nowhere: the *zero vector* $\mathbf{0}$. Also from an algebraic viewpoint:

$$\mathbf{AB}+\mathbf{BC}+\mathbf{CA}=\mathbf{B}-\mathbf{A}+\mathbf{C}-\mathbf{B}+\mathbf{A}-\mathbf{C}=\mathbf{0}.$$

(ii) The lengths of the sides are just $|\mathbf{AB}|$, $|\mathbf{BC}|$ and $|\mathbf{CA}|$ and can be found with our Pythagoras formula:

(iii) First of all we should locate the angle $\angle ABC$:

Now we can use the dot product to find the acute (smaller) angle between vectors — this is no problem we just compare the geometric and coordinate forms:

However the angle we want to find, θ, is not exactly the acute angle between **AB** and **BC**:

We have a few options here. The best, I think, is to instead look at the dot product of **BA** and **BC**:

We calculate **BA**:

Now we calculate $\mathbf{BA} \cdot \mathbf{BC}$:

Now this is equal to $|\mathbf{BA}||\mathbf{BC}|\cos\theta$. Luckily we know $|\mathbf{BA}|$ and $|\mathbf{BC}|$ they are the lengths of the sides of the triangle from above. Hence we have

(iv) We can use the following formula to find the area of the triangle:

Applied to our example we have

Exercises

1. **Summer 2012 Q. 1(f)** Find the angle between the vectors $\mathbf{x} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ and $\mathbf{y} = -4\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$ [4 Marks]. **Ans:** $180°$

2. **Autumn 2012: Question 1 (e) [4 Marks]** Given the vectors $\mathbf{x} = 3\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ and $\mathbf{y} = 2\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ find

   (i) $\mathbf{x} + \mathbf{y}$

   (ii) $\mathbf{x} - \mathbf{y}$

   (iii) $(\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y})$

   Ans: 5

3. **Autumn 2012: Question 3 (a) [10 Marks]** Given $\mathbf{v} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$, find

   (i) Find the magnitude of $\mathbf{v}$.

   (ii) Find a unit vector in the same direction as $\mathbf{v}$.

   (iii) If $\mathbf{w} = 4\mathbf{i} - 3\mathbf{j} + t\mathbf{k}$ is perpendicular to $\mathbf{v}$, find t.

   Ans: (ii) $\left(\frac{3}{7}, \frac{2}{7}, \frac{6}{7}\right)$ (iii) -1.

4. **Summer 2012 Q. 3(a) [11 Marks]** Given the three points $P(2, 4, 2)$, $Q(-2, 0, 0)$ and $R(4, 4, 4)$. Find the vectors **PQ**, **QR** and **PR**. The line segments **PQ**, **QR**, **PR** form the sides of a triangle. Determine

   (i) the lengths of all sides of the triangle

   (ii) the angle RPQ.

   Ans: (ii) $135°$

5. A triangle ABC has vertices $A(4, -2, 0)$, $B(4, 1, 1)$ and $C(3, 0, 2)$.

   (i) Find the vectors **AB**, **BC** and **CA** and hence find their sum.

   (ii) Determine the lengths of the sides of the triangle.

   (iii) Determine the angle $\angle ABC$.

   (iv) Find the area of the triangle ABC.

   Ans: (iii) $68.583°$ (iv) 2.55 square units.

2.4.2 Vector/Cross Product

The second 'vector product' that we study is the *vector* or *cross* product. In a *very* rough sense, the cross product of two vectors is a vector that describes of the torque or 'turning power' of two vectors. Vectors like this will have a large cross product:

Vectors like this will have a small cross product:

Rightie-Tightie, Leftie-Loosie

The cross product was developed to model the turning point of a force. Consider the bolt and wrench system as shown:

There are two vectors here; the force vector $\mathbf{F}$ which is applied at the displacement vector $\mathbf{r}$. We know that according to "rightie-tightie: leftie-loosie" (a slightly more cerebral name for this is the right-hand rule), that the bolt here will go in the down direction. This is accounted for in the definition of the cross product in the following way. First of all we are going to call the cross product $\mathbf{u} \times \mathbf{v}$ in words *vector* $\mathbf{u}$ *cross vector* $\mathbf{v}$ and draw it like this:

This is 'rightie-tightie' and the cross product goes down as shown:

Then, to model 'leftie-loosie', we must have that $\mathbf{v} \times \mathbf{u}$, *vector* $\mathbf{v}$ *cross vector* $\mathbf{u}$, looks as follows:

Finally, we note that only the perpendicular component of the force causes torque/twisting:

Geometric Definition

Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors. The *cross product of* $\mathbf{u}$ *and* $\mathbf{v}$ is a vector whose magnitude is given by

and whose direction is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$ and obeys the left-hand rule:

The cross product is only defined for three dimensional vectors. Putting all of this together we can write down the definition of the cross product. I sometimes call it the *toilet roll vector*:

Proposition

We have the following:

1. If $\mathbf{u} \perp \mathbf{v}$ then $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|$.
2. If $\mathbf{u} \parallel \mathbf{v}$ then $\mathbf{u} \times \mathbf{v} = \mathbf{0}$. In particular, $\mathbf{u} \times \mathbf{u} = \mathbf{0}$.
3. $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$.
4. $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = 0$.
5. The magnitude of the cross product $\mathbf{u} \times \mathbf{v}$ is equal to the area of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$.
6. We have that

$$\begin{aligned} \mathbf{i} \times \mathbf{j} &= \mathbf{k} \\ \mathbf{j} \times \mathbf{k} &= \mathbf{i} \\ \mathbf{k} \times \mathbf{i} &= \mathbf{j} \end{aligned}$$

Proof. 1. If the vectors are perpendicular then the acute angle between them is 90°:

2. If the vectors are parallel then the acute angle between them is 0°.

3. Firstly the magnitudes are equal

   The right-hand rule shows that these vectors are in the opposite direction:

4. By definition, $\mathbf{u} \times \mathbf{v}$ is perpendicular to *both* $\mathbf{u}$ and $\mathbf{v}$. Recall that

5. This parallelogram is given below. Note that we can move the triangle on the right as shown:

Now look at the sine of θ:

Hence we have that the area of the parallelogram is

$$|\mathbf{u}||\mathbf{v}|\sin\theta = |\mathbf{u}\times\mathbf{v}|.$$

6. Draw a picture

Coordinate Definition

It takes a little bit of work, but we can show from the geometric definition that if we have three vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ that the cross product is *linear*:

This allows to calculate the dot product of two coordinate form vectors $\mathbf{u}$ and $\mathbf{v}$ as follows:

Now all the like-terms are equal but we do have some cross terms. Suppose

$$\mathbf{u} = u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k} = (u_x, u_y, u_z) \text{ , and}$$

$$\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k} = (v_x, v_y, v_z).$$

$$\mathbf{u}\times\mathbf{v} = (u_y v_z - u_z v_y)\mathbf{i} + (u_z v_x - u_x v_z)\mathbf{j} + (u_x v_y - u_y v_x)\mathbf{k} \tag{2.7}$$

Luckily enough there is an easy way to remember this formula:

Example

If $\mathbf{u} = (1, 3, 4)$ and $\mathbf{v} = (2, 7, -5)$ find $\mathbf{u} \times \mathbf{v}$.

Solution:

The image of the toilet roll holder is one that helps me with examples like this one.

Example

Find a vector perpendicular to the plane that passes through the points $P(1, 4, 6)$, $Q(-2, 5, -1)$ and $R(1, -1, 1)$.

Solution: First a picture:

The vector $\mathbf{PQ} \times \mathbf{PR}$ is perpendicular to both $\mathbf{PQ}$ and $\mathbf{PR}$ and is therefore perpendicular to the plane through P, Q and R (i.e. the bathroom floor). We know how to compute $\mathbf{PQ}$ and $\mathbf{PR}$:

We compute the cross product of these two vectors

Winter 2012: Question 1 (e)

For the vectors $\mathbf{p} = 3\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$ and $\mathbf{q} = 2\mathbf{i} + 5\mathbf{j} - \mathbf{k}$, find $|\mathbf{p}|$ and $\mathbf{p} \times \mathbf{q}$.

[5 Marks]

Solution: The magnitude of $\mathbf{p}$ is found using the Pythagoras formula:

Now $\mathbf{p} \times \mathbf{q}$ is calculated using the determinant

Autumn 2011: Question 1 (e)

Find two unit vectors perpendicular to both $2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ and $\mathbf{i} - 4\mathbf{j} + \mathbf{k}$.

[5 Marks]

Solution: First we draw a picture:

This is the toilet roll vector, $\mathbf{u} \times \mathbf{v}$:

To make a unit vector we divide by the magnitude of $\mathbf{u} \times \mathbf{v}$. The magnitude is given by Pythagoras:

What is another vector perpendicular to both of these vectors?

2.4.3 Scalar Triple Product

There is another vector product that takes in *three* vectors and spits out a number. It is not as important as the other two and is given in terms of the dot and cross products:

The geometric significance of the triple scalar product can be seen by considering the parallelepiped determined by the vectors **a**, **b** and **c** as shown:

The area of the base parallelogram is given by $|\mathbf{b} \times \mathbf{c}|$. If θ is the angle between **a** and $\mathbf{b} \times \mathbf{c}$, then the height h of the parallelepiped is[2] $h = |\mathbf{a}||\cos\theta|$. Therefore the volume of the parallelepiped is

Example

Find the volume of the parallelepiped determined by the vectors $\mathbf{a} = (6, 3, -1)$, $\mathbf{b} = (0, 1, 2)$ and $\mathbf{c} = (4, -2, 5)$.

Solution: We know that the answer is given by

First we calculate the cross product

Now we calculate the dot product and hence the volume:

[2] we must use $|\cos\theta|$ rather than $\cos\theta$ in case $\cos\theta$ is negative

Autumn 2011: Question 1 (f)

If $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{c} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, find $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$.

[5 Marks]

Solution: First we calculate $\mathbf{b} \times \mathbf{c}$ using the determinant:

Now we do the dot product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ and write down the absolute value:

Exercises

1. **Autumn 2012: Question 1 (f) [4 Marks]** For the vectors $\mathbf{a} = 3\mathbf{i}-\mathbf{j}+2\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i}+3\mathbf{j}-\mathbf{k}$, find $\mathbf{a} \times \mathbf{b}$. **Ans:** $(-5, 7, 11)$

2. **Summer 2012: Question 1 (e) [4 Marks]** For the vectors $\mathbf{a} = \mathbf{i}-2\mathbf{j}-2\mathbf{k}$ and $\mathbf{b} = \mathbf{i}-3\mathbf{j}+3\mathbf{k}$, find $\mathbf{a} \times \mathbf{b}$. **Ans:** $(-12, -5, -1)$

3. Find the cross product $\mathbf{a} \times \mathbf{b}$ and verify that it is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

   (a) $\mathbf{a} = (1, 2, 0)$ and $\mathbf{b} = (0, 3, 1)$ **Ans**: $(2, -1, 3)$

   (b) $\mathbf{a} = (5, 1, 4)$ and $\mathbf{b} = (-1, 0, 2)$ **Ans**: $(2, -14, 1)$

   (c) $\mathbf{a} = \mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + \mathbf{k}$ **Ans**: $-\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

4. If $\mathbf{u} = (3, 1, 2)$, $\mathbf{v} = (-1, 1, 0)$ and $\mathbf{w} = (0, 0, -4)$, show that $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$.

5. Find a *unit* vector orthogonal to both $\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $2\mathbf{i} + \mathbf{k}$, **Ans**: $\frac{1}{\sqrt{6}}(1, 1, -2)$

6. If $\mathbf{a} = \mathbf{i} - 2\mathbf{k}$ and $\mathbf{b} = \mathbf{j} + \mathbf{k}$, find $\mathbf{a} \times \mathbf{b}$. Sketch $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$ as vectors starting at the origin.

7. Find the volume of the parallelepiped determined by $\mathbf{u} = \mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{v} = \mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{w} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$. **Ans**: 4

8. **Autumn 2011: Question 3 (c) [7 Marks]** Let $\mathbf{O} = (0, 0, 0)$, $\mathbf{A} = (1, -2, 5)$ and $\mathbf{B} = (-3, 4, -1)$. Show that the vectors $\mathbf{OA}$ and $\mathbf{OB}$ are not orthogonal. Find a unit vector which is perpendicular to the plane spanned by $\mathbf{OA}$ and $\mathbf{OB}$. **Ans**: $\frac{1}{\sqrt{131}}(-9, -7, -1)$.

9. * State whether each expression is meaningful. If not, explain why. If yes, state whether it is a vector or scalar:

   (a) $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$

   (b) $\mathbf{u} \times (\mathbf{v} \cdot \mathbf{w})$

   (c) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$

   (d) $(\mathbf{u} \cdot \mathbf{v}) \times \mathbf{w}$

   (e) $(\mathbf{a} \cdot \mathbf{b}) \times (\mathbf{c} \cdot \mathbf{d})$

   (f) $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d})$

   [HINT: $\cdot : (\mathbb{R}^3, \mathbb{R}^3) \to \mathbb{R}$ and $\times : (\mathbb{R}^3, \mathbb{R}^3) \to \mathbb{R}^3$]

2.5 Applications

2.5.1 Forces

Vectors are useful in many aspects of physics and engineering. A force is represented by a vector because it has both a magnitude (measured in Newtons) and a direction. If several forces are acting on an object, the *resultant force* experienced by the object is the sum of these forces.

Example

If a child pulls a sled through the snow with a force of 50 N exerted at an angle of 38° above the horizontal, find the horizontal and vertical components of the force.

Solution: First a picture:

Using the sine and cosine of 38°:

2.5.2 Work

One use of vectors, specifically the dot product, occurs in physics in calculating the work done by a force. In one dimension work is defined as

In MATH6015, for a variable force, we showed that the appropriate definition is given by

In more than one dimension things are a little different. For now suppose that we have a constant force. Suppose, however, that the constant force is a vector $\mathbf{F}$ pointing in some other direction than the displacement as shown:

If the force moves the vector from P to Q then the displacement vector is $\mathbf{d} = \mathbf{PQ}$. The work in this case is not done by the force $\mathbf{F}$ but rather the force in the same direction as $\mathbf{d}$ which is exactly $|\mathbf{F}|\cos\theta$. Hence we have that the work is given by

However this is nothing other than the dot product of $\mathbf{d}$ with $\mathbf{F}$ and so we have

Example

A crate is hauled 8 m up a ramp under a constant force of 200 N applied at an angle of 25° to the ramp. Find the work done.

Solution: The situation is as shown

Hence we have that the work is given by

Autumn 2012: Question 3 (b)

A force is given by a vector $\mathbf{F} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ and moves a particle from the point $A(1,1,1)$ to the point $B(2,2,2)$. Find the work done if the distance is measured in metres and the magnitude of the force is measured in Newtons.

[10 Marks]

Solution: The displacement vector $\mathbf{d}$ is given by

The work is given by $W = \mathbf{d} \cdot \mathbf{F}$:

2.5.3 Moments

The idea of the cross product occurs often in physics. In particular, we consider a force $\mathbf{F}$ acting on a rigid body at a point given by a displacement vector $\mathbf{d}$:

The *moment* (about the origin) or *torque* of the force is defined as the cross product of the displacement and force vectors

and measures the tendency of the body to rotate.

The direction of the moment/ torque vector is given by the right-hand rule and its magnitude is given by

where θ is the acute angle between the displacement and force vectors.

We may also be asked for the moment about a point other than the origin. We might be asked for the bending moment of a force $\mathbf{F}$, acting at the point Q *about the point* P.

In this case we just have $\mathbf{d} = \mathbf{PQ}$.

Autumn 2012: Question 3 (c)

A force of 18 N acts through the point $P(-1, 3, -4)$ in the direction of the vector $3\mathbf{i}+6\mathbf{j}-6\mathbf{k}$. Find its moment about the origin $(0, 0, 0)$. You may assume that displacement is measured in metres.

[10 Marks]

Solution: A little picture

The issue here is that the vector $\mathbf{v} = 3\mathbf{i} + 6\mathbf{j} - 6\mathbf{k}$ is *not* the same as the force vector: it only gives the direction of the force vector. We have two options:

1. Find the unit vector in the same direction as $\mathbf{v}$; $\hat{\mathbf{v}}$ and write $\mathbf{F} = 18\hat{\mathbf{v}}$.
2. Alternatively calculate that $|\mathbf{v}| = \sqrt{9 + 36 + 36} = 9$ so $\mathbf{F} = 2\mathbf{v}$.

Option two is smart but might not always be easy so we will go with option 1. We have already seen that $|\mathbf{v}| = 9$ hence the unit vector $\hat{\mathbf{v}}$ is given by

and so the force vector is given by

Now we calculate the bending moment

where the units are Newton-metres.

Winter 2012: Question 3 (c)

A force of 18 units acts through the point $C(-2,-1,4)$ in the direction of the vector $4\mathbf{i}+4\mathbf{j}+2\mathbf{k}$. Find its moment about the point $A(2,1,-1)$.

[10 Marks]

Solution: A little picture

Once the vector $\mathbf{v}=4\mathbf{i}+4\mathbf{j}+2\mathbf{k}$ is *not* the same as the force vector: it only gives the direction of the force vector. Note its magnitude:

Hence the force vector is three times as strong as $\mathbf{u}$

Now the displacement vector is given by $\mathbf{AC}$

Now we calculate the bending moment

Exercises

1. A force is given by a vector $\mathbf{F} = 3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ and moves a particle from the point $P(2, 1, 0)$ to the point $Q(4, 6, 2)$. Find the work done. **Ans**: 36 units

2. A constant force with vector representation $\mathbf{F} = 10\mathbf{i} + 18\mathbf{j} - 6\mathbf{k}$ moves an object along a straight line from the point $(2, 3, 0)$ to the point $(4, 9, 15)$. Find the work done if the distance is measured in metres and the magnitude of the force is measured in Newtons.

3. [**10 Marks**] A force $\mathbf{F} = 4\mathbf{i} - 5\mathbf{j} + 7\mathbf{k}$ acts at the point $A(-2, 5, -3)$. Find its moment about the point $B(3, 6, -2)$.

4. [**10 Marks**] A force of 12 units acts through the point $P(2, 3, -5)$ in the direction of the vector $4\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$. Find its moment about the point $A(1, 2, -3)$.

Chapter 3

Further Differentiation

Mathematics is a game played according to certain simple rules with meaningless marks on paper.

David Hilbert

3.1 Review of Differentiation

Here we review the main ideas and results from the differentiation section of MATH6015.

3.1.1 Functions

The primary objects that we study here are *functions*. For the purposes of this review, a function is an object f that takes as an input a real number x and outputs a real number $f(x)$ *that is given in terms of the input*:

$$f : x \mapsto f(x).$$

For example, $f(x) = x^2$ is a function

$$x \mapsto x^2.$$

The functions that we are interested in are combinations of

— constant functions; e.g. $k(x) = 1$

— lines; e.g. $\ell(x) = 3x - 2$

— quadratics; e.g. $q(x) = x^2 + 2x - 3$

— polynomials; e.g. $p(x) = x^4 + x^3 - 2x$

— powers; e.g. $s(x) = \sqrt{x} = x^{1/2}$

— trigonometric; e.g. $\sin x$, $\cos x$ and $\tan x$

— inverse trigonometric; e.g. $\sin^{-1}(x)$, $\tan^{-1}(x)$

— exponential; $f(x) = e^x = \exp(x)$

— logarithmic; $g(x) = \ln x$

By combinations we mean

— sums; e.g. $f(x) = \sin x - 3$

— scalar multiples; e.g. $f(x) = 5\sqrt[3]{x}$

— differences; e.g. $f(x) = x^2 + 23x - 4 - e^x$

— products; e.g. $f(x) = x^3 \tan x$

— quotients; e.g. $f(x) = \frac{\sin^{-1}(x)}{\sqrt{x}}$

— powers; e.g. $f(x) = (x^4 + x^3)^{-3}$

— roots; e.g. $f(x) = \sqrt{\ln x}$

— compositions; e.g. $f(x) = \sin(x^2 + 3x - 2)$

We can visualise all of these objects by looking at their graph. The graph of $f(x)$ is all of the points of the form $(x, f(x))$. More on this in the function catalogue.

3.1.2 The Derivative of a Function

We developed the idea of the *derivative* of a function that would allow to study the following problems:

1. **Tangents** Most of these elementary functions are *smooth*. This means that *locally* (near a point), they are well-approximated by lines:

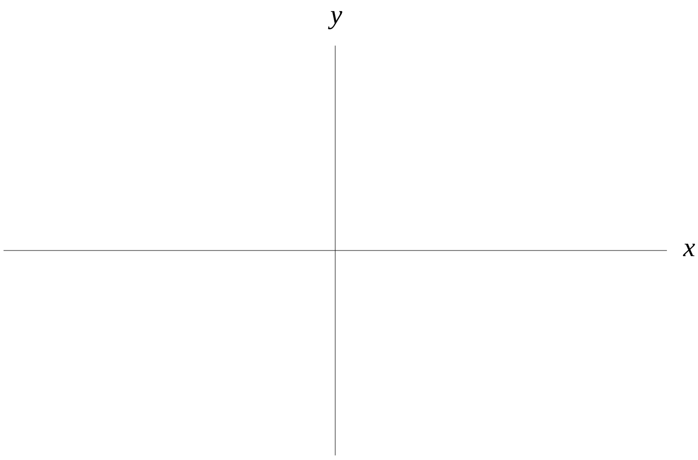


Figure 3.1: Near the point $x = 1$, the curve $y = \ln x$ is well-approximated by the line $y = x - 1$.

To find the equation of this *tangent* line, which has equation

$$y - y_1 = m(x - x_1)$$

it is necessary to find the slope of the tangent.

This led us to the following:

$$\text{slope of tangent to } y = f(x) \text{ at } x = a \;:= f'(a)$$

where

$$f'(x) \equiv \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

This allows us to approximate functions locally by lines; i.e. using their tangents.

2. **Rates of Change** Suppose we have a function that describes the temperature of an object:

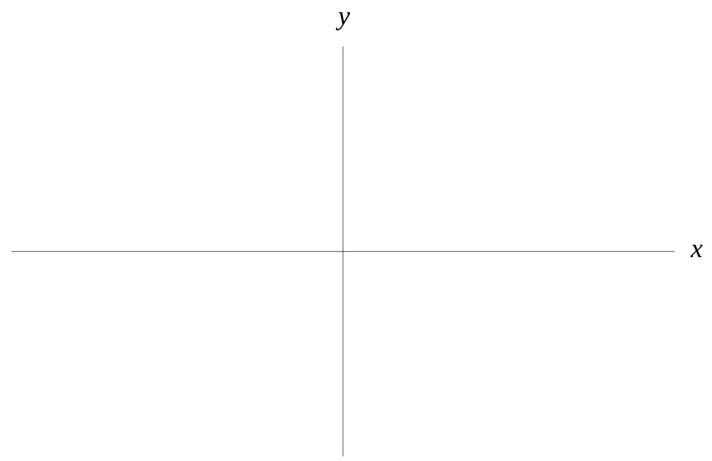


Figure 3.2: Suppose that we have a formula $y = T(x)$ for the temperature of an object after x seconds. Can we say when the temperature is increasing/decreasing?

It turns out that

$$\text{rate of change of } f(x) \;= f'(x) \equiv \frac{dy}{dx}.$$

and we can analyse the rate of change of a function like this.

3. **Local Maxima/Minima** Suppose we have a function of the form $y = f(x)$. Can we find its (local) maxima and minima? The derivative allows us to do so:

4. **Differential Equations** Suppose we have a quantity $Q(x)$ which increases in direct proportion to itself so that

$$\text{rate of change of } Q \text{ with respect to } x \;= kQ$$
$$\frac{dQ}{dx} = k \cdot Q(x),$$

This is a *differential equation.* Two simple examples include radioactive decay and exponential population growth.

The issue here is that we need to differentiate a function $f(x)$. The derivatives of the functions outlined above are given in the function catalogue. The question is how do we find the derivative of

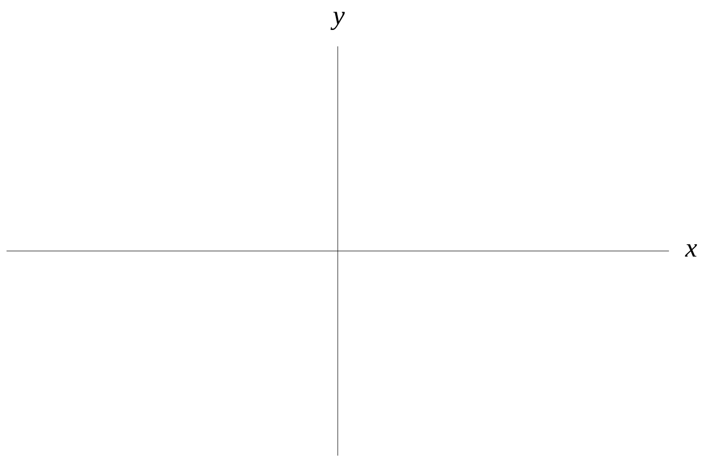


Figure 3.3: At a turning point, the slope of the tangent to the curve is zero: $f'(x) = 0$. We can use the second derivative to determine whether the turning point is a (local) maximum or minimum.

— sums

— scalar multiples

— differences

— products

— quotients

— powers

— roots

— compositions

3.1.3 "Rules of Differentiation"

The answer is by the Sum, Scalar, Product, Quotient & Chain Rules which need to be well understood to do well in MATH6040. They are not really rules but theorems/facts that describe how we should differentiate sums, products, compositions, etc.

"Rules of Differentiation"

Suppose that $f(x)$ and $g(x)$ are functions, $n \in \mathbb{Q}$ a fraction and $k \in \mathbb{R}$ a real number. Suppose also that $\frac{d}{dx}$ is the *differential operator* (i.e. it means differentiate), and $f'(x)$ and $g'(x)$ are the derivatives of $f(x)$ and $g(x)$ respectively. *Then*

$$\frac{d}{dx}(f(x)+g(x)) = f'(x)+g'(x) \quad \text{[Sum Rule]}$$

$$\frac{d}{dx}(k \cdot f(x)) = k \cdot f'(x) \quad \text{[Scalar Rule]}$$

$$\frac{d}{dx}(f(x)-g(x)) = f'(x)-g'(x) \quad \text{[Difference Rule]}$$

$$\frac{d}{dx}(f(x)\cdot g(x)) = f(x)\cdot g'(x)+g(x)\cdot f'(x) \quad \text{[Product Rule]}$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\cdot f'(x)-f(x)\cdot g'(x)}{[g(x)]^2} \quad \text{[Quotient Rule]}$$

$$\frac{d}{dx}x^n = nx^{n-1} \quad \text{[Power Rule]}$$

$$\frac{d}{dx}(f(g(x))) = f'(g(x))\cdot g'(x) \quad \text{[Chain Rule]}$$

Remark

The Difference Rule is a corollary of the Sum and Scalar Rules. The Quotient Rule can be seen as a corollary of the Product and Chain Rules. The Power Rule handles roots. These formulas are all in the tables — the functions are called u and v but they are the same ideas.

3.1.4 Function Catalogue

Constant Functions

1. **Definition** Let $k \in \mathbb{R}$. A *constant* function is of the form

$$f(x) = k.$$

   An example of a constant function is $f(x) = 2$.

2. **Main Idea/Properties** A constant function outputs the same number for all inputs and so has a rate of change of zero.

3. **Derivative** The derivative of a constant (function) is zero:

$$\frac{d}{dx}(k) = 0.$$

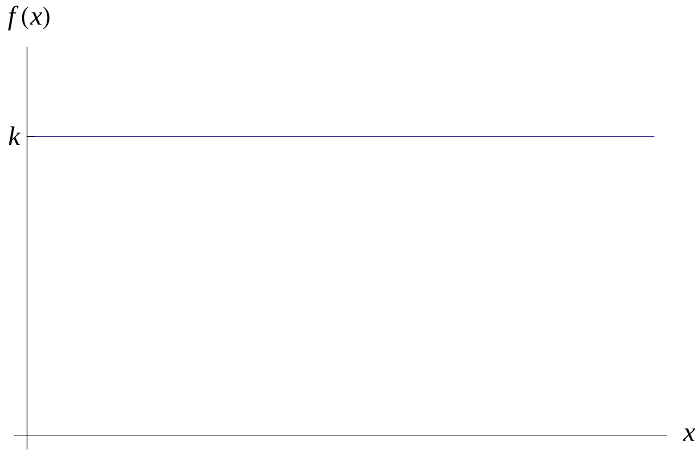


Figure 3.4: The graph of a constant function. Note that the slope morryah the derivative is zero.

Lines

1. **Definition** Let $m \in \mathbb{R}$ and $c \in \mathbb{R}$. A *line of slope m and y-intercept c* is given by

$$f(x) = mx + c.$$

   An example of a line is $f(x) = 3x - 2$.

2. **Main Idea/Properties** A line does exactly what it says on the tin. The slope/derivative of a line is a constant so the rate of change of a line is constant.

3. **Derivative** The derivative of a line is the slope:

$$\frac{d}{dx}(mx + c) = m.$$

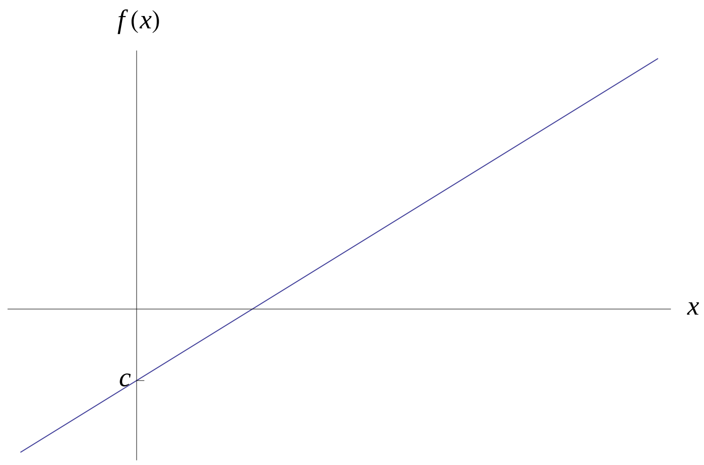


Figure 3.5: The graph of a line. Note that the slope morryah the derivative is constant.

Quadratics

1. **Definition** Let $a, b, c \in \mathbb{R}$. A quadratic is a function of the form

$$f(x) = ax^2 + bx + c.$$

An example of a quadratic is $f(x) = x^2 + 1$.

2. **Main Idea/Properties** A quadratic either has a $\bigcup$ shape (when $a > 0$) or a $\bigcap$ shape (when $a < 0$). It has two *roots* given by the $\dfrac{-b \pm \sqrt{\cdots}}{2a}$ formula. If they are both *real* (when $b^2 - 4ac > 0$), then the graph cuts the x-axis at two points. The graph is symmetric about the max/min. Hence the max/min can be found by looking at $f'(x) = 0$ or else be found at the midpoint of the roots. If $b^2 - 4ac < 0$ then the roots contain a $\sqrt{(-)}$ — *complex roots.*

3. **Derivative** The derivative of a quadratic is a line!

$$\frac{d}{dx}(ax^2 + bx + c) = a(2x) + b(1) + 0 = 2ax + b.$$

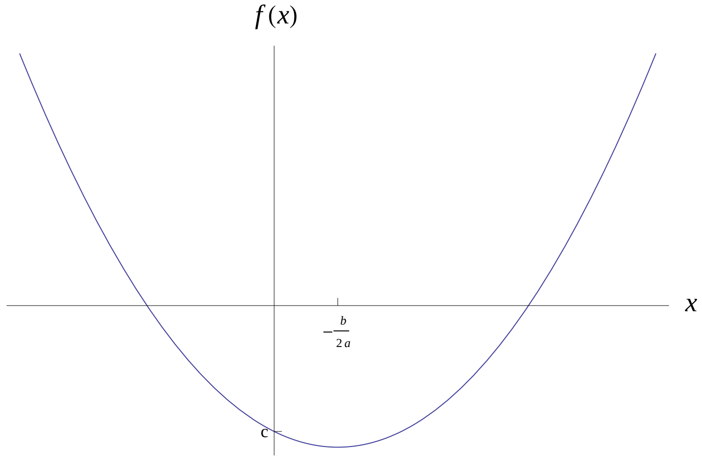


Figure 3.6: The graph of a quadratic with $a > 0$. Note that the slope goes from negative to zero to positive — like a line. This quadratic has two real roots and the minimum occurs at $-\dfrac{b}{2a}$. At this point the tangent is horizontal. This point can be found by differentiating $ax^2 + bx + c$, e.g. getting the slope, and setting it equal to zero.

Polynomials

1. **Definition** Let $a_n, a_{n-1}, \dots, a_2, a_1, a_0 \in \mathbb{R}$. A *polynomial of degree n* is a function of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0.$$

As example of a polynomial is $f(x) = x^5 - 3x^3 + 2x^2 + 1$.

2. **Main Idea/Properties** A polynomial of degree n has n roots — some of which may be complex, some of which may be repeated. However if all the roots are real and distinct then the polynomial cuts the x-axis n times. The derivative of a polynomial of degree n is a polynomial of degree $n - 1$;

$$\text{e.g.} \quad \frac{d}{dx}(x^5 - 3x^3 + 2x^2 + 1) = 5x^4 - 3(3x^2) + 2(2x) = 5x^4 - 9x^2 + 4x,$$

which has potentially $n-1$ real roots and hence $n-1$ points where $f'(x) = 0$ — potentially $n-1$ turning points. As an example note that quadratics are degree two polynomials and have one turning point.

3. **Derivative** We differentiate a polynomial using the Sum, Scalar & Power Rules:

$$\frac{d}{dx}(ax^n) = a\frac{d}{dx}x^n = a(nx^{n-1}) = anx^{n-1}.$$

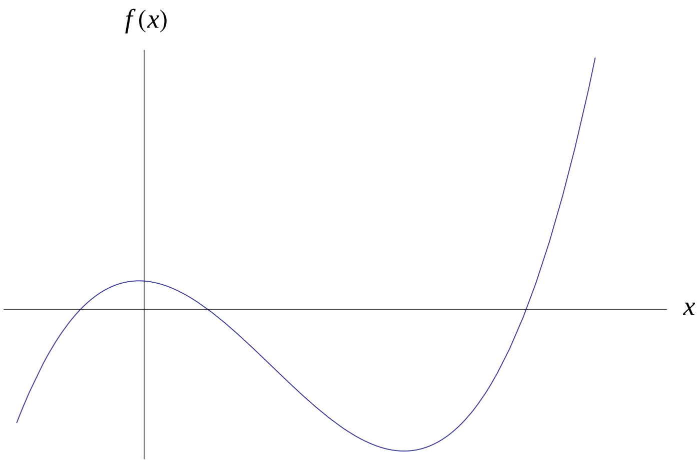


Figure 3.7: This is an example of a cubic: $ax^3 + bx^2 + cx + d$. Note that it has *three* real roots and *two* turning points. In some sense this is typical behaviour of polynomials.

Roots

1. **Definition** Let $n \in \mathbb{N}$. The *nth root function* is a function of the form:

$$f(x) = \sqrt[n]{x},$$

the *positive n-th root* of x.

2. **Main Idea/Properties** We can show that if we define

$$x^{1/n} = \sqrt[n]{x}$$

then all of the theorems of indices and differentiation work properly with this definition and it turns out that $\sqrt[n]{x} = x^{1/n}$ written as a power can be differentiated using the Power Rule.

3. **Derivative** Using the power rule

$$\frac{d}{dx}\sqrt[n]{x} = \frac{d}{dx}x^{1/n} = \frac{1}{n}x^{1/n-1}.$$

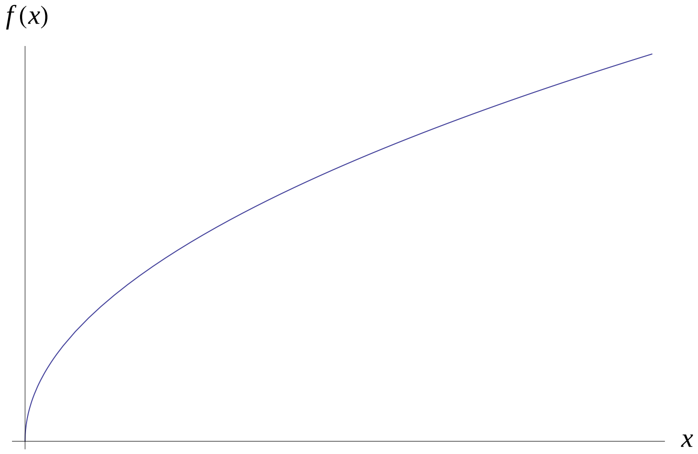


Figure 3.8: A plot of the square root function $f(x) = \sqrt{x}$. Note that roots are only defined for *positive* values of x.

Trigonometric

1. **Definition** Let $0 \leq \theta \leq 2\pi$ be an angle. Consider now the unit circle

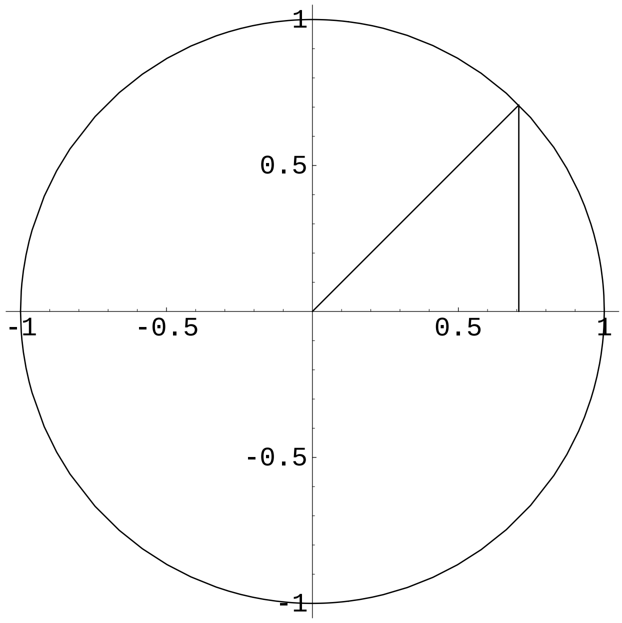


Figure 3.9: If the angle made with the positive x-axis is θ, then the coordinate of the point on the circle is $(\cos\theta, \sin\theta)$.

This defines cosine and sine for angles between 0 and 2π. The definition is extended by periodicity to the whole of the number line by

$$\cos(\theta + 2\pi) = \cos(\theta)$$
$$\sin(\theta + 2\pi) = \sin\theta$$

We also define

$$\tan x = \frac{\sin\theta}{\cos\theta}$$

2. **Main Idea/Properties** Sine and Cosine are waves that oscillate between ± 1. Sine begins at zero ($\sin 0 = 0$) while cosine begins at one ($\cos 0 = 1$). Apart from this they are very similar: the graph of sine is got by shifting the graph of cosine $\pi/2$ units to the right.

3. **Derivative** We can show that

$$\frac{d}{dx}\sin x = \cos x$$

$$\frac{d}{dx}\cos x = -\sin x.$$

Using the Quotient Rule we have

$$\frac{d}{dx}\tan x = \sec^2 x = (\sec x)^2,$$

where

$$\sec x := \frac{1}{\cos x}. \tag{3.1}$$

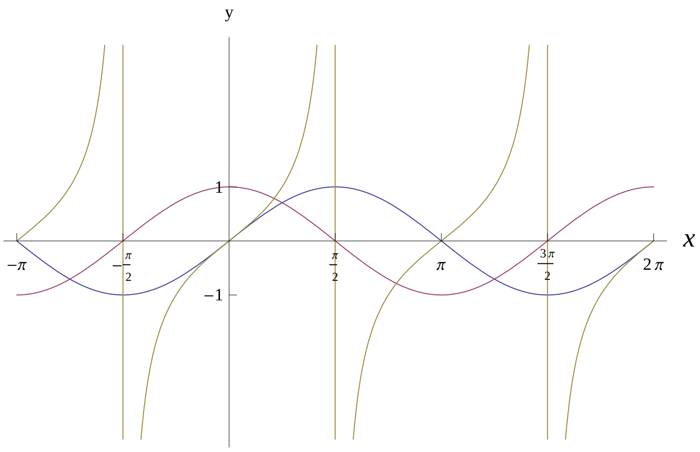


Figure 3.10: Note that $|\sin x|, |\cos x| \leq 1$ while $\tan x \to \pm\infty$ at $\pi/2$.

Inverse Trigonometric

1. **Definition** These functions are *inverse functions* of the trigonometric functions.

Let $y \in [-\pi/2, \pi/2]$ and $x \in [-1, 1]$:

$$y = \sin^{-1}(x) \Leftrightarrow x = \sin y. \tag{3.2}$$

Let $y \in [-\pi/2, \pi/2]$:

$$y = \tan^{-1}(x) \Leftrightarrow x = \tan y. \tag{3.3}$$

2. **Main Idea/Properties** These are the inverse functions of $\sin x$ and $\tan x$. For example

$$\sin\theta = \frac{1}{2} \Rightarrow \sin^{-1}(\sin\theta) = \sin^{-1}(1/2) \Rightarrow \theta = \frac{\pi}{6}.$$

In other words, $\sin^{-1}(x)$ asks for the angle — *between* $\pm\pi/2$ — which has a sine of x.

3. **Derivative** We can show that

$$\frac{d}{dx}\sin^{-1}\left(\frac{x}{a}\right) = \frac{1}{\sqrt{a^2 - x^2}} \tag{3.4}$$

and

$$\frac{d}{dx}\tan^{-1}\left(\frac{x}{a}\right) = \frac{a}{a^2 + x^2}. \tag{3.5}$$

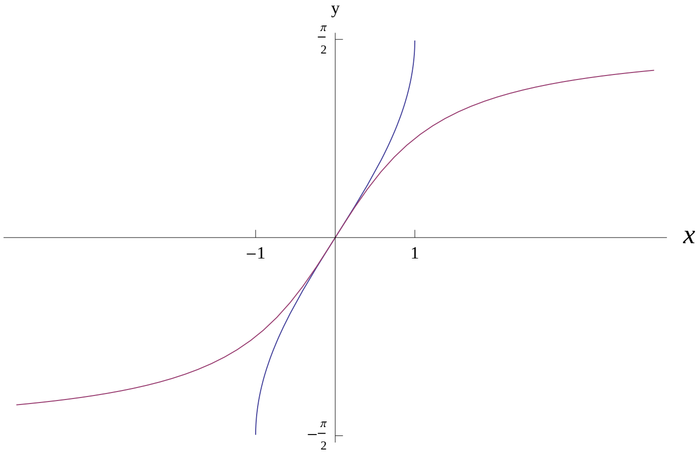


Figure 3.11: Inverse Sine can only take inputs between ± 1. However we have $\tan^{-1}(x) \to \pi/2$ as $x \to \infty$.

Exponential

1. **Definition** The *exponential function* can be defined as a power series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \tag{3.6}$$

2. **Main Idea/Properties** The exponential function is the unique function that is equal to its own derivative.

3. **Derivative** We have that

$$\frac{d}{dx}e^x = e^x, \tag{3.7}$$

and using the Chain Rule

$$\frac{d}{dx}e^{ax} = a \cdot e^{ax}. \tag{3.8}$$

Logarithmic

1. **Definition** The natural logarithm is the inverse function of e^x:

$$y = \ln x \Leftrightarrow x = e^y \tag{3.9}$$

2. **Main Idea/Properties** As $e^y > 0$, the natural logarithm can only take strictly positive inputs. They can be used to solve exponential equations:

$$e^x = 2 \Rightarrow \ln(e^x) = \ln 2 \Rightarrow x = \ln 2.$$

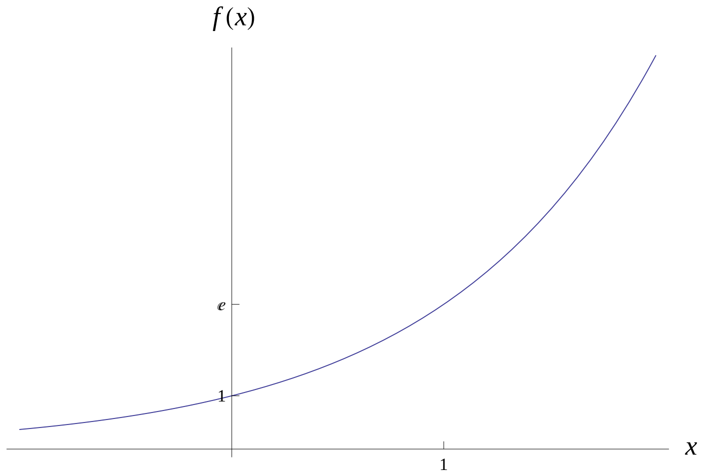


Figure 3.12: We have that $e^0 = 1$. $e^x \to \infty$ as $x \to +\infty$. When the input is negative, say $x = -N$, then $e^x = e^{-N} = \frac{1}{e^N} \to 0$ as $N \to \infty \Leftrightarrow x \to -\infty$.

Note that we have

$$\ln 1 = 0$$
$$\ln(xy) = \ln x + \ln y$$
$$\ln(x^n) = n \ln x.$$

3. **Derivative** We can show that

$$\frac{d}{dx} \ln x = \frac{1}{x}.$$

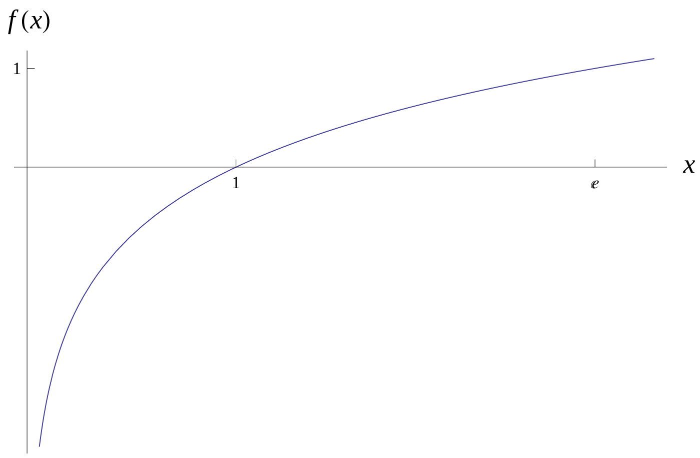


Figure 3.13: As $x \to 0$, $\ln x \to -\infty$; $\ln 1 = 0$ and $\ln x \to \infty$ — slowly — as $x \to \infty$.

3.2 Parametric Differentiation

In MATH6015 we learned how to find the slopes of tangents to curves that are graphs of functions: In MATH6040 we learn how to find the slopes of two more types of curves. The first of these are *parametrically equations*:

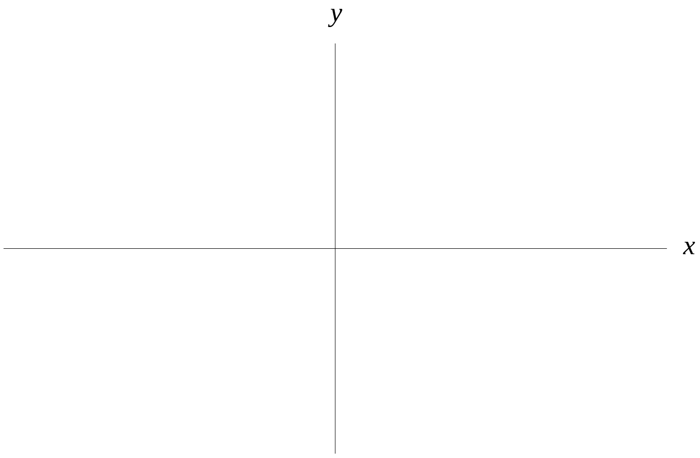


Figure 3.14: The graph of a function.

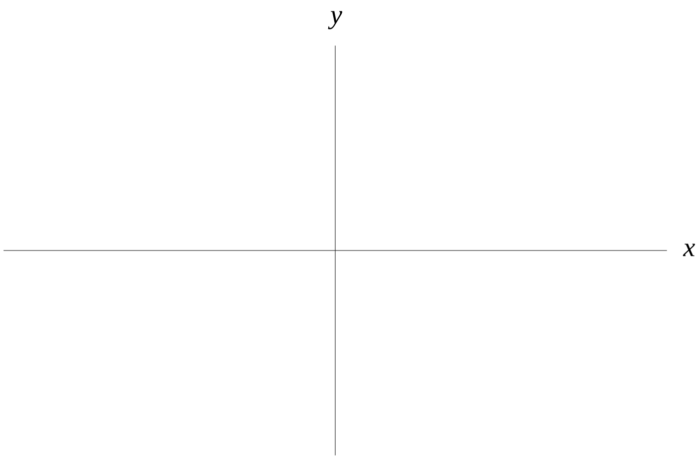


Figure 3.15: Parametric Curves can describe the motion of a particle in two dimensions.

Suppose that $x(t)$ and $y(t)$ are functions of t for some $t \in A \subset \mathbb{R}$. Then the set of coordinates

describes a parametric curve.

Suppose that we want to find the slope of the curve at a point given by t. As long as the curve is not vertical[1], we can do this by zooming in on the point:

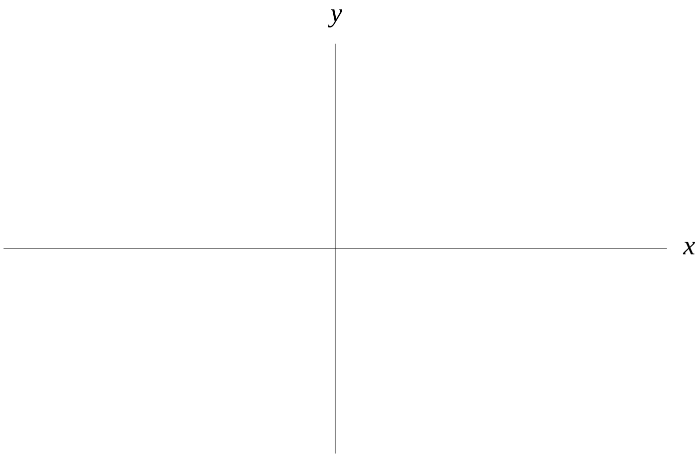


Figure 3.16: Locally — near $(x(t), y(t))$ — the parametric graph looks like the graph of a function: $y = F(x(t))$.

We can use the Chain Rule to differentiate

We can also find the second derivative with respect to x. We first note that $\dfrac{dy}{dx}$ is a function of t so can be differentiating with respect to t. However we have $x = x(t)$ so we have

This is our 'formula' for finding the second derivative of a parametric curve.

[1] in which case the slope is undefined

Examples

1. **Winter 2012: Question 4 (c)**
   Given $x = t^2$ and $y = 2t$, find

   (a) $\dfrac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the point $(1, -2)$.

   (b) the equation of the tangent line to this curve at $(1, -2)$.

   [11 Marks]

   Solution:

   (a) We have

   We calculate

   Hence we have

   Now we find $\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)$. We can write $\dfrac{1}{t} = t^{-1}$ and use the Power Rule:

   so we have

   Now we want to evaluate these at $(t^2, 2t = (1, -2) \Rightarrow t = -1$:

   (b) For the equation of a line we need a point (x_1, y_1) and a slope m:

   We know the slope is given by the derivative and we have a point $(1, -2)$:

2. **Summer 2012: Question 4 (a)**
   Given $x = -t^2$ and $y = \frac{1}{3}t^3$, find

   (i) $\dfrac{dy}{dx}$

   (ii) the equation of the tangent line to this curve at $\left(-4, \dfrac{8}{3}\right)$

   (iii) $\dfrac{d^2y}{dx^2}$

   [11 Marks]

   Solution:

   (a) We use

   We calculate

   (b) For the equation of a line we need a point (x_1, y_1) and a slope m:

   We know the slope is given by the derivative and we have a point $(-4, 8/3)$. It remains to evaluate the derivative. What is t?

   So we have

   and we have

   (c) We use

   and hence calculate

3. **Autumn 2012: Question 1 (b)**
A curve is described parametrically by the equations

$$x = 3t^2 \quad y = 3t.$$

Find the slope of the tangent line to the curve at the point $(24, 6)$.

[5 Marks]

Solution: The slope of the tangent line is given by

We calculate

Now we have

Hence the slope is given by

4. **Autumn 2011: Question 1 (b)**
A curve is defined by the equations

$$x = t - t^2$$
$$y = t - t^3$$

Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

[5 Marks]

Solution: We use

We hence calculate

Now we use

We hence calculate $\frac{d}{dt}\left(\frac{dy}{dx}\right)$. This needs a *Quotient Rule*:

So we have

Exercises:

1. Find $\frac{dy}{dx}$

$$x = t - t^3, \;\; y = 2 - 5t \;\; \textbf{Ans}: -\frac{5}{1-3t^2}.$$

2. Find the equation of the tangent to the curve

$$x = t^4 + 1, \;\; y = t^3 + t$$

at the point $(2, -2)$. **Ans:** $y = -x$

3. Find the equation of the tangent to the curve

$$x = \cos\theta + \sin 2\theta, \;\; y = \sin\theta + \cos 2\theta$$

at the point $\theta = 0$. **Ans:** $y = \frac{1}{2}x + \frac{1}{2}$.

4. Find the equation of the tangent to the curve

$$x = e^t, \;\; y = (t-1)^2$$

at the point $(1, 1)$. **Ans:** $y = -2x + 3$.

5. Find the equation of the tangent to the curve

$$x = 2\sin 2t, \;\; y = 2\sin t$$

at the point $(\sqrt{3}, 1)$. **Ans:** $y = \frac{3}{2}x - \frac{1}{2}$.

6. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$:

   (a) $x = 4 + t^2, \;\; y = t^2 + t^3$. **Ans:** $\frac{2t+3t^2}{2t}, \;\frac{3}{4t}$

   (b) $x = t - e^t, \;\; y = t + e^{-t}$. **Ans:** $\frac{1-e^{-t}}{1-e^t}, \;\frac{e^{-t}}{1-e^{-t}}$

   (c) $x = 2\sin t, \;\; y = 3\cos t$. **Ans:** $-\frac{3}{2}\tan t, \; -\frac{3}{4}\cdot\frac{1}{\cos^3 t}$

7. Find the points on the curve where the tangent is horizontal or vertical:

$$x = 10 - t^2, \;\; y = t^3 - 12t, \;\; \textbf{Ans}: \text{hor. } (6, -16), (6, 16), \text{ ver. } (10, 0)$$

3.3 Implicit Differentiation

Suppose there is an equation in terms of x, y given by:

If the x and y are combined in an 'appropriate' way, then this defines a curve in space and this is the *equation of the curve*. The equation of the curve asks the question; is a general point on the plane (x_0, y_0) on the curve? If it satisfies the equation then it is on the curve

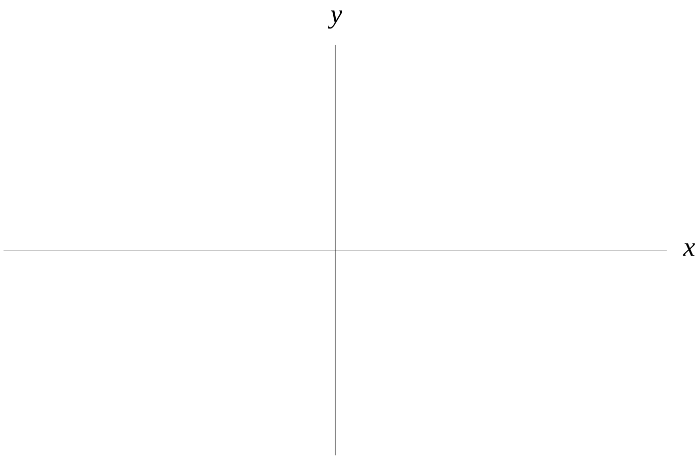


Figure 3.17: A point is on the curve if and only if it satisfies the equation of the curve.

Example

Consider the equation

This is the equation of the unit circle. It is not a function: but is rather is comprised of *two* functions (branches):

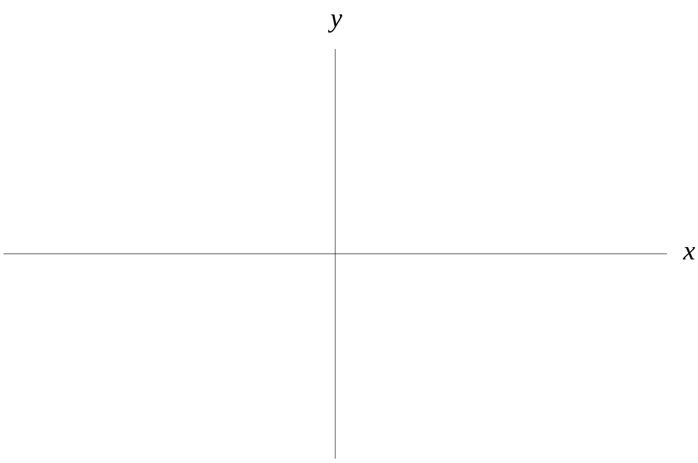


Figure 3.18: A circle is two functions f_1 and f_2, *glued* together.

Suppose we want to find the equation of the tangent to the point $(1/\sqrt{3}, \sqrt{2/3})$. We *could* differentiate $f_1(x) = \sqrt{1-x^2}$ and find the equation of the tangent that way, but what if the curve is not as simple as the circle — with many, many branches?

Luckily there is a technique called *implicit differentiation*, which allows us to differentiate away and not worry so much about branches etc. Briefly, we implicitly say that $y = f(x)$, so when we differentiate, for example y^2, we must see it as $[y(x)]^2$ so it would have derivative:

Given the equation of a curve

then if both sides are differentiated with respect to x then the equation reads:

which can be solved for $y' \equiv dy/dx$ in terms of x and y. This will give the slope of the tangent to the curve at (x, y). If (x, y) is not on the curve, dy/dx in this context is meaningless.

Just remember:

- The derivative of y with respect to x is just dy/dx.
- The derivative of a function of y must be differentiated as a chain rule.
- The product rule for a term of the form:

Examples

1. Use implicit differentiation to find the equation of the tangent line to the curve

$$16x^4 + y^4 = 32$$

at the point $(1, 2)$.

Solution: First it is a good idea to check that the point $(1, 2)$ is indeed on the line ✓ It is also a good idea to rewrite the curve replacing y with $[y(x)]$ — this reminds you to use the Chain Rule. Differentiating across with respect to x, and plugging in $(x_0, y_0) = (1, 2)$:

Now using the line formula:

2. The *Folium of Descartes* is a plane curve with the equation

$$x^3 + y^3 - 3xy = 0$$

Find the x-coordinate of a point where it has a horizontal tangent.

Solution: For a horizontal tangent we must have

We find dy/dx by implicitly differentiating with respect to x:

The only time a fraction is zero is when the numerator is zero:

To see which points on the curve satisfy this condition, substitute into the equation of the curve:

3. Use implicit differentiation to find the equation of the tangent line to the curve $x^2+3xy^2+y^3 = 5$ at the point $(1,1)$.

   Solution: The tangent is a line so we need to use the equation of a line 'formula':

   We have $(x_1, y_1) = (1,1)$ so we need $m = \frac{dy}{dx}$. First we note that $y = y(x)$ and we write

   Now we differentiate with respect to x:

   Now we evaluate this, the slope, at $(x,y) = (1,1)$

   Now we use the formula for the equation of a line:

   Alternative solution: If we are careful, we can skip a few steps. From the point where we differentiated with respect to x:

   $$2x + 3x \cdot 2y \cdot \frac{dy}{dx} + y^2 \cdot 3 + 3y^2 \cdot \frac{dy}{dx} = 0,$$

   we can, *after we have differentiated*, substitute in our x and y values and solve for $\frac{dy}{dx}$:

4. **Winter 2012: Question 1 (b)** Given the ellipse with the equation

$$2x^2 + y^2 = 48$$

(a) Find a general expression for $\dfrac{dy}{dx}$.

(b) Find the two points on the ellipse with $x = 4$.

(c) Find the value of $\dfrac{dy}{dx}$ at each of these points.

[6 Marks]

Solution:

(a) First we note that $y = y(x)$ so we write

Now we differentiate both sides with respect to x:

Now we solve for $\dfrac{dy}{dx}$:

(b) *A point is on the curve if and only if it satisfies the equation of the curve.* Suppose $(4, y)$ is on the curve: what must y equal? Let $x = 4$:

(c) We calculate

5. **Summer 2012: Question 1 (a)** Use implicit differentiation to find the slope of the tangent line to the curve $2x^3 - y = 3y^2$ at the point $(1, -1)$.

   Solution: The slope of the tangent is exactly the derivative so we must find $\frac{dy}{dx}$. First we note that $y = y(x)$ so we write

   Now we differentiate both sides with respect to x:

   Now we solve for $\frac{dy}{dx}$:

   Take out the common factor $\frac{dy}{dx}$ on the right-hand side:

   Now we evaluate at $(x, y) = (1, -1)$

Exercises

1. Find $y' = \frac{dy}{dx}$:

   (a) $xy + 2x + 3x^2 = 4$ **Ans:** $-\dfrac{y + 6x + 2}{x}$

   (b) $\dfrac{1}{x} + \dfrac{1}{y} = 1$ **Ans**: $-\dfrac{y^2}{x^2}$

   (c) $x^2 + y^2 = 1$ **Ans**: $-\dfrac{x}{y}$

   (d) $x^3 + x^2y + 4y^2 = 6$ **Ans:** $-\dfrac{3x + 2y}{x}$

   (e) $x^2y + xy^2 = 3x$ $\dfrac{3 - 2xy - y^2}{x(x + 2y)}$

   (f) $x^2y^2 + x\sin(y) = 4$ **Ans**: $-\dfrac{\sin y + 2xy^2}{x(\cos y + 2xy)}$

   (g) $4\cos x \sin y = 1$ **Ans**: $\tan x \tan y$

   (h) * $\tan\left(\frac{x}{y}\right) = x + y$ **Ans**: $\dfrac{\left(\sec^2\left(\frac{x}{y}\right) - y\right)y}{x\sec^2\left(\frac{x}{y}\right) + y^2}$

2. Use implicit differentiation to find the equation of the tangent line to the curve at the given point

   (a) $x^2 + xy + y^2 = 3$, $(1, 1)$ **Ans**: $y = -x + 2$

   (b) $x^2 + 2xy - y^2 + x = 2$, $(1, 2)$ **Ans**: $y = \frac{7}{2}x - \frac{3}{2}$

   (c) * $x^2 + y^2 = (2x^2 + 2y^2 - x)^2$, $(0, 1/2)$

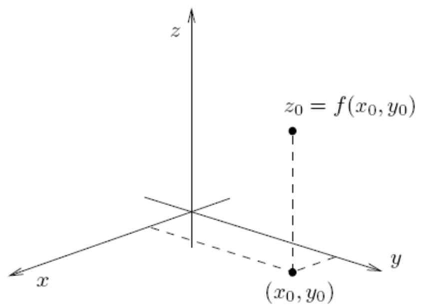
   (d) * $x^{2/3} + y^{2/3} = 4$, $(-3\sqrt{3}, 1)$

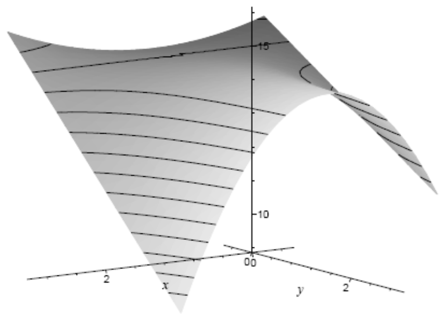
3.4 Multivariable Calculus

Functions of Several Variables: Surfaces

Many equations in engineering, physics and mathematics tie together more than two variables. For example Ohm's Law ($V = IR$) and the equation for an ideal gas, $PV = nRT$, which gives the relationship between pressure (P), volume (V) and temperature (T). If we vary any two of these then the behaviour of the third can be calculated:

How P varies as we change T and V is easy to see from the above, but we want to adapt the tools of one-variable calculus to help us investigate functions of more than one variable.
For the most part we shall concentrate on functions of two variables such as $z = x^2 + y^2$ or $z = xsin(y + e^x)$. Graphically $z = f(x, y)$ describes a surface in 3D space — varying the x- and y-coordinates gives the z-coordinate, producing the surface:





As an example, consider the function $z = x^2 + y^2$. If we choose a positive value for z, for example $z = 4$, then the points (x, y) that can give rise to this value are those satisfying $x^2 + y^2 = 4 = 2^2$, i.e. those on the circle centred on the origin of radius 2. Note that at $(x, y) = (0, 0)$, $z = 0$, but if $x \neq 0$ or $y \neq 0$, then $x^2 > 0$ or $y^2 > 0$, and it follows that $z > 0$. Thus the minimum value taken by this function is $z = 0$, at the origin:

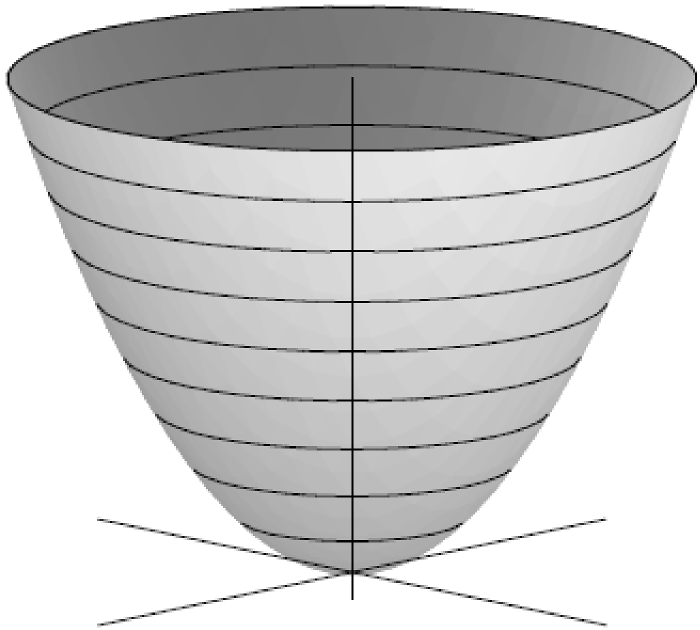
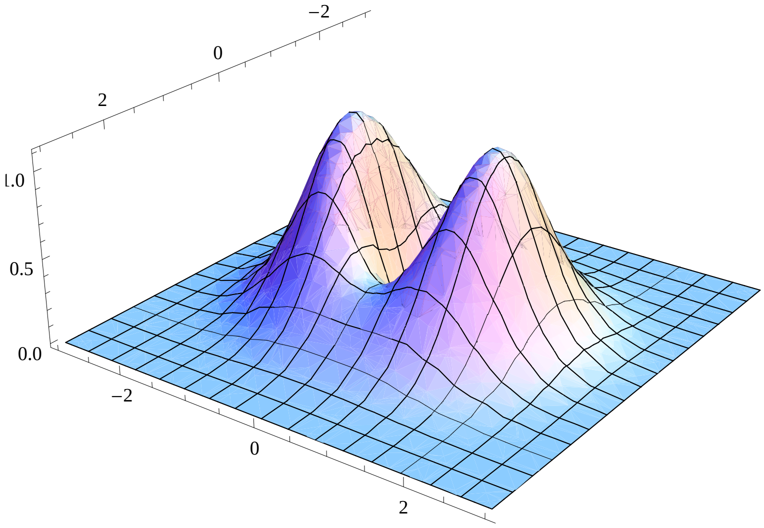
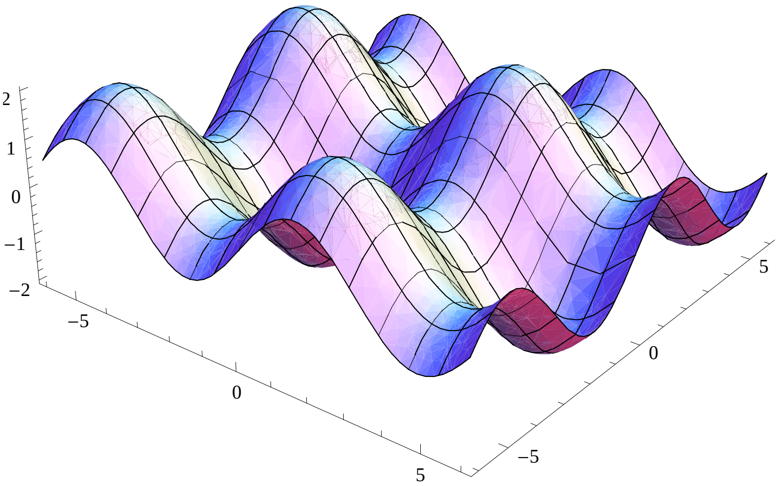


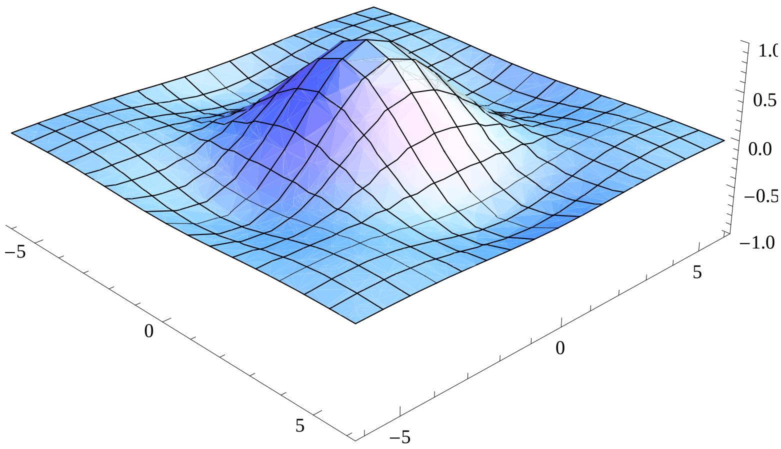
Figure 3.19: The surface defined by the relation $x^2 + y^2 = 2$.

Three examples. Which are which?

$$f(x, y) = (x^2 + 3y^2)e^{-x^2-y^2}$$
$$g(x, y) = \frac{\sin x \sin y}{xy}$$
$$h(x, y) = \sin x + \sin y$$



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5

Partial Derivatives

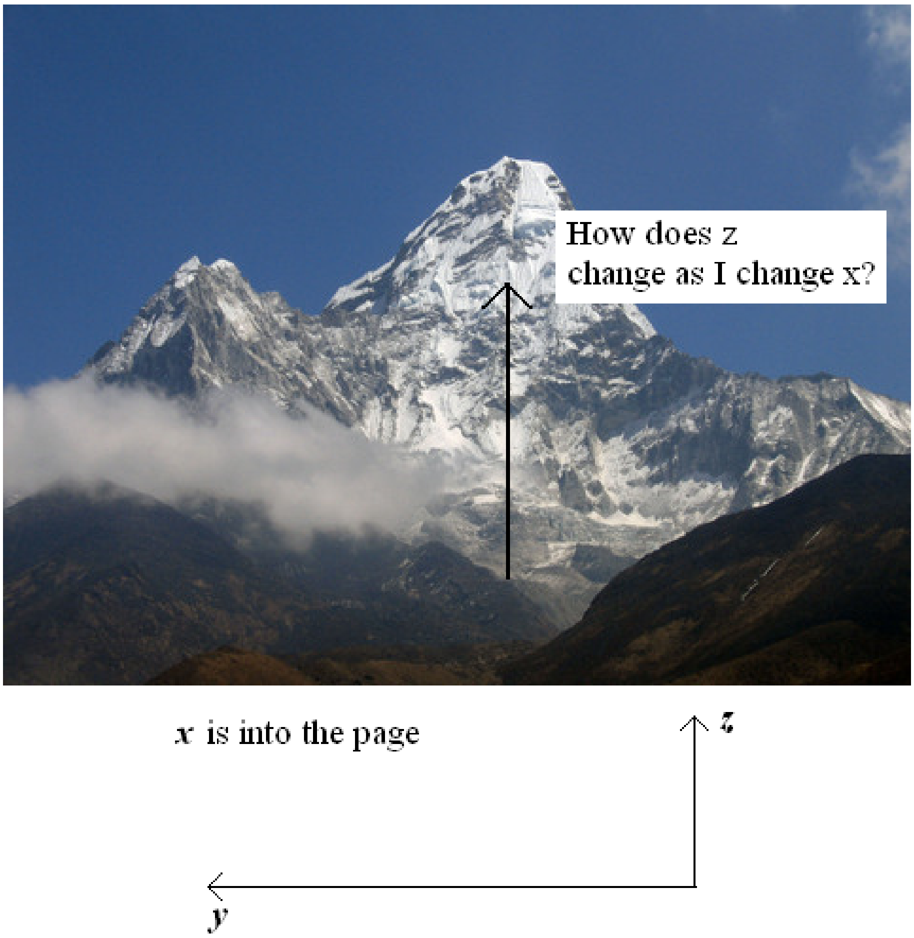

Figure 3.20: What is the rate of change in z as I keep y constant

If we were to look at this from side on:

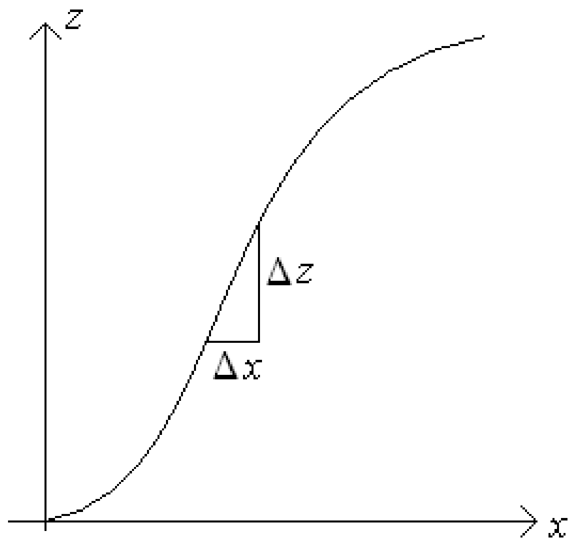

Figure 3.21: When y is a constant z can be considered a function of x only.

In general we have that $z = f(x, y)$; but if $y = b$ is fixed (constant):

We can view $f(x, b)$ as a function of x alone. Now what is the rate of change of a single-variable function $g(x)$ with respect to x:

Which is also the slope of the tangent to f at x. Hence the rate of change of $f(x, y)$ with respect to x at $x = a$ when y is fixed at $y = b$ is the slope of the surface in the x-direction.

Example

Let $z = f(x, y) = x^3 + x^2y^3 - 2y^3$. What is the rate of change of z with respect to x when $y = 2$?

Solution:

Hence the rate of change of z with respect to x, when y is fixed at $y = b$, is given by:

More generally, we fix $y = y$ and define

as *the partial derivative of f with respect to x.*

We define the partial derivative of f with respect to y in exactly the same way.

Example

What are the partial derivatives of

$$z = x^2 + xy^5 - 6x^3y + y^4$$

with respect to x and y respectively?

Solution:

There are many alternative notations for partial derivatives. For instance, instead of $\frac{\partial f}{\partial x}$ we can write f_x or f_1. In fact,

$$\frac{\partial f}{\partial x} \equiv \frac{\partial z}{\partial x} \equiv f_x(x, y) \equiv f_1(x, y)$$
$$\frac{\partial f}{\partial y} \equiv \frac{\partial z}{\partial y} \equiv f_y(x, y) \equiv f_2(x, y)$$

To compute partial derivatives, all we have to do is remember that the partial derivative of a function with respect to x is the same as the *ordinary* derivative of the function g of a single variable that we get by keeping y fixed. Thus we have the following:

1. To find $\frac{\partial f}{\partial x}$, regard y as a constant and differentiate $f(x, y)$ with respect to x.
2. To find $\frac{\partial f}{\partial y}$, regard x as a constant and differentiate $f(x, y)$ with respect to y.

Example

If $f(x, y) = 4 - x^2 - 2y^2$, find $f_x(1, 1)$ and $f_y(1, 1)$ and interpret these numbers as slopes.

Solution:

Using this technique we can make use of known results from one-variable theory such as the product, quotient and chain rules (Careful — the Chain rule only works if we are differentiating with respect to one of the variables — we may have more to say on this in the next section).

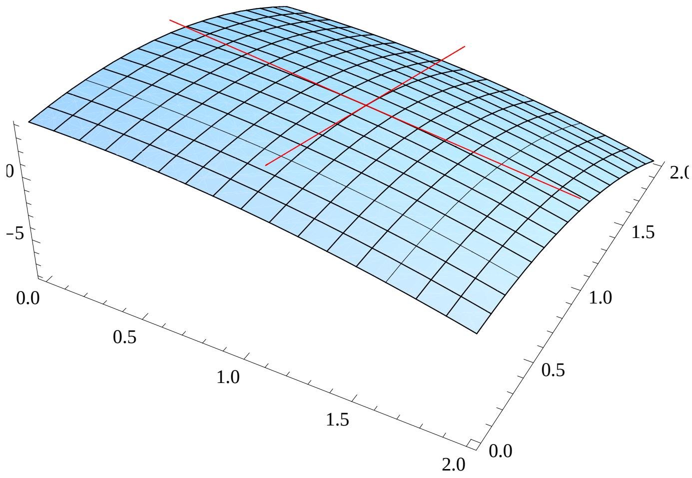

Figure 3.22: The slope in the y-direction at $(1,1)$ is $f_y(1,1) = -2$ whilst the slope in the x-direction at $(1,1)$ is $f_x(1,1) = -4$.

Examples

Find the partial derivative with respect to y of the function

$$f(x,y) = \sin(xy)e^{x+y}$$

Solution:

Compute f_1 and f_2 when $z = x^2y + 3x\sin(x-2y)$.

Solution:

Functions of More Variables

We can extend the notion of partial derivatives to functions of any (finite number) of variables in a natural way. For example if $w = \sin(x + y) + z^2 e^x$ then:

Higher Order Derivatives

Suppose $z = x \sin y + x^2 y$. Then

Both of these partial derivatives are again functions of x and y, so we can differentiate both of them, either with respect to x, or with respect to y. This gives us a total of four *second order partial derivatives:*

Remark

The mixed partial derivatives in this case are equal:

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}.$$

This is not something special about our particular example — it will be true for all *reasonably well-behaved functions.* This is the *symmetry of second derivatives.* Note the confusing notation:

$$\frac{\partial}{\partial x \partial y} = f_{yx} \quad \text{etc.} \tag{3.10}$$

Examples

Compute

$$\frac{\partial z}{\partial x}\,,\ \frac{\partial z}{\partial y}\ ,\ \text{and}\ \frac{\partial^2 z}{\partial x^2}$$

when $z = x^3y + e^{x+y^2} + y\sin x$.

Solution:

Compute all the second order partial derivatives of the function $f(x, y) = \sin(x + xy)$.

Solution:

Winter 2012: Question 1 (a)

Given $s = -5x^3 + 3x^2y - 2y^2$ find

$$\frac{\partial s}{\partial x}, \frac{\partial s}{\partial y} \text{ and } \frac{\partial^2 s}{\partial y \partial x}.$$

[5 Marks]

Solution: We calculate

Summer 2012: Question 1 (b)

Given $z = x^3y + y^2$ find $\frac{\partial z}{\partial x}$, $\frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial x \partial y}$.

Solution: We calculate

Summer 2012: Question 5 (c)

Given the function $z = \ln(x^2 + y^2)$ show that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0.$$

[8 Marks]

Solution: We calculate

Exercises

1. Find all the first order derivatives of the following functions:

$$(i)\ f(x,y) = x^3 - 4xy^2 + y^4 \qquad (ii)\ f(x,y) = x^2e^y - 4y$$
$$(iii)\ f(x,y) = x^2 \sin xy - 3y^2 \quad (iv)\ f(x,y,z) = 3x\sin y + 4x^3y^2z$$

2. Find the indicated partial derivatives: (i) $f(x,y) = x^3 - 4xy^2 + 3y$: f_{xx}, f_{yy}, f_{xy}
(ii) $f(x,y) = x^4 - 3x^2y^3 + 5y$: f_{xx}, f_{xy}, f_{xyy}
(iii) $f(x,y,z) = e^{2xy} - \frac{z^2}{y} + xz\sin y$: f_{xx}, f_{yy}, f_{yyzz}

Selected Solutions:

1. (i)

$$\frac{\partial f}{\partial x} = 3x^2 - 4y^2(1) + 0.$$

$$\frac{\partial f}{\partial y} = -4x(2y) + 4y^3 = 4y^4 - 8xy.$$

(ii)

$$\frac{\partial f}{\partial x} = e^y(2x) + 0 = 2xe^y.$$

$$\frac{\partial f}{\partial y} = x^2(e^y) - 4 = x^2e^y - 4.$$

(iii) This one needs a product and a chain rule for f_x and a chain rule for f_y.

$$\begin{aligned}\frac{\partial f}{\partial x} &= x^2 \times \frac{\partial \sin xy}{\partial x} + \sin xy \times \frac{\partial x^2}{x} + 0\\ &= x^2 \times \cos xy \times \frac{\partial xy}{\partial x} + \sin xy \times 2x\\ &= x^2 \cos xy \times y + 2x\sin xy = x^2y\cos xy + 2x\sin xy.\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial y} &= x^2 \times \cos xy \times \frac{\partial xy}{\partial y} - 6y\\ &= x^2 \times \cos xy \times x - 6y = x^3\cos xy - 6y.\end{aligned}$$

(iv)

$$\frac{\partial f}{\partial x} = \sin y(3) + 4y^2z(3x^2) = 3(\sin y + 4x^2y^2z).$$

$$\frac{\partial f}{\partial y} = 3x(\cos y) + 4x^3z(2y) = 3x\cos y + 8x^2yz.$$

$$\frac{\partial f}{\partial z} = 0 + 4x^3y^2(1) = 4x^3y^2.$$

2. (i)

$$f_x = 3x^2 - 4y^2(1) + 0 = 3x^2 - 4y^2$$

$$f_{xx} = 6x + 0 = 6x.$$

$$f_y = 0 - 4x(2y) + 3 = -8xy + 3.$$

$$f_{yy} = -8x(1) + 0 = -8x.$$

$$f_{xy} = (f_x)_y = 0 - 8y = -9y.$$

(ii)

$$f_x = 4x^3 - 3y^3(2x) + 0 = 4x^3 - 6xy^3 = .$$

$$f_{xx} = 12x^2 - 6y^3(1) = 6(2x^2 - y^3).$$

$$f_{xy} = (f_x)_y = 0 - 6x(3y^2) = -18xy^2.$$

$$f_{xyy} = (f_{xy})_y = -18x(2y) = -36y.$$

(iii) This example is going to require the chain rule when we differentiate e^{2xy}. Also to ease differentiation, we use the fact that $1/a^n = a^{-n}$ to write:

$$f(x, y, z) = e^{2xy} - z^2y^{-1} + xz\sin y.$$

$$f_x = e^{2xy} \times \frac{\partial 2xy}{\partial x} + 0 + z\sin y = e^{2xy} \times 2y + z\sin y = 2ye^{2xy} + z\sin y.$$

$$f_{xx} = 2ye^{2xy} \times \frac{\partial 2xy}{\partial y} + 0 = 2ye^{2xy} \times 2y = 4y^2e^{2xy}.$$

$$\begin{aligned} f_y &= e^{2xy} \times \frac{\partial 2xy}{\partial y} - z^2(-1y^{-2}) + xz(\cos y) = e^{2xy} \times 2x + z^2y^{-2} + xz\cos y, \\ &= 2xe^{2xy} + z^2y^{-2} + xz\cos y. \end{aligned}$$

$$\begin{aligned} f_{yy} &= 2xe^{2xy}\frac{\partial 2xy}{\partial y} + z^2((-2)y^{-3}) + xz(-\sin y), \\ &= 2xe^{2xy} \times 2x - 2\frac{z^2}{y^3} - xz\sin y = 4x^2e^{2xy} - 2\frac{z^2}{y^3} - xz\sin y. \end{aligned}$$

Well $f_{yyzz} = (f_{yy})_{zz}$ so first we evaluate:

$$(f_{yy})_z = 0 - \frac{4z}{y^3}(1) - x\sin y(1) = \frac{4z}{y^3} - x\sin y,$$

$$(f_{yy})_{zz} = \frac{4}{y^3}.$$

3.5 Applications to Error Analysis

Differentials

For a differentiable function $y = f(x)$ of a single variable x, we define the differential 'dx' to be an independent variable; that is, dx can be given the value of any real number. Differentiable functions are *locally approximately linear*: the tangent at x approximates the function well near x. The differential of y is then defined by:c

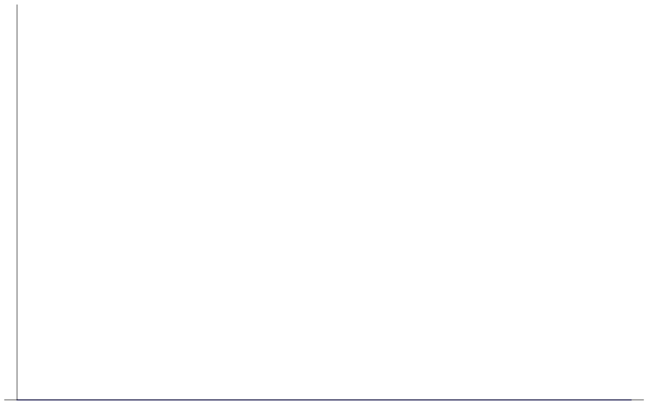

Figure 3.23: The differential estimates the actual change in y, Δy, due to a change in x: $x \to \Delta x$. For small changes in x, the differential is approximately equal to the actual change in y: $dy \approx \Delta y$.

For a differentiable function of two variables $z = f(x, y)$, we define the differentials dx and dy to be independent variables and the differential dz estimates the change in z when x changes to $x + \Delta x$ and y changes to $y + \Delta y$:

Example

If $z = f(x, y) = x^2 + 3xy - y^2$, find the differential dz. If x changes from 2 to 2.05 and y changes from 3 to 2.96, compute the values of dz and Δz (the actual change in z).

Solution:

Example

The pressure, volume and temperature of a mole of an ideal gas are related by the equation $PV = 8.31T$, where P is measured in kilopascals, V in litres and T in kelvins. Use differentials to find the approximate change in the pressure if the volume increases from 12 L to 12.3 L and the temperature decreases from 310 K to 305 K.

Solution:

Exercise: Compare this with the actual change

$$\Delta P = P(305, 12.3) - P(310, 12).$$

Propagation of Errors

Suppose we have a physical property P related to two other properties A and B by:

Now suppose we measure A and B and record values A_0 and B_0 with associated errors ΔA and ΔB. We can now keep track of the errors in P due to errors in A and B by knowing "*how much P will change due to small changes in A (and/ or B) between $A - \Delta A$ and $A + \Delta A$ (and $B - \Delta B$ and $B + \Delta B$)*". The differential of P gives an estimate of this:

Now we don't want errors to cancel each other out so we write:

Example

The base radius and height of a right circular cone are measured as 10 cm and 25 cm, respectively, with a possible error in measurement of as much as 0.1 cm in each. Use differentials to estimate the maximum error in the calculated volume of the cone.

Solution:

This procedure generalises in the obvious way.

Example

The dimensions of a rectangular box are measured to be $h = 75$ cm, $w = 60$ cm, and $l = 40$ cm, and each measurement is correct within 0.2 cm. Use differentials to estimate the largest possible error when the volume of the box is calculated from these measurements.

Solution:

Note that $V(75, 60, 40) = 180,000$ cm^3 so this error is of the order of 1%.

Example

The power P consumed in a resistor is given by

$$P = \frac{V^2}{R},$$

where V is the voltage and R is the resistance across the resistor.

(i) Use partial derivatives and differentials to determine an approximate expression for ΔP, the change in the power P.

(ii) Find the *approximate* change in P when V is changed by 5% and R is decreased by 0.5%.

[8 Marks]

Solution

(i) The differential dP approximates the change in P, ΔP in terms of dV and dR, the changes in V and R respectively:

Hence we write $P(V, R) = V^2R^{-1}$ and calculate the partial derivatives:

(ii) To find the percentage change in P we look at $\dfrac{\Delta P}{P} \approx \dfrac{dP}{P}$. Hence we divide the differential by P:

Now we have that $\dfrac{dV}{V} = 0.05$ and $\dfrac{dR}{R} = 0.005$. So we have

The answer is 9.5%.

Exercises

1. Use differentials to estimate the amount of tin in a closed tin closed tin can with diameter 8 cm and height 12 cm if the can is 0.04 cm thick.

2. Use differentials to estimate the amount of metal in a closed cylindrical can that is 10 cm high and 4 cm is diameter if the metal in the wall is 0.05 cm thick and the metal in the top and bottom is 0.1 cm thick.

3. If R is the total resistance of three resistors, connected in parallel, with the resistances R_1, R_2 and R_3, then
$$\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}$$
. If the resistances are measured as $R_1 = 25\ \Omega$, $R_2 = 40\ \Omega$ and $R_3 = 50\ \Omega$, with possible errors of 5% in each case, estimate the maximum error in the calculated value of R.

4. The moment of inertia of a body about an axis is given by $I = kbD^3$ where k is a constant and B and D are the dimensions of the body. If B and D are measured as 2 m and 0.8 m respectively, and the measurement errors are 10 cm in B and 8 mm in D, determine the error in the calculated value of the moment of inertia using the measured values, in terms of k.

5. The volume, V, of a liquid of viscosity coefficient η delivered after a time t when passed through a tube of length l and diameter d by a pressure p is given by
$$V=\frac{pd^4t}{128\eta l}.$$
If the errors in V, p and l are 1%, 2% and 3% respectively, determine the error in η. HINT: If the error in A is x% then the error is $xA_0/100$ when $A = A_0$.

Selected Solutions:

1. Assuming that the measurements of 8 cm and 12 cm are taken from the outside of the can, then we could estimate the change in volume of a cylinder if the radius were increased by 0.04 cm to 4 cm and the height increased by 0.08 cm to 12 cm (convince yourself of this with a picture.). Now the tin in the can comprises the difference between a $(r, h) = (3.96, 11.92)$ cylinder and a $(r, h) = (3, 12)$ cylinder. Now the volume of a cylinder is given by
$$V=\pi r^2 h. \tag{3.11}$$
We can use the differential of V, dV (evaluated at $(r, h) = (3.96, 11.92)$ — although the other way around would also be a good estimate) to estimate the change in volume:
$$dv=\frac{\partial V}{\partial r}dr+\frac{\partial V}{\partial h}dh,$$
where $dr = +0.04$ cm and $dh = +0.08$ cm. Now
$$\frac{\partial V}{\partial r}=2\pi rh\bigg|_{(r,h)=(3.96,11.92)}=2\pi(3.96)(11.92)\text{ , and}$$
$$\frac{\partial V}{\partial h}=\pi r^2\bigg|_{(r,h)=(3.96,11.92)}=\pi(3.96)^2.$$
$$\Rightarrow dP=2\pi(3.96)(11.92)\times(+0.04)+\pi(3.96^2)\times(+0.8)\approx 15.805\text{ cm}^3.$$

2. Using the same method, the differential is a good estimate:

$$dv = \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial h}dh,$$

where in this case we take $(r,h) = (1.95, 9.8)$, and $dh = +0.2$ and $dr = 0.05$. Now

$$\frac{\partial V}{\partial h} = \pi r^2\Big|_{(r,h)=(1.95,9.8)} = \pi(1.95)^2,$$

$$\frac{\partial V}{\partial r} = 2\pi r h\Big|_{(r,h)=(1.95,9.8)} = 2\pi(1.95)(9.8).$$

Hence

$$dP = \pi(1.95^2) \times (+0.2) + 2\pi(1.95)(9.8) \times (0.05) \approx 8.393 \text{ cm}^3.$$

3. Now first we want to get an expression for $R(R_1, R_2, R_3)$:

$$\frac{1}{R} = \frac{1}{R_1}\cdot\frac{R_2R_3}{R_2R_3} + \frac{1}{R_2}\cdot\frac{R_1R_3}{R_1R_3} + \frac{1}{R_3}\cdot\frac{R_1R_2}{R_1R_2},$$

$$\Rightarrow R(R_1,R_2,R_3) = \frac{R_1R_2R_3}{R_1R_2 + R_1R_3 + R_2R_3}.$$

We can approximate the error in R, ΔR by:

$$\Delta R \approx dR = \left|\frac{\partial R}{\partial R_1}\right| dR_1 + \left|\frac{\partial R}{\partial R_2}\right| dR_2 + \left|\frac{\partial R}{\partial R_3}\right| dR_3,$$

where dR_i corresponds to the error in R_i, ΔR_i. Now the errors are 5 % hence:

$$dR_1 = (0.05)(25) = 1.25.$$

$$dR_2 = (0.05)(40) = 2.$$

$$dR_3 = (0.05)(50) = 2.5.$$

Also, using the quotient rule:

$$\begin{aligned}\frac{\partial R}{\partial R_1} &= \frac{(R_1R_2 + R_1R_3 + R_2R_3)(R_2R_3) - (R_1R_2R_3)(R_2+R_3)}{[R_1R_2 + R_1R_3 + R_2R_3]^2}.\\ &= \frac{R_1R_2^2R_3 + R_1R_2R_3^2 + R_2^2R_3^2 - R_1R_2^2R_3 - R_1R_2R_3^2}{[R_1R_2 + R_2R_3 + R_1R_3]^2}.\end{aligned}$$

Similarly,

$$\frac{\partial R}{\partial R_2} = \frac{R_1^2R_3^2}{[R_1R_2 + R_1R_3 + R_2R_3]^2}.$$

$$\frac{\partial R}{\partial R_3} = \frac{R_1^2 R_2^2}{[R_1R_2 + R_1R_3 + R_2R_3]^2}.$$

Note that these will be evaluated at $(R_1, R_2, R_3) = (25, 40, 50)$, using a calculator;

$$\frac{\partial R}{\partial R_1} = \frac{(40)^2(50)^2}{[(25)(40) + (25)(50) + (40)(50)]^2} = \frac{64}{289}.$$

Similarly,

$$\frac{\partial R}{\partial R_2} = \frac{25}{289}.$$

$$\frac{\partial R}{\partial R_3} = \frac{16}{289}.$$

Hence,

$$\Delta R \approx \frac{64}{289} \times (1.25) + \frac{25}{289} \times (2) + \frac{16}{289} \times (2.5) \approx 0.588\,\Omega.$$

4. In class we did an example where we estimated the change in I due to changes in B and D (from the sample test). The only difference between that example and this one is that errors are always positive and we take absolute values; i.e:

$$\Delta I = \left|\frac{\partial I}{\partial B}\right| \Delta B + \left|\frac{\partial I}{\partial D}\right| \Delta D.$$

5. First we solve for a function $\eta(V, P, l, d, t)$:

$$\eta = \frac{Pd^4t}{128Vl} = \frac{d^4t}{128} PV^{-1}l^{-1}.$$

Again we use the differential to estimate the error (as errors in d and t were not mentioned we will assume they don't have errors):

$$\Delta\eta \approx \left|\frac{\partial \eta}{\partial V}\right| \Delta V + \left|\frac{\partial \eta}{\partial P}\right| \Delta P + \left|\frac{\partial \eta}{\partial l}\right| \Delta l.$$

Now we must look at the partial derivatives (verify the last steps yourself):

$$\left|\frac{\partial \eta}{\partial V}\right| = \left|-\frac{d^4t}{128} PV^{-2}l^{-1}\right| = \frac{\eta}{V}.$$

$$\left|\frac{\partial \eta}{\partial P}\right| = \left|\frac{d^4t}{128} V^{-1}l^{-1}\right| = \frac{\eta}{P}.$$

$$\left|\frac{\partial \eta}{\partial l}\right| = \left|\frac{d^4t}{128} PV^{-1}l^{-2}\right| = \frac{\eta}{l}.$$

Now by the hint, the errors in V, P and l $\Delta V = \frac{V}{100}$, $\Delta P = \frac{2P}{100}$, and $\Delta l = \frac{3l}{100}$. Hence,

$$\begin{aligned}\Delta\eta &= \frac{\eta}{V} \times \frac{V}{100} + \frac{\eta}{P} \times \frac{2P}{100} + \frac{\eta}{l} \times \frac{3l}{100} \\ &= \frac{6}{100}\eta.\end{aligned}$$

That is the error in η is 6 %.

3.6 Related Rates

If we are pumping a balloon, both the volume and radius of the balloon are increasing and their rates of increase are related to each other. In a related rates problem the idea is to compute the rates of change of one quantity in terms of the rate of change of another quantity which, possibly, may be more easily measured. The idea is to find equations relating the two quantities and then use the Chain Rule to differentiate both sides with respect to time, t:

Examples

1. The area of a circle is decreasing at 2 cm^2/min. How fast is the radius of the circle changing when the area is 100 cm^2?

   Solution: We know that the area is related to the radius by

   Note that we can now differentiate both sides with respect to t — using the Chain Rule on the right-hand-side:

   Note that we actually have $\frac{dA}{dt} = -2$ and we could evaluate $\frac{dA}{dr} = 2\pi r$ if we knew r. Note however that we have $A = 100$ so can find r:

   Hence the only unknown is $\frac{dr}{dt}$ which is exactly what we are looking for:

2. Air is being pumped into a spherical balloon so that its volume is increasing at a rate of 100 cm^3/s. How fast is the radius of the balloon increasing when the diameter is 50 cm?

   Solution: We know that the volume is related to the radius by

   Note that we can now differentiate both sides with respect to t — using the Chain Rule on the right-hand-side:

   Note that we actually have $\dfrac{dV}{dt} = 100$ and can evaluate $\dfrac{dV}{dr} = 4\pi r^2$ at $r = 50$ cm. Hence the only unknown is $\dfrac{dr}{dt}$ which is exactly what we are looking for:

3. **Summer 2012: Question 4 (b) [11 Marks]** The surface area of a sphere is increasing at a constant rate of 500 cm^2/s. Find the rate of increase of the volume when the radius equals 20 cm.

   Solution: OK firstly we know that the surface area of the sphere and volume of the sphere are given by

   We know that the rate of change of surface area with respect to time, $\dfrac{dA}{dt}$ is given by 500. We use the Chain Rule to differentiate with respect to time, t:

   Note that we can calculate $\dfrac{dA}{dr}$:

   In particular we are interested in when $r = 20$ cm so we have

In particular we can find $\dfrac{dr}{dt}$:

Now we try to calculate $\dfrac{dV}{dt}$. We use the Chain Rule:

However we can calculate $\dfrac{dV}{dr}$ at $r = 20$ cm and we already know $\dfrac{dr}{dt}$:

Exercises:

1. (a) If A is the area of a circle with radius r and the circle expands as time passes, find $\dfrac{dA}{dt}$ in terms of $\dfrac{dr}{dt}$. **Ans**: $2\pi r \cdot \dfrac{dr}{dt}$.

   (b) Suppose oil spills from a ruptured tanker and spreads in a circular pattern. If the radius of the oil spill increases at a constant rate of 1 m/s, how fast is the area of the spill increasing when the radius is 30 m? **Ans**: 60π.

2. **Autumn 2012: Question 4 (b) [10 Marks]** The surface area A of a spherical balloon is increasing at the rate of 40 cm^2/s. Find the rate of change of the radius R of the balloon when the radius equals 20 cm. **Ans**: $\dfrac{1}{4\pi}$ cm/s.

3. If a snowball melts so that its surface area decreases at a rate of 1 cm^2/min, find the rate at which the diameter decreases when the diameter is 10 cm. **Ans**: $\dfrac{1}{20\pi}$ cm/min.

4. If V is the volume of a cube with edge length x and the cube expands as time passes, find $\dfrac{dV}{dt}$ in terms of $\dfrac{dx}{dt}$. **Ans**: $3x^2 \cdot \frac{dx}{dt}$.

5. How fast is the surface area of a cube changing when the volume of the cube is 64 cm^3 and increasing at 2 cm^3/s? **Ans**: $\dfrac{1}{24}$ cm^2/s.

6. * At a certain instant the length of a rectangle is 16 m and the width 12 m. The width is increasing at 3 m/s. How fast is the length decreasing if the area of the rectangle is not changing? **Ans**: -24 m/s.

7. * Sawdust is falling onto a pile at a rate of $\frac{1}{2}$ m^3/min. If the pile of sawdust maintains the shape of a right circular cone with height equal to half the diameter of the base, how fast is the height of the pile increasing when the pile is 3 m high? **Ans**: $\dfrac{1}{18\pi}$ m/min.

Chapter 4

Integration

The essence of mathematics is not to make simple things complicated, but to make complicated things simple.

S Gudder

4.1 Integration by Parts

The chain rule for differentiation leads to the substitution method for integration. The product rule for differentiation leads to a new integration technique: *integration by parts*:

Integrating, we get the integration by parts formula:

$$\int u\,dv = uv - \int v\,du.$$

To use this formula to evaluate an integral, you must make a double substitution: choose u and dv so that $u\,dv$ equals the integrand, then apply the formula. (Clearly this is more complicated than the ordinary substitution method, which you would normally try first.)

It can help if one follows the LIATE guideline in choosing u. The reason for this is once you choose u, dv is determined and the LIATE u.

Examples

Evaluate each of the following:

1. $I = \int x \ln x\,dx$. You may assume the derivative of $\ln x$ is $1/x$:

Solution:

Choose $u = \ln x$ by LIATE and therefore $dv = x\,dx$:

Check this answer by differentiating it—observe that this needs the product rule, as one would expect. This example is typical: one does not need to insert "$+C$" at the early stage of going from dv to v, but it is needed when the final integral is evaluated.

Sometimes one needs to apply integration by parts more than once, as the next example illustrates.

2. $I = \displaystyle\int x^2 e^x\,dx.$

   Solution: Let $u = x^2,\ dv = e^x\,dx$:

   We do not know immediately how to evaluate $\displaystyle\int xe^x\,dx$. Has the integration by parts failed?

   No, because we have replaced $\displaystyle\int x^2 e^x\,dx$ by the simpler integral $\displaystyle\int xe^x\,dx$.

   Thus continue down the same road by applying integration by parts to this new integral, hoping to simplify it still further: set

   $J = \int xe^x\,dx$. Let $u = x,\ dv = e^x\,dx$, Substituting this formula into I gives

since $-2C$ is again "any constant" and can be written more simply as C.

Integration by parts, which comes from the product rule, is usually applied to integrands that are products of different types of functions. Our three examples above are such products: a polynomial times a log function, a polynomial times a trigonometric function, and a polynomial times an exponential function. But sometimes the product nature of the integrand is not immediately apparent, as in the next example.

3. $I = \int \arccos x \, dx$. Use the fact that $\frac{d}{dx} \arccos(x) = -\frac{1}{\sqrt{1-x^2}}$.

   Solution: The hint should tell us to try $u = \arccos(x)$:

   Now apply the formula

   Now for J, let $w = 1 - x^2$ (function-derivative)

One has to choose u and dv correctly for the method to work. In the last example, if we had taken $u = 1$ and $dv = \arccos(x)\,\mathrm{dx}$, we would have been stuck because to continue we need to know v, and finding this is the same problem as evaluating the original integral, so we cannot proceed further.

Similarly, a poor choice of u and dv can make things worse instead of better. Consider $J = \int xe^x\,dx$. If we set $u = e^x,\ dv = x\,dx$, then $du = e^x\,dx,\ v = x^2/2$, so

$$J = uv - \int v\,du = \frac{1}{2}x^2e^x - \int \frac{1}{2}x^2e^x\,dx.$$

While this equation is true, it is of no help to us since we have replaced the original integral by a more difficult one. If instead we had started by choosing $u = x$ and $dv = e^x$, then integration by parts works.

4. Use *integration by parts* to evaluate $\displaystyle\int_0^{\pi/2} x\sin 2x\,dx$.

Solution: Firstly we will find an anti-derivative by finding

$$I = \int x\sin 2x\,dx,$$

and worry about the limits later. Prompted to use integration by parts we choose $u = x$ by LIATE. Hence we have $dv = \sin 2x\,dx$. We want to use $\displaystyle\int u\,dv = uv - \int v\,du$ so will need v and du. Differentiating u and integrating dv does this for us:

Now we use the formula:

We have that $\displaystyle\int \cos 2x\,dx = \frac{1}{2}\sin 2x$ so we have

$$I = -x\frac{\cos 2x}{2} + \frac{1}{4}\sin 2x.$$

Now we must plug in the limits to evaluate the integral:

where we used the fact that $\cos\pi = -1$, $\sin\pi = 0$ and $\sin 0 = 0$.

5. Use integration by parts to evaluate $\displaystyle\int x\sec^2 x\,dx$.

   Solution: By LIATE choose[1] $u = x$ and $dv = \sec^2 x\,dx$:

Winter 2012: Question 1 (d)

Find the following integral

$$\int 5te^{-3t}\,dt.$$

[5 Marks]

Solution: In the exam we will not be prompted to use integration by parts. In MATH6040, once you see the totality of questions, it should be clear that this requires integration by parts. Otherwise... Direct? Manipulation? Substitution? Integration by Parts in now another technique that you have if none of these work. By LIATE we have no log, no inverse trig, algebraic...

Now putting everything back together:

[1]the aware among you should note that $\sec^2$ is the derivative of tan so should definitely be in the dv as we know how to integrate it

Summer 2012: Question 1 (d)

Find the following integral

$$\int 2x\sin(5x)\,dx.$$

[6 Marks]

Solution: By LIATE we have no log, no inverse trig, algebraic...

Now putting everything back together:

Exercises

1. Find $\int x\cos x\,dx$ and check your solution. **Ans**: $x\sin x+\cos x+C$
2. Find $\int xe^{2x}\,dx$ and check your solution. **Ans**: $\frac{1}{2}xe^{2x}-\frac{1}{4}e^{2x}+C$
3. Evaluate
$$\int_0^{\pi/2} x\cos 2x\,dx$$
Ans: $\frac{\pi}{2}-1$
4. Evaluate
$$\int_1^2 \ln x\,dx,$$
giving your answer in the form $\ln p+q$ where $p,q\in\mathbb{Q}$. **Ans**: $\ln 4-1$
5. Find $\int \ln 2x\,dx$. **Ans**: $x\ln(2x)-x+C$
6. * Integrate $\int \sin^{-1}x\,dx$. **Ans**: $x\sin^{-1}(x)+\sqrt{1-x^2}+C$
7. Integrate $\int x\ln x\,dx$. **Ans**: $\frac{1}{2}x^2\ln x-\frac{1}{4}x^2+C$
8. * Integrate $\int \theta\sec^2\theta\,d\theta$. **Ans**: $\theta\tan(\theta)-\ln|\sec\theta|+C$
9. * Evaluate
$$\int_1^4 \sqrt{t}\,\ln t\,dt.$$
Ans: $\frac{16}{9}\ln 64-\frac{28}{9}$
10. $\displaystyle\int x\sec^2 x\,dx$ Ans: $x\tan x+\ln|\cos x|+C$

11. $\displaystyle\int_0^2 xe^{2x}\,dx$ Ans: $\dfrac{1}{4}(3e^4+1)$

12. * $\displaystyle\int x\arctan x\,dx$ Ans: $\dfrac{1}{2}(x^2+1)\arctan x - \dfrac{x}{2} + C$

13. $\displaystyle\int \frac{\ln x}{x^2}\,dx$ Ans: $-\dfrac{1+\ln x}{x} + C$

14. * $\displaystyle\int \arcsin x\,dx$ Ans: $x\arcsin x + \sqrt{1-x^2} + C$

15. $\displaystyle\int \ln x\,dx$ Ans: $x(\ln x - 1) + C$

16. * $\displaystyle\int \sec^3 x\,dx$ Ans: $\dfrac{1}{2}\sec x\tan x + \dfrac{1}{2}\ln|\sec x + \tan x| + C$

17. $\displaystyle\int x^2\sin x\,dx$ Ans: $(2-x^2)\cos x + 2x\sin x + C$

18. * $\displaystyle\int_1^4 \sec^{-1}\sqrt{x}\,dx$ Ans: $\dfrac{4\pi}{3} - \sqrt{3}$

19. * $\displaystyle\int \frac{x^3}{\sqrt{1-x^2}}\,dx$ Ans: $-x^2\sqrt{1-x^2} - \dfrac{2}{3}(1-x^2)^{3/2} + C$

4.2 Completing the Square

The substitution method replaces complicated indefinite integrals by simpler ones, but one must then be able to evaluate those simpler integrals. This is often done by using a table of standard integrals such as the list in the mathematical tables. It is not necessary to memorize all of these, but one should recognize each one of them if it arises when attempting to integrate some function.

In a table of standard integrals, quadratic expressions always appear in one of the forms $x^2 + a^2$, $x^2 - a^2$ or $a^2 - x^2$, where $a \in \mathbb{R}$ is some constant. These are related to the inverse trigonometric functions.

We use implicit differentiation to find the derivative of $\sin^{-1}(x)$ and $\tan^{-1}(x)$.

4.2.1 Proposition

We have that

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$$

Proof. We start with $y(x) = \sin^{-1}(x)$. This reads,

Therefore we have that $x = \sin(y(x))$. Now differentiate implicitly with respect to x:

We want our derivative to be in terms of x. Note that y is the angle whose sine is x so we can place it is a right-angled triangle as shown:

Now we can use Pythagoras to write $\cos y$ in terms of x:

Therefore $\cos y = \sqrt{1-x^2}$ and we are done.

The derivative of $\tan^{-1}(x)$ is left as an exercise.

These are the formulae that appear in your mathematical tables:

4.2.2 Proposition

We have that

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right)+C$$
$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$$

Proof. We can show this by differentiating the right-hand side using a Chain Rule

The second integral is left as an exercise.

Examples

1. Find $\displaystyle\int_0^{1/4} \frac{1}{\sqrt{1-4x^2}}\,dx$.

   Solution: There are two ways to approach this integral. First note that we can't integrate directly from the tables. Is there a manipulation? If we take out the four from the bottom and manipulate the integrand as follows:

   which we can actually integrate directly:

If we didn't see this manipulation we would try a substitution. Note that $4x^2 = (2x)^2$ so that the integral is

so is like an inverse sine except that it is $2x$ rather than x...

Now put everything back together suppressing the limits for now:

A lot of texts use this second approach but I prefer the second but it is up to you which you prefer.

2. **Summer 2012: Question 1** Evaluate

$$\int_0^{5/2} \frac{1}{4x^2+25}\,dx.$$

[6 Marks]

Solution: Can we do it directly? A manipulation? How about taking out a four from the denomoninator?

Now we can integrate directly:

Alternatively, if we don't see this manipulation we would try a substitution. Note that $4x^2 = (2x)^2$ so that the integral is

so is like an inverse tan except that it is $2x$ rather than x...

Now put everything back together suppressing the limits for now:

If you have a quadratic that is not a sum or difference of two squares (i.e., if an x term appears also) then complete the square — that is write

$$ax^2 + bx + c = \pm(px + q)^2 \pm r^2$$

for some p, q, $r \in \mathbb{R}$, where the $\pm$ depend on whether we will be integrating an integrand with $x^2 + a^2$, $x^2 - a^2$ or $a^2 - x^2$.

Examples

1. Complete the square:

$$4x^2 + 8x + 28$$

Solution: Do what you want to do:

Now equate coefficients:

2. Evaluate

$$\int \sqrt{5 - 4x - x^2}\, dx,$$

given that

$$\int \sqrt{a^2 - u^2}\, du = \frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\arcsin\frac{u}{a} + C, \qquad \text{for } a > 0.$$

Solution: Here we want to make the integrand look like $\sqrt{a^2 - u^2}$ so we begin by completing the square thus:

Now equate coefficients and rewrite the integrand:

Now make the substitution $u = x + 2$:

3. *Evaluate*

$$\int \frac{dx}{\sqrt{15+2x-x^2}}.$$

Solution: There is no direct integration but there is a possible manipulation. There is a quadratic under a square-root in the denominator. Complete the square:

Now equate coefficients:

Now rewrite the integral and note that it looks like the integral of $\arcsin(x)$:

Winter 2012 Question 5(a)

Determine the following integral

$$\int_{-2}^{1} \frac{1}{x^2+6x+16}\,dx$$

[7 Marks]

Solution: Can this integral be done directly? Manipulation? We could try and complete the square:

Now this yields to the substitution $u = x+3$. We have $\frac{du}{dx} = 1$ and so $du = dx$. Putting everything back together and suppressing the limits for now:

This can be integrated directly:

Summer 2012 Question 5(a)

Evaluate

$$\int_{-1}^{1} \frac{1}{x^2+8x+25}\,dx$$

[7 Marks]

Solution:
Can this integral be done directly? Manipulation? We could try and complete the square:

Now this yields to the substitution $u = x+4$. We have $\frac{du}{dx} = 1$ and so $du = dx$. Putting everything back together and suppressing the limits for now:

This can be integrated directly:

Exercises

1. Prove that $\dfrac{d}{dx}\tan^{-1}(x) = \dfrac{1}{1+x^2}$.
2. Prove that $\displaystyle\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$.
3. Find/evaluate the integral
   (a) $\displaystyle\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{6}{\sqrt{1-t^2}}\,dt$.
   (b) $\displaystyle\int_0^1 \frac{4}{t^2+1}\,dt$.
   (c) $\displaystyle\int_0^{\frac{\sqrt{3}}{4}} \frac{dx}{1+16x^2}$.
   (d) $\displaystyle\int \frac{dt}{\sqrt{1-4t^2}}$.

Q. 1-3: Complete the Square. Q. 4-6 Evaluate the integral.

1. x^2+x+1.
2. $-x^2+5x-2$.
3. $3x^2-2x+4$.
4. $\displaystyle\int \frac{dx}{\sqrt{15+2x-x^2}}$ Ans: $\arcsin\left(\dfrac{x-1}{4}\right)+C$
5. $\displaystyle\int \frac{dx}{x^2-x+2}$ Ans: $\dfrac{2}{\sqrt{7}}\arctan\left(\dfrac{2x-1}{\sqrt{7}}\right)+C$
6. $\displaystyle\int \frac{dx}{3x^2-2x+5}$ Ans: $\dfrac{1}{\sqrt{14}}\arctan\left(\dfrac{3x-1}{\sqrt{14}}\right)+C$

More exercises, as always, to be found in past exam papers.

4.3 Work

4.3.1 Work done by a Force

We saw in MATH6015 that if we have a force $F(x)$ that is given as a function of position, that the work done by the force in moving the object from $x = a$ to $x = b$ is given by

$$W = \int_a^b F(x)\,dx. \tag{4.1}$$

This is also examinable in MATH6040.

Examples

1. **Winter 2012 Question 5 (c)**
   A force F kN acting on an object is described by

   $$F = \frac{24}{9+4x^2}.$$

   where x m is the displacement of the object from a point O. The object is moved from O to a point 1.5 m from O. Determine the work done stating the units.

   [7 Marks]

   Solution: We have that

   $$W = \int_0^{3/2} \frac{24}{9+4x^2}\,dx.$$

   To integrate we can either write the denominator as $3^2+(2x)^2$ and use the substitution $u = 2x$ or else manipulate by dividing above and below by four as follows:

   However this is exactly an inverse tan with $a = \frac{3}{2}$:

   The units are kN m or kJ.

2. The force acting on an object is given by

   $$F = \frac{12}{4x^2+9}\,\text{N}$$

   where x is the displacement of the object in metres. Find the work done in moving the object R m from the origin. Hence show that

   $$\lim_{R\to\infty} \text{Work}$$

   is finite.

Solution: We have that the work is given by

$$W = \int_0^R \frac{12}{4x^2+9}\,dx.$$

We can integrate by either writing the bottom as $(2x)^2 + 3^2$ and making the substitution $u = 2x$ or manipulating by dividing above and below by four:

where the units are N m or Joules.

Now we find the work done as $R \to \infty$:

which is finite.

Exercises

1. A particle is moved along the x-axis by a force that measures $\dfrac{10}{1+x^2}$ N at a point x m from the origin. Find the work done in moving the particle from the origin to a distance of 9 m. **Ans**: $10\tan^{-1}(9)$ J ≈ 14.601 J.

2. A particle is moved along the x-axis by the following forces; that are given at a point x m from the origin and are measured in kN. Find the work done in moving a particle from the origin to a distance of 2.5 m in each case:

$$F_1 = \frac{4}{x^2+1},\ F_2 = \frac{1}{1+16x^2},\ F_3 = \frac{1}{x^2-2x+5}$$

Ans: $4\tan^{-1}(3/2)$ kJ ≈ 3.931 kJ, $\frac{1}{4}\tan^{-1}(6)$ kJ ≈ 0.351 kJ, $\frac{1}{4}\tan^{-1}(84/13)$ kJ ≈ 0.354 kJ
In the case of forces F_1 and F_2 find the work done in moving an object from the origin to a point arbitrarily far from the origin. **Ans**: 2π, $\dfrac{\pi}{8}$

4.3.2 Work done by an Expanding Gas

Consider the work done by a gas expanding from a volume of V to a volume of $V + dV$:

Now we have that pressure, $P = \dfrac{F}{A}$ and hence that $F = PA$. Now whilst the area doesn't change during the expansion the pressure could. Suppose that initially the pressure is P and after the expansion the pressure is $P + dP$. Now we want to find the work done over the distance dx. We can approximate the force as the average of the force before and after the expansion:

Now the work done is given by

However note that $A\,dx$ is just the change in volume, dV so we have that the work done by a gas in expanding from a volume V_1 to V_2 is given by

$$W = \int_{V_1}^{V_2} P\,dV. \tag{4.2}$$

Summer 2012 Question 5 (b)

A gas expands according to the relation

$$PV^2 = 1200.$$

The initial volume of the gas is 0.1 m^3. Find the work done by the gas as it expands to a volume of 0.2 m^2.

Solution: We first find P:

Now we integrate with respect to volume from 0.1 to 0.2 m^2:

The units of work are Joules.

Exercises:

1. A gas expands according to the relation

$$PV^2 = 600.$$

   The initial volume of the gas is 0.2 m^3. Find the work done by the gas as it expands to a volume of 0.4 m^3. **Ans:** 1.5 kJ

2. A gas expands according to the relation

$$PV^3 = 800.$$

   The initial volume of the gas is 0.1 m^3. Find the work done by the gas as it expands to a volume of 0.5 m^3. **Ans:** 38.4 kJ