MS 3011 - Dynamical Systems

J.P. McCarthy

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Introduction

Lecturer:

J.P. McCarthy

Office:

Mathematics Research, Western Gateway Building
Meetings by appointment via email only.

Email & Web:

jippo@campus.ie and *http://irishjip.wordpress.com*
This page will comprise the webpage for this module and as such shall be the venue for course announcements including definitive dates for the test and the homework. This page shall also house resources such as supplementary material. Please note that not all items here are relevant to MS 3011; only those in the category 'MS 3011'. Feel free to use the comment function therein as a point of contact.

Module Objective:

To provide an overview of dynamical systems and their applications.

Module Content:

Discrete time systems, fixed point and stability analysis, complex dynamics, applications.

Assessment:

Total Marks 100: End of Year Written Examination 75 marks; Continuous Assessment 25 marks.

Continuous Assessment:

The continuous assessment consists of a test and a homework. The test will take place on or about February 20 2012. You will also be given a homework. The final date for submission is 24 April 2012.
The webpage will contain the latest and definitive information about these.
Absence from a test will not be considered unless accompanied by a reasonable excuse (requiring medical cert or similar), in which case special arrangements will come into force.
The marks you obtain for the continuous assessment will be carried forward to the Autumn exam.

Lectures:

Monday 10-11, WGB G 18; Wednesday 10-11 WGB G 14.

Tutorials:

There will be a weekly tutorial starting in the third week of term (19/01/12): Thursday 11-12 ORB 123. Please email me if this time clashes with another lecture, etc. There are many ways to learn maths. Two methods which aren't going to work are

1. *reading your notes and hoping it will all sink in*
2. *learning off a few key examples, solutions, etc.*

By far and away the best way to learn maths is by *doing exercises*, and there are two main reasons for this. The best way to learn a mathematical fact/ theorem/ etc. is by using it in an exercise. Also the doing of maths is a skill as much as anything and requires practise. After the summer lay off you may find yourself rusty in terms of algebra. Regularly doing exercises will eliminate small slips and mistakes.
There is no shortage of exercises for you to try. Past exam papers are fair game. Also during lectures there will be some things that will be *left as an exercise.* How much time you can or should devote to doing exercises is a matter of personal taste, however tutorials will be far more productive for both you and I if you have at least attempted some exercises.
The format of tutorials is that those of you who have questions shall have them answered by me. No secrets will be divulged at tutorials and they are primarily for students who have questions about exercises. More general questions on course material shall be answered also. If there are no questions you shall be asked to do some exercises in class. Feel free at this point to put your hand up for some one-to-one attention.

THINK OF TUTORIALS AS FREE GRINDS!

Reading:

Your primary study material shall be the material presented in the lectures. Exercises done in tutorials may comprise further worked examples. The webpage may contain supplementary material, and contains links and pieces about topics that are at or beyond the scope of the course. Finally the internet provides yet another resource.

Exam:

The exam will have four questions of which three must be answered.

Motivation: What are Dynamical Systems?

> *Mathematics is facts; just as houses are made of stones, so is mathematics made of facts; but a pile of stones is not a house and a collection of facts is not mathematics.*
>
> Henri Poincaré

A dynamical system describes the evolution of a 'system' over time. A set S comprises the possible *states* of the system. In a *discrete* dynamical system the times are $i = 0$ (the initial state), $1, 2, 3, \ldots$. The system changes has a state *evolution* between times i and $i+1$. The evolution is *Markov*. This means that the state of the system at time i, s_i, only depends on the state of the system at the previous time, s_{i-1}:

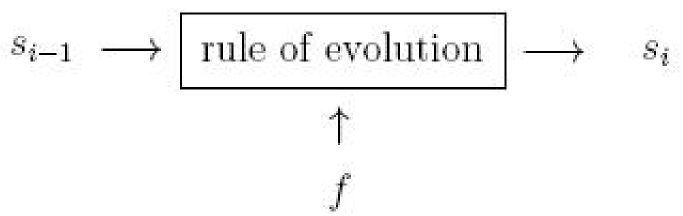


Figure 1: If the state of the system at time i is determined fully by the state at time $i-1$, then the dynamical system can be realised by repeated iteration of a function — the *iterator function*; i.e. suppose that system is such that the state 1 always evolves to the state 2 then we can say that the iterator function is such that $f(1) = 2$.

Real-world examples of dynamical systems include savings schemes and population growth models.

Definition

Let S be a set and $f : S \to S$. Define inductively

$$x_{n+1} = f(x_n)\ ,\ \ n \in \mathbb{N},$$

for some choice of $x_0 \in S$. Then (S, f) is a *dynamical system* with *state space* S, *initial state* x_0 and *iterator function* f. The *state of the dynamical system at time* n is given by x_n.

Note that we will nearly always have S as a subset of the real numbers, $\mathbb{R}$.

There are a number of natural questions to ask about dynamical systems:

- Can we find an expression for x_n?
- Does the system exhibit limiting behaviour? Does it tend to zero? To infinity? To a constant?
- Does the system exhibit periodic (repeating) behaviour?
- For which choices of x_0 is the dynamical system fixed?
- For which choices of x_0 is the behaviour eventually periodic or limiting? Or does chaotic behaviour exist?
- Suppose x is a fixed point. If we disturb $x \to x + \varepsilon$; does the system return to x or move away from x (stability)?
- Does a small change in initial conditions signal a large change in longtime behaviour? This is the famous *butterfly effect* of chaos theory.

Discrete Dynamical Systems

We study processes that evolve in time in which the changes occur at specific times, rather that continuously. The study of discrete dynamical systems often involves the process of *iteration*. To iterate means to repeat a procedure numerous times. In dynamics the process that is repeated is the application of a mathematical function. We begin with two examples.

1 Discrete Growth Models

1.1 Savings Scheme

Invest €1,000 in a savings scheme which pays 4% interest per year, compounded annually.

Let A_n be the amount in the account after n years. Then

$$A_0 = 1000$$
$$A_1 = 1000 + (0.04)1000 = 1000(1.04) = A_0(1.04)$$

$$A_2 = A_1 + (0.04)A_1 = A_1(1.04) = A_0(1.04)^2$$

$$A_3 = A_2 + (0.04)A_2 = A_2(1.04) = A_0(1.04)^3$$

and, by induction, one can show that $A_n = A_0(1.04)^n$.

So we can determine the balance at any time in the future.

The two ingredients required were

- **initial state** i.e. value of A_0,
- **rule of evaluation** $\quad A_{n+1} = A_n(1.04)$

Write this rule in the form

$$f(n+1) = f(n)(1.04)$$

then $f(n)$ in the **iterator function**.

If, in general, x_n denotes the state at time n, then

$$x_{n+1} = f(x_n).$$

This last equation is the associated *difference equation*.

1.2 Population Growth

Assume population size in a given generation is directly proportional to the population size in the previous generation

So

$$P_{n+1} = k\,P_n$$

where k is the proportionality constant that determines the growth rate.

If initial size (state) is P_0, then

$$P_1 = k\,P_0$$
$$P_2 = k\,P_1 = k^2\,P_0$$
$$P_3 = k\,P_2 = k^3\,P_0,$$

and, by induction, $P_n = k^n\,P_0$.

Then the iterator function is $f(x) = k\,x$ and the difference equation is

$$x_{n+1} = f(x_n) = k\,x_n.$$

We consider 3 cases

- $k = 1$.
- $k > 1$.
- $k < 1$.

Case $\mathbf{k = 1}$

$$P_n = k^n\,P_0 = P_0,$$

so population size remains constant.

Case $\mathbf{k > 1}$

$$P_n = k^n\,P_0.$$

Since $k^n \longrightarrow \infty$ as $n \longrightarrow \infty$, then we have a population of *unbounded growth.*

Case $\mathbf{k < 1}$

$$P_n = k^n\,P_0.$$

Now $k^n \longrightarrow 0$ as $n \longrightarrow \infty$, so population decays.

In all three cases, the iteration function is

$$f(x) = k\,x$$

and $x_{n+1} = k\,x_n$ is the difference equation.

1.3 Logistic Equation

To make population growth more realistic, we add the following assumptions.

- When the population size is very small, then the population size at the end of next generation is proportional to the population size at the end of the current generation.

- If population is too large, then all the resources will be used and the entire population will die out in the next generation.

The second assumption is that there is a maximum level M that when reached, the result is extinction. The number M is called the annihilator parameter.

A mathematical model (equation) that reflects these assumptions is

$$P_{n+1} = k\,P_n(1 - P_n/M).$$

Then if P_n is very small, $P_{n+1} \approx k\,P_n$ and if P_n is close to M, then $1 - P_n/M \approx 0$, so population becomes extinct.

Rather than deal with large numbers in a population, we take P_n as the proportion (or fraction) of the maximum population size. Thus we assume that $M = 1$ and that $0 < P_n < 1$. Then the model equation becomes

$$P_{n+1} = k\,P_n\,(1 - P_n).$$

The iteration function is

$$f(x) = k\,x\,(1 - x),$$

and the difference equation is

$$x_{n+1} = k\,x_n\,(1 - x_n).$$

This is the **logistic difference equation.**

1.4 Some predictions of the model

Suppose we begin with a population that is exactly half the maximum population allowed, that is, $P_0 = 0.5$. Then depending on the value of k, we find very different results when we compute successive values of P_n.

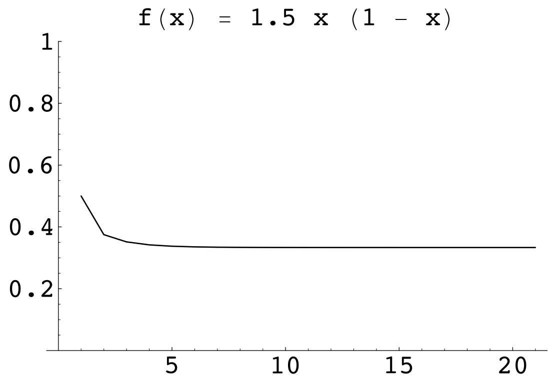
Case $\mathbf{k = 0.5}$. The successive values of P_n are

$$\{0.5,\ 0.125,\ 0.0546,\ 0.0258,\ 0.0125,\ 0.0062,\ 0.0030,\ 0.0015,\ \ldots\}.$$

So the population tends to extinction.

Case $\mathbf{k = 1.5}$. The successive values of P_n are

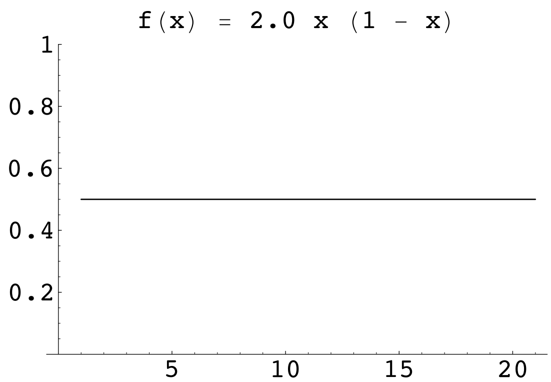
$$\{0.5,\ 0.3750,\ 0.3515,\ 0.3419,\ 0.3375,\ 0.3354,\ 0.3343, 0.3338,\ 0.3335, 0.3334, 0.3333,\ \ldots,\ 0.3333, \ldots, \}.$$



So the population tends to an equilibrium state.

Case $\mathbf{k = 2.0}$. The successive values of P_n are

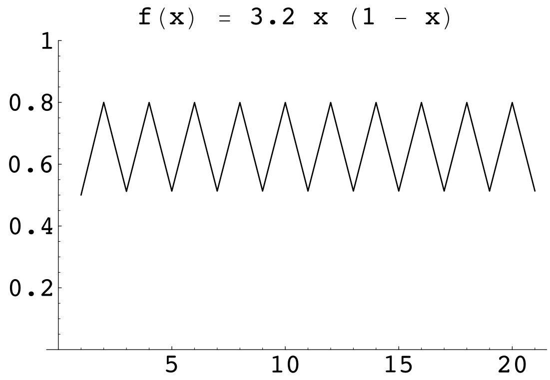
$$\{0.5,\ 0.5,\ 0.5,\ 0.5,\ \ldots, \}.$$



So there is no change in the size of the population.

Case **k = 3.2**. The successive values of P_n are

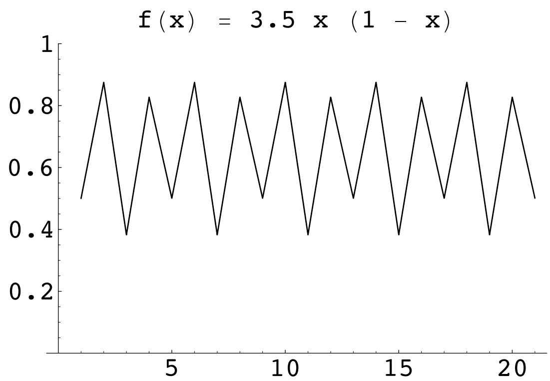
$$\{0.5,\ 0.8000,\ 0.5120,\ 0.7995,\ 0.5128,\ 0.7995,\ 0.5130, 0.7995,\ 5130,\ 0.7995,\ \ldots,\}.$$



The population size oscillates back and forth between two different values.

Case **k = 3.5**. The successive values of P_n are

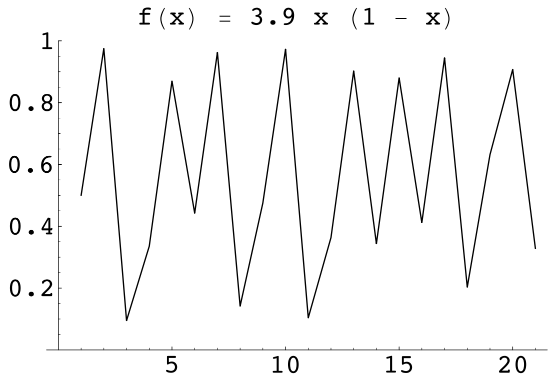
$$\{0.5,\ 0.8750,\ 0.3828,\ 0.8269,\ 0.5008,\ 0.8749,\ 0.3828,\ 0.8269,\ 0.5008,\ 0.8749,\ 0.3828,$$
$$0.8269,\ 0.5008,\ 0.8749,\ 0.3828,\ 0.8269,\ 0.5008,\ 0.8749,\ 0.3828,\ 0.8269,\ 0.5008,\ldots\}.$$



The population size oscillates back and forth between four different values.

Case **k = 3.9**. The successive values of P_n are

$$\{0.5,\ 0.9750,\ 0.0950,\ 0.3355,\ 0.8694,\ 0.4426,\ 0.9621, 0.1419,\ 0.4750,\ 0.9725,$$
$$0.1040,\ 0.3634,\ 0.9022,\ 0.3438,\ 0.8799,\ 0.4120,\ 0.9448,\ 0.2033,\ 0.6316,\ 0.9073,\ 3278,\ldots\}.$$



There is no apparent pattern to the successive population sizes.

EXAMPLE

Consider the difference equation

$$x_{n+1} = \sqrt{x_n}.$$

If $x_0 = 64$, then

$$x_1 = 8, \qquad x_2 = \sqrt{8}, \qquad x_3 = \sqrt{\sqrt{8}}, \ldots.$$

Find a formula for x_n and show that $x_n \longrightarrow 1$ as $n \longrightarrow \infty$.

$$\begin{aligned} x_1 &= 64^{1/2} \\ x_2 &= (64^{1/2})^{1/2} = 64^{1/4} = 64^{1/2^2} \\ x_3 &= (64^{1/2^2})^{1/2} = 64^{1/2^3} \end{aligned}$$

so $x_n = 64^{1/2^n}$.

As $n \longrightarrow \infty$, $1/2^n \longrightarrow 0$ so $x_n = 64^{1/2^n} \longrightarrow 64^0 = 1$.

EXAMPLE

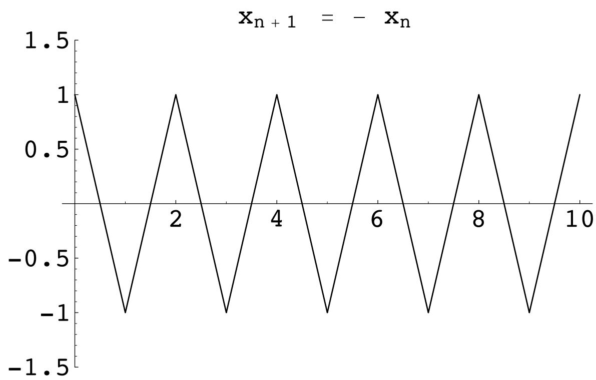
Find the solution of the the difference equation

$$x_{n+1} = -x_n \qquad n = 0, 1, 2, \ldots,$$

with $x_0 = 1$. Sketch the graph of the solution and describe its behaviour.

$$\begin{aligned} x_0 &= 1 \\ x_1 &= -x_0 = -1 \\ x_2 &= -x_1 = -(-1) = +1 \\ x_3 &= -x_2 = -1, \end{aligned}$$

so solution is $\{+1, -1, +1, -1, \ldots\}$.



Values oscillate between $+1$ and -1.

2 Iteration

If $x_{n+1} = f(x_n)$ is a difference equation, then a solution $\{x_0, x_1, x_2, \ldots\}$ consists of a sequence of values got from the initial value x_0 and then iterating the function f to get successively $x_1, x_2, x_3, \ldots$.

EXAMPLE

Let $f : \mathbf{R} \longrightarrow \mathbf{R}$ be given by $f(x) = 2x + 1$. Find the first 4 iterates of $x_0 = 0$ under f. Find a formula for the n^{th} iterate $f^n(0)$.

$$\begin{aligned}
x_0 &= 0 = & 2^0 - 1 \\
x_1 &= f(0) = \ 1 = & 2^1 - 1 \\
x_2 &= f(1) = \ 3 = \ 4 - 1 = & 2^2 - 1 \\
x_3 &= f(3) = \ 7 = \ 8 - 1 = & 2^3 - 1 \\
x_4 &= f(7) = \ 15 = \ 16 - 1 = & 2^4 - 1.
\end{aligned}$$

Thus it looks like

$$x_n = f^n(0) = 2^n - 1.$$

Prove by induction that this is true.

EXAMPLE

Decide whether or not there is an iterator mapping for the sequence

$$\{1, 1/2, 1/3, 1/4, 1/5, \ldots\}.$$

Solution.

We want a function $f(x)$ such that $x_{n+1} = f(x_n)$.
We have $x_n = 1/n$ and $x_{n+1} = 1/n+1$. Then

$$\begin{aligned} f(1/n) &= \frac{1}{n+1} \\ &= \frac{1/n}{1+1/n}. \end{aligned}$$

So take $f(x) = \frac{x}{1+x}$.

EXAMPLE

Decide whether or not there is an iterator function for the sequence

$$\{1, 2, 3, 1, 3, 2, 1, 2, 3, \ldots\}.$$

Solution. Since $x_0 = 1, \quad x_3 = 1$, then $x_1 = f(x_0)$ and $x_4 = f(x_3)$ should have the same value. But $x_1 = 2$ and $x_4 = 3$ so there is no iterator function.

2.1 Orbits

Given an iterator mapping

$$f : \mathbf{S} \longrightarrow \mathbf{S},$$

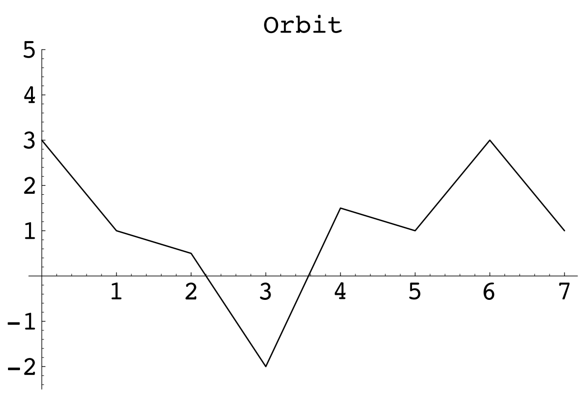
where $\mathbf{S}$ is a set and $x_0 \in S$, iteration produces a sequence

$$\{x_0, \; x_1, \; x_2, \; \ldots\}$$

of elements of S. The sequence of iterates is called the **orbit** of x_0 under f.
So the orbit is a mapping

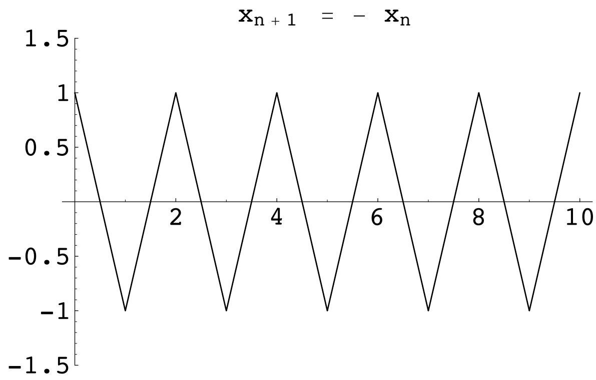
$$n \longrightarrow x_n$$

where n is a non-negative integer. We can then graph the orbit as follows: plot the points $\{(n, x_n) : n = 0, 1, \ldots\}$ and join the successive points (n, x_n) and $(n+1, x_{n+1})$.



EXAMPLE

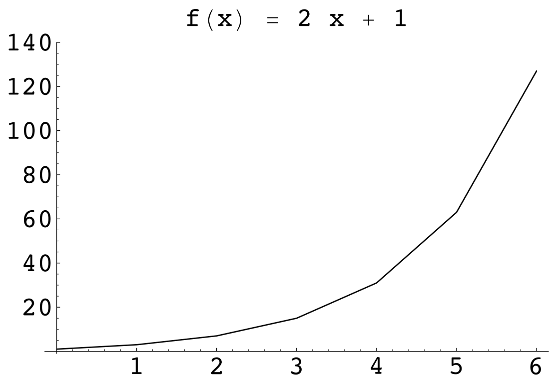
Plot the graph of the difference equation $x_{n+1} = -x_n$ with $x_0 = 1$.
Sequence is $\{1, -1, +1, -1, \ldots\}$



EXAMPLE

Let $f(x) = 2x + 1$. Find the orbit of $x_0 = 1.$ under f and sketch the graph.

Orbit is $\{1,\ 3,\ 7,\ 15,\ 31,\ \ldots\}$.



The process of repeatedly applying the function f can be expressed in terms of compositions of functions.

$$f^2 = f \circ f \quad \text{that is} \quad f^2(x) = f(f(x)),$$

$$f^3 = f \circ f \circ f \quad \text{that is} \quad f^3(x) = f(f(f(x))).$$

Thus

$$\begin{aligned} x_1 &= f(x_0) \\ x_2 &= f^2(x_0) \\ x_3 &= f^3(x_0) \\ x_n &= f^n(x_0) \end{aligned}$$

Thus the orbit can also be written as

$$\{x_0,\ f(x_0),\ f^2(x_0),\ f^3(x_0), \ldots\}.$$

EXAMPLE

Find f^2 if $f(x) = 2x + 1$.

we have

$$f^2(x) = f(f(x)) = f(2x+1) = 2(2x+1) + 1 = 4x + 3.$$

EXERCISE
Show that

(i) $f^3(x) = 8x + 7$.

(ii) Find a formula for $f^n(x)$.

2.2 Periodic orbits

Let $\mathbf{S}$ be a set and $f : \mathbf{S} \longrightarrow \mathbf{S}$ a mapping. An element $x \in \mathbf{S}$ is a **fixed point** of f if $f(x) = x$.
So the orbit of a fixed point x_0 is

$$\{x_0, x_0, x_0, x_0, x_0, \ldots\}.$$

Fixed points are obtained by solving the equation

$$x = f(x).$$

EXAMPLE
Find the fixed points of

$$f(x) = x^2 - 2x - 4.$$

Solution: Solve $x = f(x)$, that is

$$\begin{aligned} x^2 - 2x - 4 &= x, \\ x^2 - 3x - 4 &= 0. \end{aligned}$$

The fixed points are $x = 4$ and $x = -1$.

A point $x \in \mathbf{S}$ is a periodic point of $f : \mathbf{S} \longrightarrow \mathbf{S}$ if $f^n(x) = x$ for some positive integer n. When this happens, the integer n is called a **period** of x. The smallest such positive integer is called the **prime period**. A point of period n is called a **period-n point**.

So a period-n point is a fixed point of f^n.

A period-1 point is a fixed point.

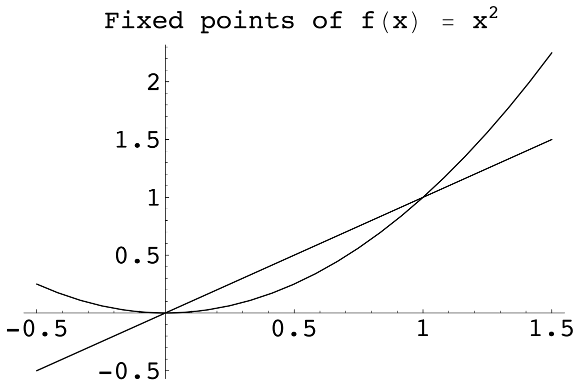
A period-1 point is a period-n point for every positive integer.

EXAMPLE

Find the fixed points of

$$f(x) = x^2.$$

Solution: Solve $f(x) = x$, that is $x^2 = x$. Get $x = 0$ and $x = 1$.



Note that the fixed points can be found graphically by finding the x-coordinate of the points where the graphs of $y = f(x)$ and $y = x$ intersect.

EXAMPLE

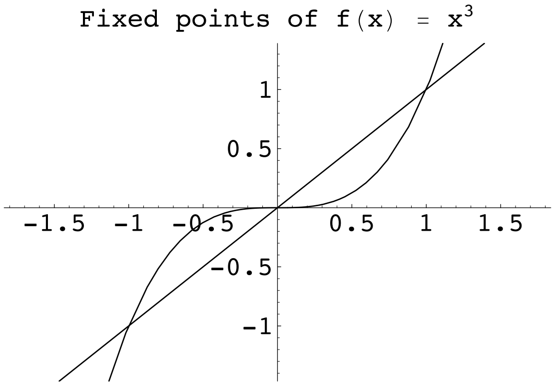
Find the fixed points and the period-2 points of the function

$$f(x) = x^3.$$

Solution: For fixed points,

$$x^3 = x,$$

$$x(x^2 - 1) = 0.$$



So $x = 0,\ +1,\ -1,$ are the fixed points.

Period-2 points are fixed points of $f^2(x) = x$. So

$$f^2(x) = f(f(x)) = f(x^3) = (x^3)^3 = x^9.$$

Period-2 points satisfy $x^9 - x = 0$.

$$\begin{aligned} x(x^8-1) &= 0 \\ x(x^2-1)(x^2+1)(x^4+1) &= 0 \\ x(x-1)(x+1)(x^2+1)(x^4+1) &= 0. \end{aligned}$$

The only period-2 points are $0,\ +1,\ -1$. (They are also fixed points.)

EXAMPLE

Let $\{x_0,\ x_1,\ x_2\ \}$ be a set of 3 distinct points in the domain of f. Suppose f maps this set to itself and that f has no points of period 2. What are the possible orbits of x_0?

Solution

$$\{x_0,\ x_1,\ x_2,\ x_0,\ x_1,\ x_2,\ x_0,\ x_1,\ x_2, \ldots\}.$$

$$\{x_0,\ x_2,\ x_1,\ x_0,\ x_2,\ x_1,\ x_0,\ x_2,\ x_1,\ \ldots\}.$$

EXAMPLE

Suppose that x_0 is a periodic point of $f : \mathbf{S} \longrightarrow \mathbf{S}$ with prime period p. Suppose that x_0 is also a period-n point of f. Show that n is an integer multiple of p.

Solution: We have $x_0 = f^n(x_0)$. By the division algorithm, $n = pq + r$ where $0 \leq r < p$. Then

$$x_0 = f^n(x_0) = f^r(f^{pq}(x_0) = f^r(x_0).$$

If $r > 0$, we have a contradiction as p is a prime period and $r < p$. Thus $r = 0$ and so $n = pq$ as required.

EXAMPLE

Let x_0 be a periodic-n point. Show that each of the points in the orbit of x_0 also has period n.

Solution: Let

$$\{x_0,\ x_1,\ x_2,\ \dots,\ x_{n-1},\ x_n,\ x_{n+1},\ \dots\}$$

be the orbit of x_0. Consider x_j for $0 < j \leq n-1$. Claim that $f^n(x_j) = x_j$.

We have

$$f^n(x_j) = f^n(f^j(x_0)) = f^{n+j}(x_0) = f^j(f^n(x_0)) = f^j(x_0) = x_j,$$

as required.

EXAMPLE

Let $f : \mathbf{R} \longrightarrow \mathbf{R}$ be given by

$$f(x) = x^2 - 2.$$

(a) Find the period-1 points of f.

(b) Find the period-2 points of f.

(c) Give 2 different orbits of prime period 2.

Solution:

(a) Solve $x = f(x) = x^2 - 2$ to get the fixed points. Get $x = 2,\ x = -1$.

(b) We have

$$f^2(x) = f(x^2-2) = (x^2-2)^2 - 2 = x^4 - 4x^2 + 2$$

so period-2 points satisfy $x = x^4 - 4x^2 + 2$ that is $x^4 - 4x^2 - x + 2 = 0$.

By part (a), $x = 2$ and $x = -1$ are solutions so $(x-2)(x+1)$ divides $x^4 - 4x^2 - x + 2 = (x^2 - x - 2)q(x)$ where $q(x)$ is of the form $q(x) = ax^2 + bx + c$ for some real numbers $a,\ b,\ c$. Equate coefficients to get $a = 1,\ b = 1,\ c = -1$.

Thus $q(x) = x^2 + x - 1$. The remaining 2 solutions are then

$$\alpha = \frac{-1-\sqrt{5}}{2}, \quad \text{and} \quad \beta = \frac{-1+\sqrt{5}}{2}.$$

So the full set of period-2 points is

$$-1, \quad 2, \quad \alpha, \quad \text{and} \quad \beta.$$

Note that $f(\alpha) = \beta$ and $f(\beta) = \alpha$

(c) The prime period-2 orbits are

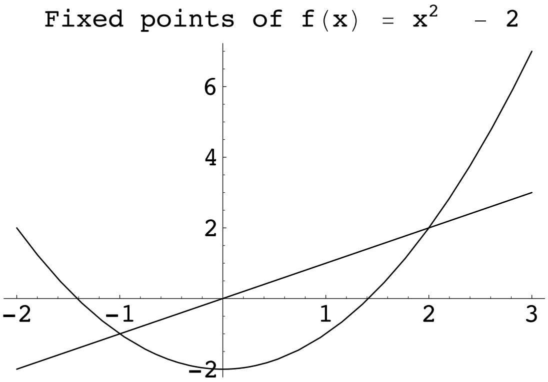
$$\{\alpha,\ \beta,\ \alpha,\ \beta,\ \alpha,\ \beta, \alpha,\ \beta, \dots\},$$

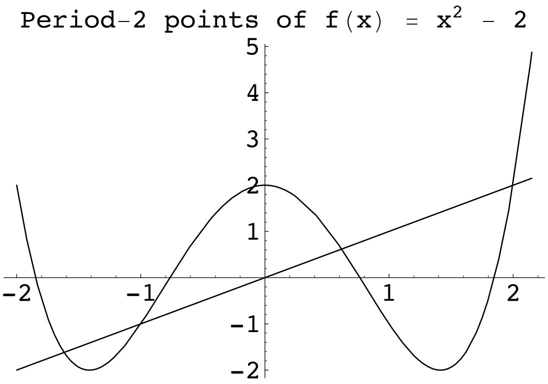
and

$$\{\beta,\ \alpha,\ \beta,\ \alpha,\ \beta.\ \alpha,\ \beta,\ \alpha, \dots\}.$$

Period-2 points are given by the intersections of the graphs of $y = f^2(x)$ and $y = x$.

EXAMPLE
Find the period-1 and period-2 points of $f(x) = x^2 - 2$ by graphing $f(x)$ and $f^2(x)$.





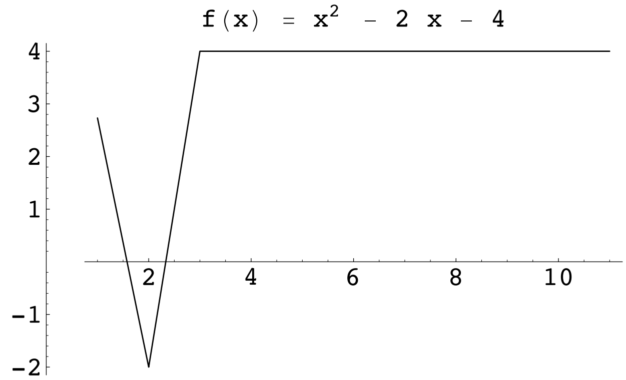
2.3 Eventually Periodic Points

Let $f : \mathbf{S} \longrightarrow \mathbf{S}$ be a function. A point $x_0 \in \mathbf{S}$ is said to be eventually periodic if some iterate x_k of x_0 is a periodic point of f.

EXAMPLE
Show that $1+\sqrt{3}$ is an eventually periodic point of $f(x) = x^2 - 2x - 4$. Graph the orbit.
Solution:

$$\begin{aligned} f(1+\sqrt{3}) &= \; -2 \\ f(-2) &= 4 \\ f(4) &= 4. \end{aligned}$$

Then $1-\sqrt{3}$ is eventually fixed (or period-1).



EXAMPLE

Let $f(x) = 4x(1-x)$. Show that $x = 0.5$ is eventually fixed.

Solution: We have $f(0.5) = 1, \quad f(1) = 0, \quad f(0) = 0.$ So orbit is

$$\{0.5,\ 1,\ 0,\ 0,\ 0,\ \ldots\}.$$

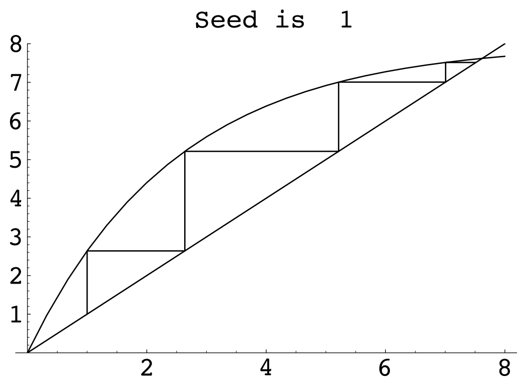
2.4 Cobweb Diagrams

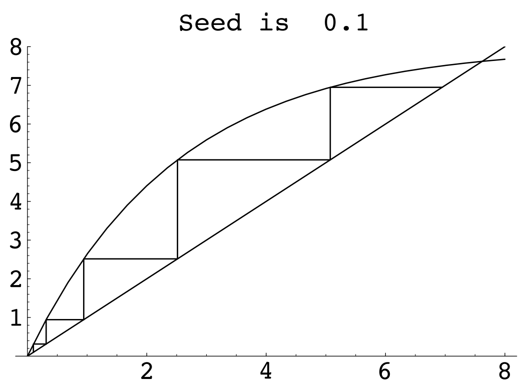
Let $f : \mathbf{S} \longrightarrow \mathbf{S}$ where $\mathbf{S}$ is an interval of $\mathbf{R}$. Given the graph of f, how can we produce the orbit

$$\{x_0, x_1, x_2, \ldots, \quad \}?$$

It is done by drawing a **cobweb diagram**. The steps are as follows.

- Plot the point (x_0, x_0) on the graph of $y = x$.
- From this point on the graph of $y = x$ move in a vertical direction to the graph of $y = f(x)$ to reach the point
$$(x_0,\ f(x_0)\,) = (x_0,\ x_1\,).$$
- From this point on the graph of f move in a horizontal direction to the graph of $y = x$ to reach the point $(x_1,\ x_1\,)$.
- Now repeat the process.





In studying the dynamics of a mapping we are interested in such questions as the following.

- What is the long-term behaviour of the orbits of the mapping?

- What is their ultimate fate?
- Do they keep close to the point where they started or do they move away?
- Does an orbit get close to a periodic orbit?
- Is it eventually periodic?

To express our answers correctly, we use the language of sequences.

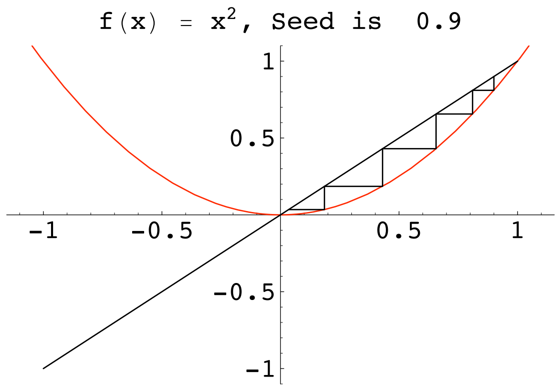
We say that a property of a sequence $\{x_0,\ x_1,\ x_2,\ \ldots\ \}$ holds *eventually* if for some $n \geq 0,$ the property holds for the sequence $\{x_n,\ x_{n+1},\ x_{n+2},\ \ldots,\quad\}$.

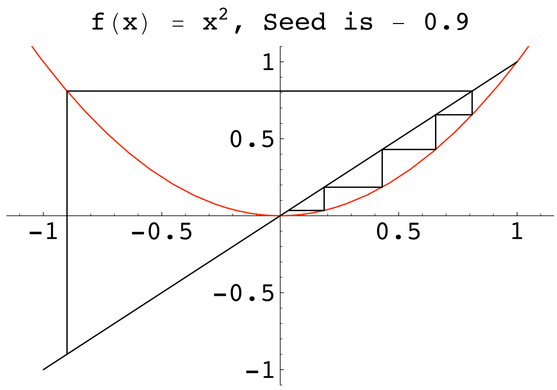
The sequence is said to converge to a limit L if for every $\varepsilon\ >\ 0,$ the elements of the sequence are all eventually within a distance ε of L. We then write

$$\mathrm{Lim}_{n\longrightarrow\infty}\, x_n = L.$$

EXAMPLE

Let $f(x) = x^2$. Use a cobweb diagram to find all points (seeds) x_0 such that the orbit of x_0 converges to 0.





The set $\{x_0 :\ \text{orbit of } x_0 \text{ converges to } 0\}$ is the set $(-1,\ 1)$.

3 Hyperbolic Behaviour

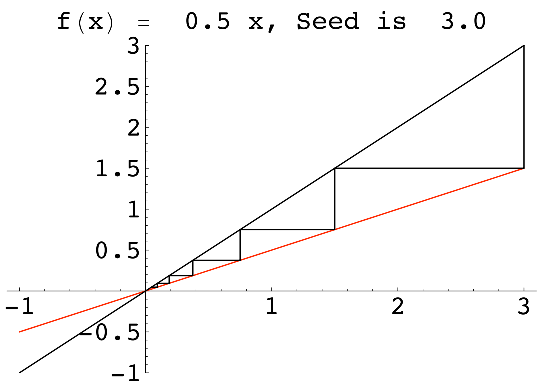
In this chapter we will study the hyperbolic behaviour of fixed and periodic points. That is, we will determine when they are attracting or repelling. We begin with the simplest case of linear functions and affine functions and then apply the results to tangent functions at a point of a differentiable functions.

3.1 Linear Mappings

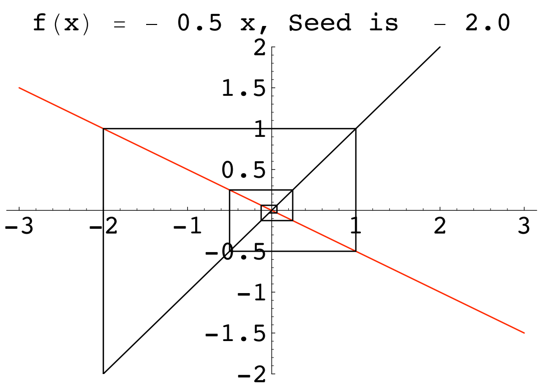
A mapping $f : \mathbf{R} \longrightarrow \mathbf{R}$ is **linear** if for some $a \in \mathbf{R}$

$$f(x) = a\,x.$$

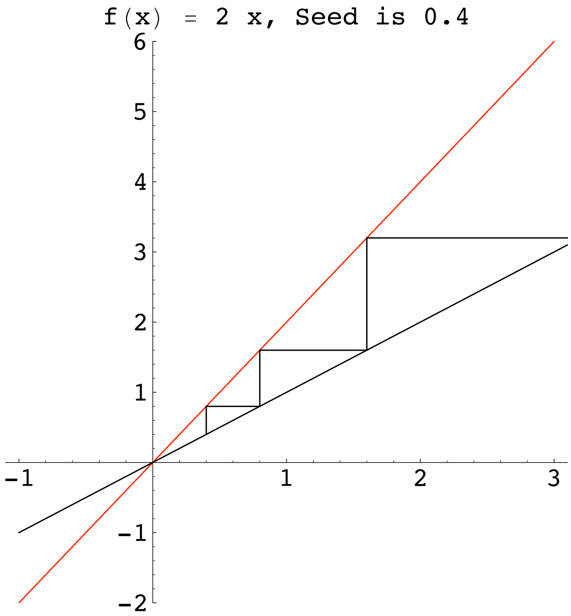
The graph is a straight line through the origin with slope a. Clearly the value $x = 0$ is a fixed point.



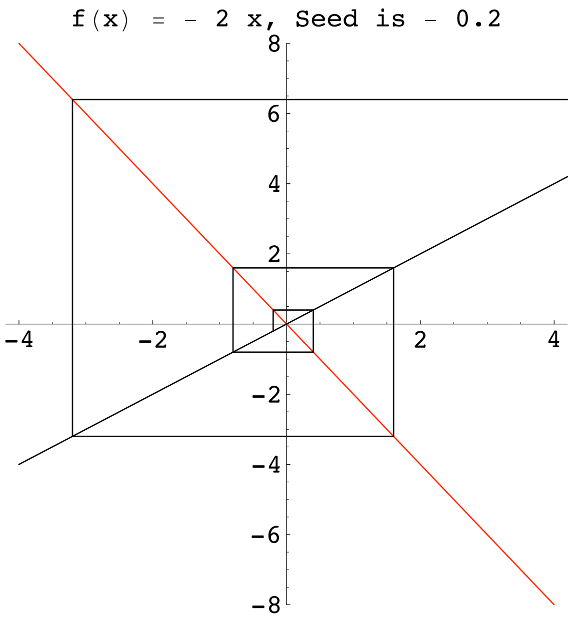
Sequence converges to $x = 0$. Typically the case if $0 < a < 1$.



Sequence converges to $x = 0$. Typically the case if $-1 < a < 0$.



Sequence diverges to ∞. Typically the case if $1 < a$.



Sequence diverges to $-\infty$. Typically the case if $a < -1$.

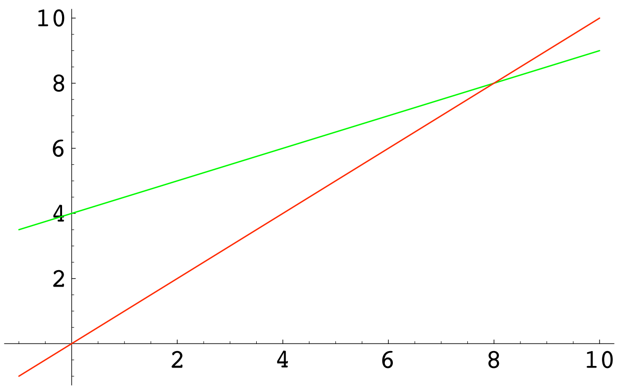
3.2 Affine mappings

A mapping $f : \mathbf{R} \longrightarrow \mathbf{R}$ is called **affine** if for some a and $b \in \mathbf{R}$

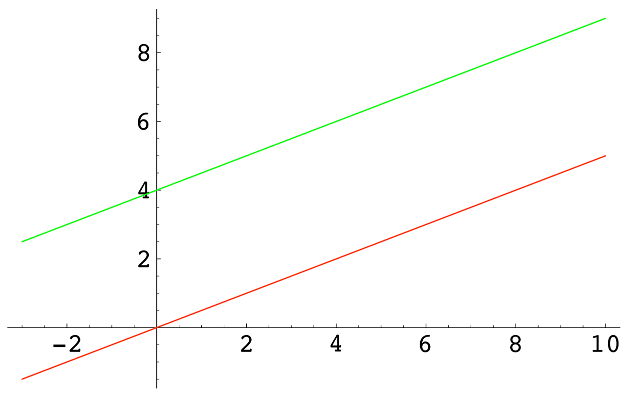
$$f(x) = a\,x + b.$$

The graph is a line which cuts the y-axis at the point $(0,\, b)$.

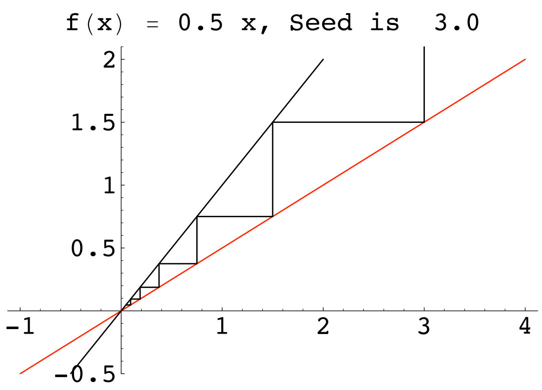
Now suppose that $a \neq 1$. Then the graphs of $f(x) = a\,x + b$ and $y = x$ intersect at a point $(p,\, p)$.

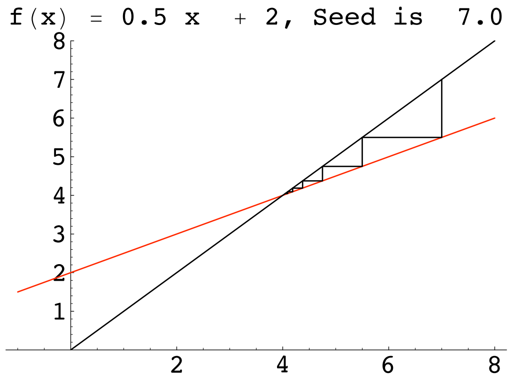


The function $l(x) = a\,x$ is the *linear part* of $f(x)$. The graph of $l(x)$ is a line through the origin and is parallel to the line which is the graph of $f(x) = a\,x + b$. The graph of $f(x)$ is got from the graph of $l(x) = a\,x$ by translating it through the vector $(p,\, p)$ taking the fixed point $(0,\, 0)$ of $l(x)$ to the fixed point $(p,\, p)$ of $f(x) = a\,x + b$.



From the geometry, the nature of the fixed point $(p,\, p)$ of $f(x)$ is the same as the nature of the fixed point $(0,\, 0)$ of $l(x)$.





THEOREM (Dynamics of Affine Mappings)

Let $f : \mathbf{R} \longrightarrow \mathbf{R}$ be an affine mapping with slope a such that $|a| \neq 1$. Let p be the fixed point of the affine mapping. For the orbit of $x_0 \in \mathbf{R}$ under f, either

$|a| \ < \ 1,$ and the orbit converges to the fixed point p, or

$|a| \ > \ 1,$ and the sequence of distances of the orbit elements from p diverges to ∞ provided $x_0 \neq p$.

Proof: Since $f(x)$ is affine with slope a,

$$f(x) = a\,x + b.$$

Then

$$\begin{aligned} x_{k+1} - p &= f(x_k) - p, \\ &= f(x_k) - f(p), \\ &= a\,x_k + b - (a\,p + b), \\ &= a(x_k - p). \end{aligned}$$

Now apply this with $k = 0,\ 1,\ 2,\ \ldots$ to get

$$\begin{aligned} x_1 - p &= & a(x_0 - p), \\ x_2 - p &= a(x_1 - p) = & a^2(x_0 - p), \\ x_3 - p &= a(x_2 - p) = & a^3(x_0 - p), \end{aligned}$$

giving

$$x_n - p = a^n(x_0 - p).$$

Then

$$\begin{aligned} &\text{if} \ \ |a| \ < \ 1, \ \ a^n \longrightarrow 0 \ \ \text{so} \ \ x_n \longrightarrow p \ \ \text{as} \ \ n \longrightarrow \infty, \\ &\text{if} \ \ |a| \ > \ 1, \ \ |a|^n \ \ \text{diverges so} \ \ x_n - p \longrightarrow \infty. \end{aligned}$$

If $a = 1$, there are two cases to consider, $b \neq 0$, or $b = 0$. If $b \neq 0$, the affine map has no fixed points and if $b = 0$, then every point is a fixed point.

If $a = -1$, then $f(x) = -x + b$, and

$$f^2(x) = f(-x + b) = -(-x + b) + b = x$$

so every point, except the fixed point, has period 2.

3.3 Differentiable Mappings

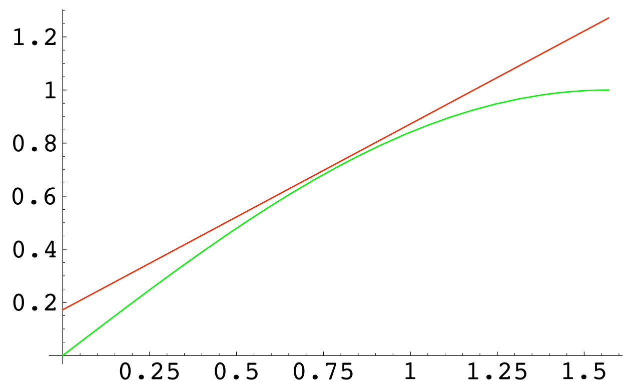
For a mapping which is differentiable, the graph has a tangent at each point. Near the point of tangency the graph stays very close to the tangent. But the tangent is the graph of an affine mapping and so the dynamics of the differentiable mapping should be close to that of the affine mapping. We use this to predict the dynamics of a differentiable mapping near a fixed point.

Since we wish to emphasize the tangent itself rather that just its slope, we define the tangent mapping at p, $\tau_p : \mathbf{R} \longrightarrow \mathbf{R}$ by

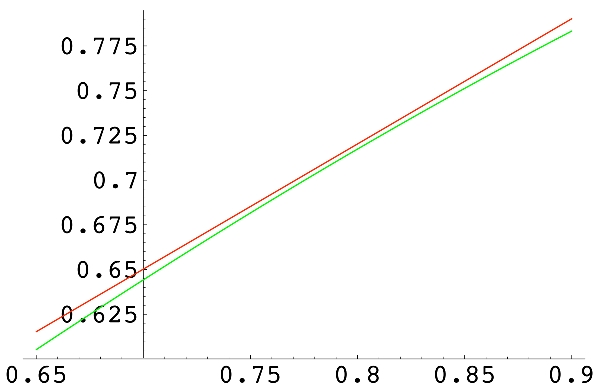
$$\tau_p(x) = f(x) + f'(p)(x - p).$$

Then

- $\tau_p(x)$ is an affine mapping,
- $\tau_p(p) = f(p)$,
- ${\tau_p}'(p) = f'(p)$.

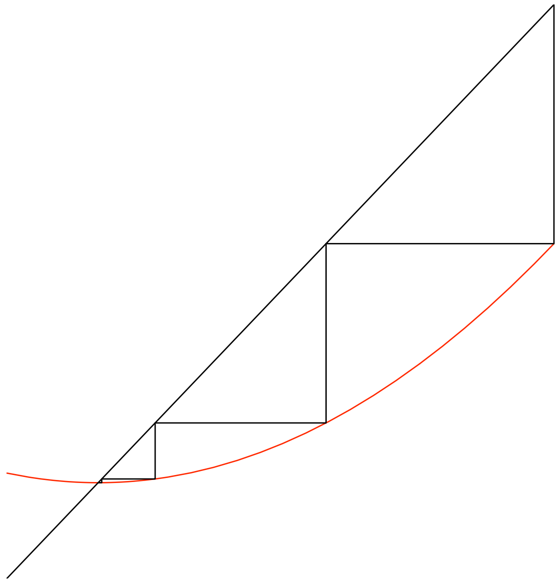


If we zoom in on a point of the graph of a differentiable function $f(x)$, then the magnified portion of the graph of $f(x)$ approaches (a segment of) the tangent to the graph at the point.

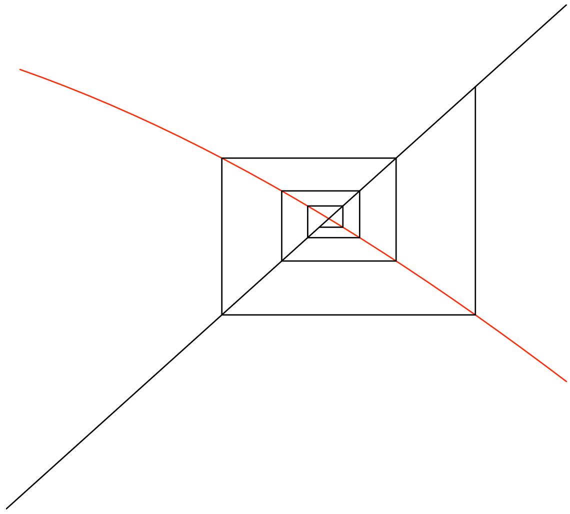


Now consider the dynamics near a fixed point. The next four diagrams illustrate the possibilities.

`Attracting fixed point with positive slope`

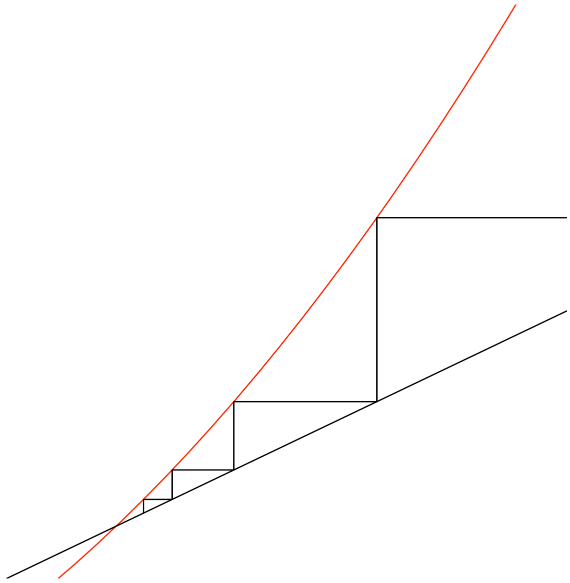


`Attracting fixed point with negative slope`

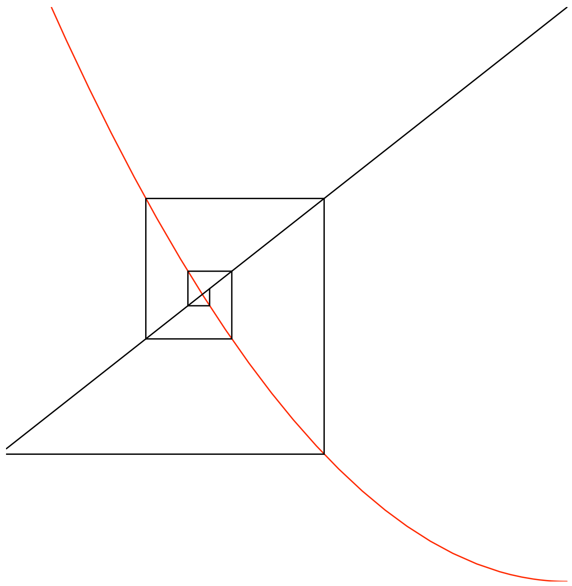


The above two diagrams show examples where $|f'(p)| < 1$.

Repelling fixed point with positive slope



Repelling fixed point with negative slope



The above two diagrams show examples where $|f'(p)| > 1$.

From the diagrams, we see that he orbits of a mapping $f(x)$ near a fixed point should be determined by the magnitude of $|f'(p)|$ near a fixed point p.

DEFINITION

Let $f : \mathbf{I} \longrightarrow \mathbf{R}$, where $\mathbf{I}$ is an interval. A fixed point p is called

- an **attractor** (or an attracting fixed point) if $|f'(p)| \; < \; 1$,
- a **repellor** (or a repelling fixed point) if $|f'(p)| \; > \; 1$,

The number $f'(p)$ is called the **multiplier** of the fixed point p.

Attractors should attract nearby orbits and repellors should repel them.

THEOREM (Dynamics near a fixed point)

Let $f : \mathbf{S} \longrightarrow \mathbf{S}$ where $\mathbf{S}$ is an interval. Suppose that $p \in \mathbf{S}$ is a fixed point of f and that $f(x)$ is differentiable at p with $|f'(p)| \; \neq \; 1$.

(i) $|f'(p)| \; < \; 1$. There is an interval $\mathbf{I}$ containing p such that every orbit $\{x_0, \, x_1, \, x_2, \, \dots \; \}$ with initial point $x_0 \in \mathbf{I}$ converges to the fixed point p. If $x_n \; \neq \; p$, then x_{n+1} is closer to p that x_n is.

(ii) $|f'(p)| \; > \; 1$. There is an interval $\mathbf{I}$ containing p such that no orbit with initial point $x_0 \in \mathbf{I} \setminus \{p\}$ remains in $\mathbf{I}$. If $x_n \in \mathbf{I} \setminus \{p\}$, then the next point x_{n+1} of the orbit is further from p that x_n is.

Before we prove this, we prove a preliminary lemma.

LEMMA

Suppose $f : \mathbf{S} \longrightarrow \mathbf{S}$ is differentiable at $p \in \mathbf{S}$ and that $|f'(p)| \; < \; 1$. Then there is a positive number $a < 1$ and an interval $\mathbf{I}$ such that for all $x \in \mathbf{I}$

$$|f(x) - f(p)| \quad < \quad a\,|x - p|.$$

Proof We have

$$f'(p) = \lim_{x \longrightarrow p} \frac{f(x) - f(p)}{x - p}.$$

Suppose $|f'(p)| \; < \; 1$. Choose a to satisfy

$$|f'(p)| \quad < \quad a \quad < \quad 1,$$

so that

$$-a \quad < \quad f'(p) \quad < \quad a.$$

Then there is an interval $\mathbf{I}$ containing p such that

$$-a < \quad \frac{f(x) - f(p)}{x - p} \quad < \quad a,$$

and so

$$\left|\frac{f(x) - f(p)}{x - p}\right| \quad < \quad a.$$

Thus $|f(x) - f(p)| \quad \leq \quad a\,|x - p|$.

Proof of theorem

Case(i) Suppose $|f'(p)| < 1$. Since p is a fixed point, from the lemma it follows that

$$|f(x) - p| \leq a\,|x - p|.$$

Now suppose $x_k \in \mathbf{I}$. Then with $x_{k+1} = f(x_k)$

$$|x_{k+1} - p| \leq a\,|x_k - p|$$

and so $x_{k+1} \in \mathbf{I}$ since $a < 1$.

By hypothesis, $x_0 \in \mathbf{I}$. Then

$$\begin{aligned}
|x_1 - p| &\leq a\,|x_0 - p| & &= a\,|x_0 - p| \\
|x_2 - p| &\leq a\,|x_1 - p| & &\leq a^2, |x_0 - p| \\
|x_3 - p| &\leq a\,|x_2 - p| & &\leq a^3, |x_0 - p|.
\end{aligned}$$

Thus

$$|x_n - p| \leq a\,|x_{n-1} - p| \leq a^n\,|x_0 - p|.$$

But $0 \leq a < 1$ so $a^n \longrightarrow 0$ as $n \longrightarrow \infty$. Thus

$$\lim_{n \longrightarrow \infty} |x_n - p| = 0,$$

and so

$$\lim_{n \longrightarrow \infty} = p.$$

Case(i) Exercise.

EXAMPLE

Consider the function

$$Q_{2.8}(x) = 2.8\,x(1 - x).$$

The fixed points are solutions of $2.8\,x(1 - x) = x$.

Clearly, $x = 0$ is one solution and the other is

$$x = 1.8/2.8 \approx 0.64.$$

Then ${Q_{2.8}}'(0) = 2.8$ and ${Q_{2.8}}'(0.64) \approx -0.78$. Thus $x = 0$ is a repelling fixed point and $x = 0.64$ is an attracting fixed point.

EXERCISE If $Q_{3.2}(x) = 3.2\,x(1 - x)$, show that the fixed points are $x = 0$ and $x = 0.6875$. What is the nature of these fixed points?

EXAMPLE

Consider the family of mappings

$$Q_\mu = \mu\,x(1 - x) \quad \text{for} \quad 0 < \mu \leq 4.$$

The fixed points are the solutions of

$$Q_\mu(x) = x,$$

that is

$$\mu\, x(1-x) = x.$$

The solutions are

$$x = 0 \quad \text{and} \quad x = \frac{\mu - 1}{\mu}.$$

The nature of these fixed points now needs to be determined for various values of μ.

$$\begin{aligned} Q_\mu{}'(x) &= x - 2\,\mu\, x \\ Q_\mu{}'(0) &= \mu \\ Q_\mu{}'(\frac{\mu-1}{\mu}) &= 2 - \mu \end{aligned}$$

Thus $x = 0$ is attracting for $0 < \mu < 1$ and it is repelling for $1 < \mu < 4$. The point $x = \frac{\mu-1}{\mu}$ is attracting for $|2-\mu| < 1$, that is $1 < \mu < 3$ and repelling for $3 < \mu < 4$.

3.4 Indifferent Fixed Points

The behaviour of orbits in the neighbourhood of indifferent points can have attracting and/or repelling behaviour. The following examples illustrate this.

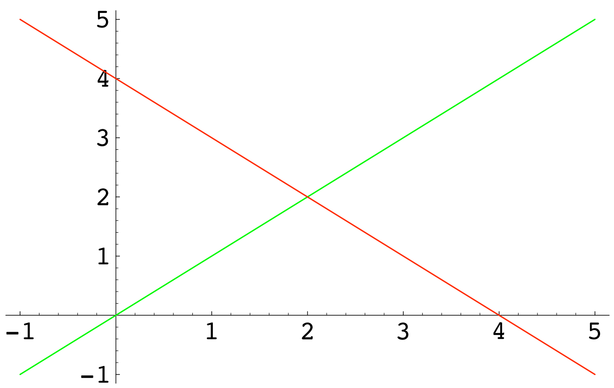
EXAMPLES

1. Consider the function $f(x) = -x + 4$.

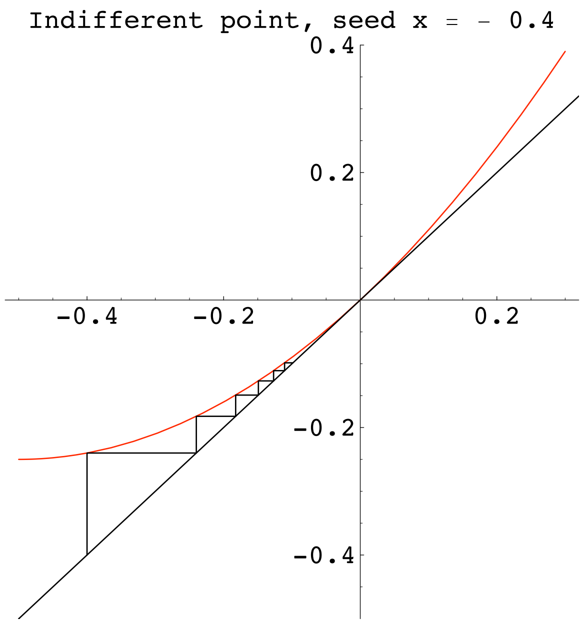
   Then $x = 2$ is a fixed point and $f'(2) = -1$.

   All points $x \neq 2$ have orbits of length 2 since

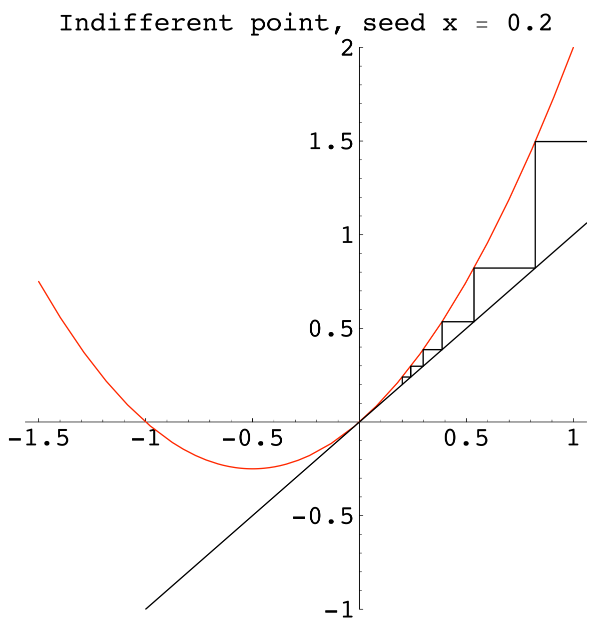
$$f^2(x) = f(-x+4) = -(-x+4) + 4 = x.$$



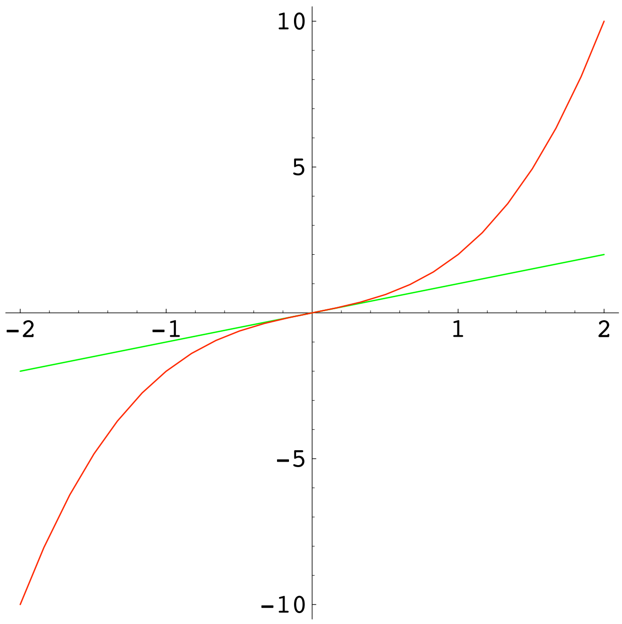
2. Consider $g(x) = x + x^2$. Then $x = 0$ is a fixed point and $g'(0) = 1$. Points on the left of $x = 0$ (and close to $x = 0$) have orbits converging to $x = 0$.



However, points on the right of $x = 0$ are repelled.



3. Consider $h(x) = x + x^3$. Then $x = 0$ is the only fixed point and $h'(0) = 1$. Show that $x = 0$ is repelling.



3.5 Classification of Periodic Points

Periodic points can also be classified as being attracting, repelling, or indifferent.

If x_0 lies on an n-orbit (n-cycle) of a function then $x_0 = f(x_0)$ and so the graph of $f^n(x)$ meets the diagonal line $y = x$ at the point (x_0, x_0). Then $f^n(x)$ has a fixed point at x_0. We say that the periodic point x_0 is an attracting, repelling, or indifferent fixed point for $f^n(x)$ according as $|f^{n\prime}(x_0)|$ takes a value less that 1, greater than 1, or equal to 1.

EXAMPLE

Let $f(x) = x^2 - 1$.

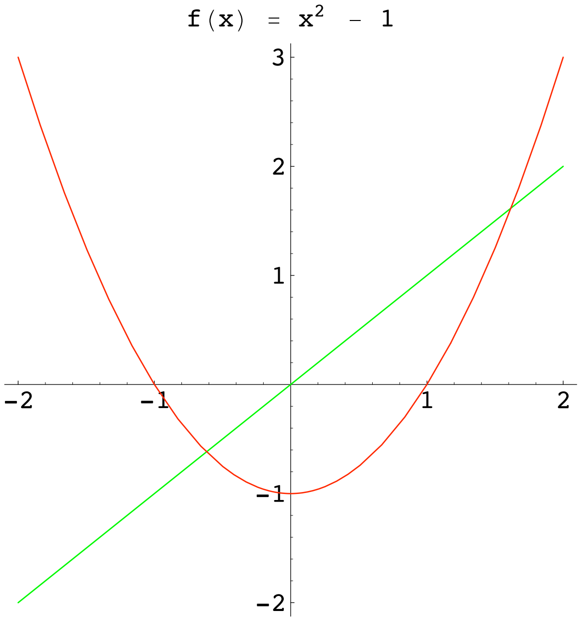
The fixed points are solutions of

$$x^2 - 1 = x$$
$$x^2 - x - 1 = 0$$

The solutions are

$$x = \frac{-1 \pm \sqrt{5}}{2}.$$

Since $f'(x) = 2\,x$ both of these fixed points are repelling.



Next we consider the period-2 points.

$$f^2(x) = f(x^2-1) = (x^2-1)^2 - 1 = x^4 - 2\,x^2.$$

$$(f^2)' = 4\,x^3 - 4\,x.$$

The fixed points of $f^2(x)$ are solutions of

$$x^4 - 2\,x^2 = x$$
$$x(x^3 - 2\,x - 1) = 0.$$

Clearly $x = 0$ is a solution. Furthermore the fixed points

$$x = \frac{-1 \pm \sqrt{5}}{2}$$

must be roots so $x^2 - x - 1$ must be a factor of $x^3 - 2\,x - 1$. By division, get $x + 1$ is the other factor so

$$x^4 - 2\,x^2 - x = x(x+1)(x^2 - x - 1).$$

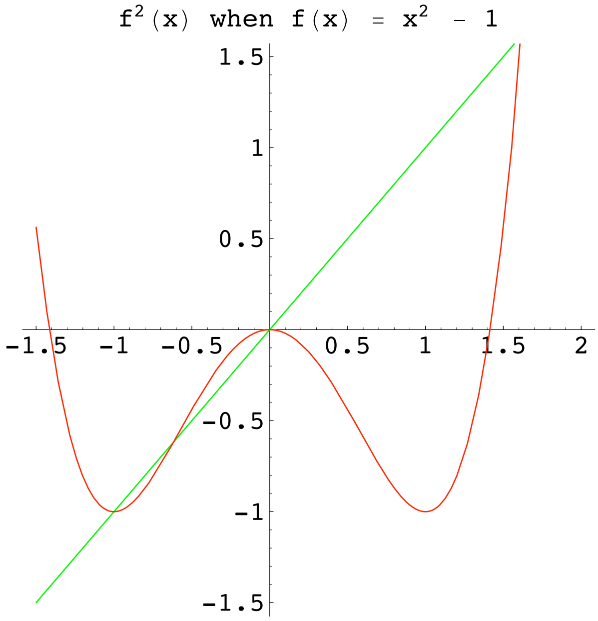
Thus $x = 0$ and $x = -1$ are the period-2 points which are not fixed points. They lie on the same orbit because

$$f(0) = -1 \quad \text{and} \quad f(-1) = 0.$$

This orbit is attracting as

$$(f^2)'(x) = 4\,x^3 - 4\,x,$$

so $(f^2)'(0) = 0$ and $(f^2)'(-1) = 0$.



In the last example we got that $(f^2)'(x)$ had the same value at each point along the period-2 orbit. This is always true, that is if $\{x_0, x_1\}$ is a period-2 orbit, then

$$(f^2)'(x_0) = (f^2)'(x_1),$$

and the common value is $f'(x_0).f'(x_1)$. To see this, we use the chain rule.

$$\begin{aligned}
(f^2)'(x_0) &= f'(f(x_0))f'(x_0) \\
&= f'(x_1)f'(x_0) \\
(f^2)'(x_1) &= f'(f(x_1))f'(x_1) \\
&= f'(x_0)f'(x_1)
\end{aligned}$$

Thus the derivative of $f^2(x)$ at a period-2 point x_0 is the product of the derivatives of $f(x)$ at each point along the period-2 orbit of x_0.

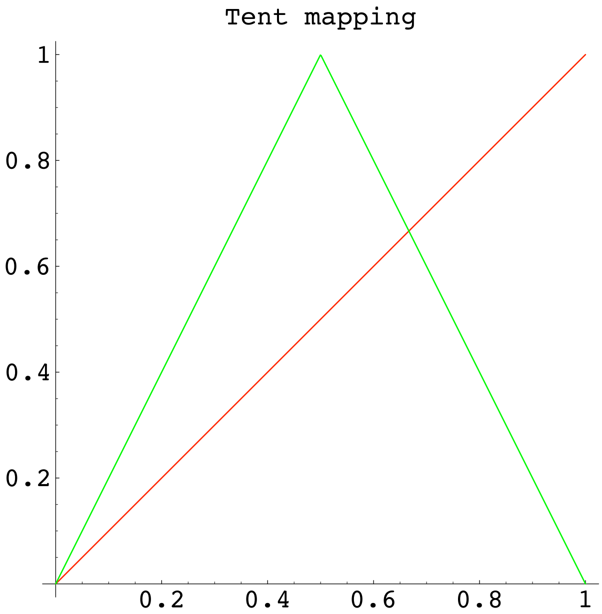
In fact, we have a similar result for period-n points. Suppose $\{x_0, x_1, x_2, \ldots, x_{n-1}\}$ is a period-n orbit. Then $f^n(x_j) = x_j$ for $0 \le j \le n-1$ and

$$(f^n)'(x_j) = f'(x_0).f'(x_1).\ldots f'(x_{n-1}).$$

EXAMPLE Let

$$\begin{aligned}
T(x) &= 2x \qquad \text{if } 0 \le x \le 1/2 \\
&= 2-2x \ \text{ if } 1/2 \le x \le 1.
\end{aligned}$$

This is the so-called **tent mapping**.



Consider $x_0 = 2/7$. Then

$$f(2/7) = 4/7,$$
$$f(4/7) = 6/7,$$
$$f(6/7) = 2/7.$$

so $\{2/7,\, 4/7,\, 6/7\}$ is a period-3 orbit. To determine if this orbit is attracting/repelling;

$$\begin{aligned} (T^3)'(2/7) &= T'(2/7).T'(4/7).T'(6/7) \\ &= 2.(-2)(-2) \\ &= 8. \end{aligned}$$

and so this orbit is repelling.

4 Families of Mappings and Bifurcation

Let f_μ denote a mapping which depends on a parameter μ. As μ varies so does the mapping f_μ. This gives rise to a function

$$\mu \longrightarrow f_\mu,$$

which assigns a mapping f_μ to each μ value in some set.

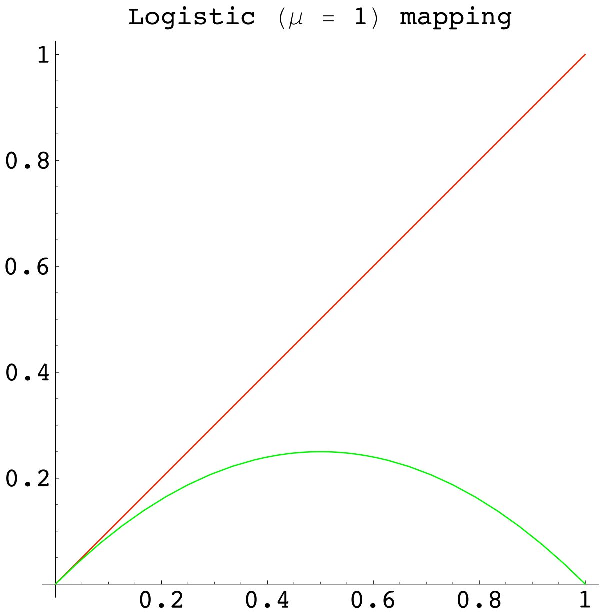
We call the function $\mu \longrightarrow f_\mu$ a **family of mappings**.

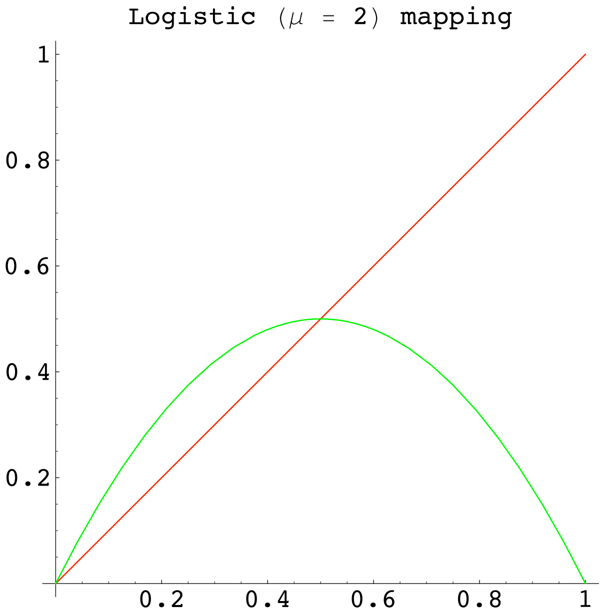
We will now consider two families of mappings, the **logistic family** and the **tent family**.

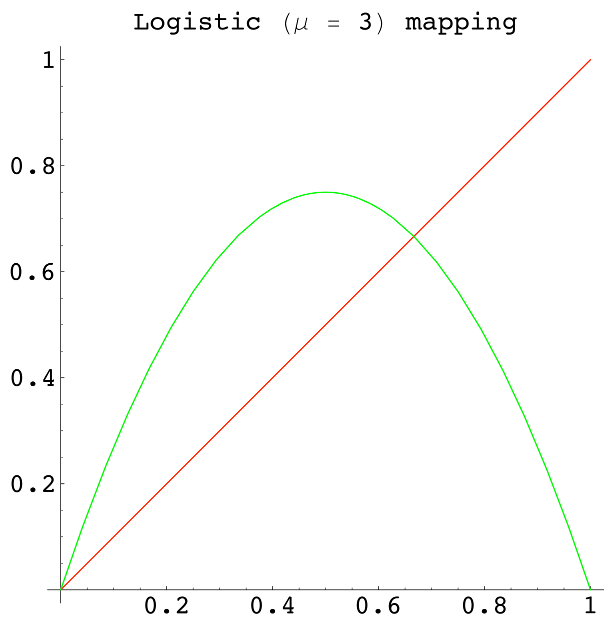
DEFINITION The **logistic family** of mappings $\mu \longrightarrow Q_\mu$ is obtained by putting

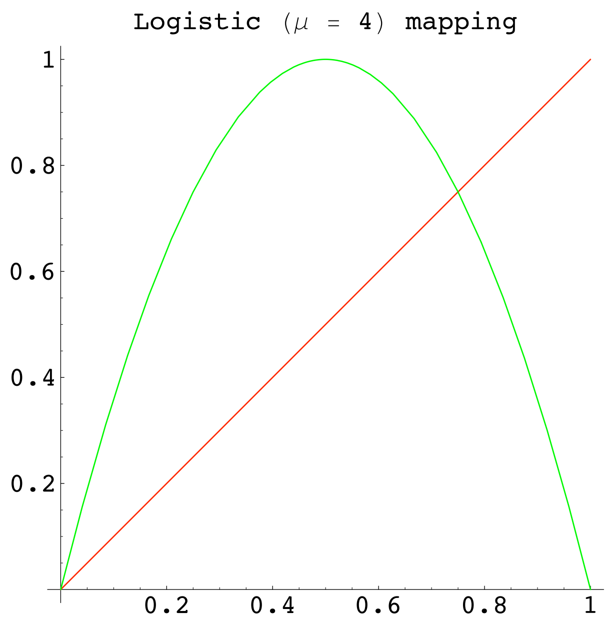
$$Q_\mu(x) = \mu\, x\,(1-x),$$

where $0 \leq \mu \leq 4$ and $0 \leq x \leq 1$.


Logistic (μ = 1) mapping
1
0.8
0.6
0.4
0.2
0.2
0.4
0.6
0.8
1


Logistic (μ = 2) mapping
1
0.8
0.6
0.4
0.2
0.2
0.4
0.6
0.8
1





The choice of the interval $[0, 1]$ for the domain of $Q_\mu(x)$ is suggested by the discrete logistic model of population growth.

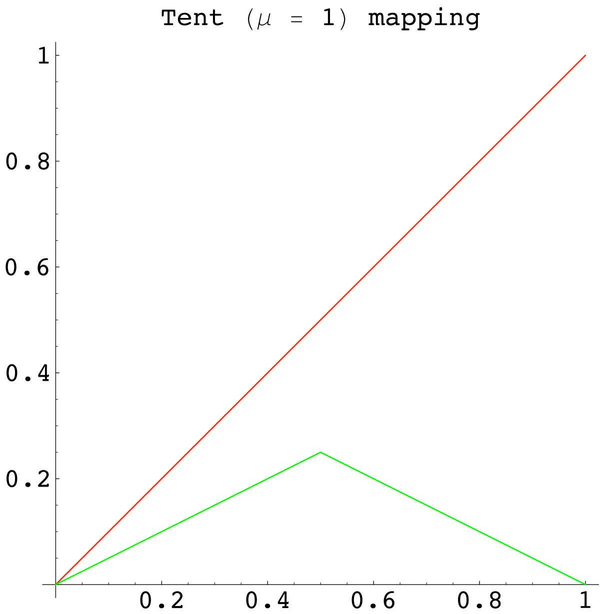
For $0 \leq \mu \leq 4$, it is easy to check that Q_μ maps $[0, 1]$ into itself. As μ increased, so does the complexity of the dynamics of the mapping Q_μ.

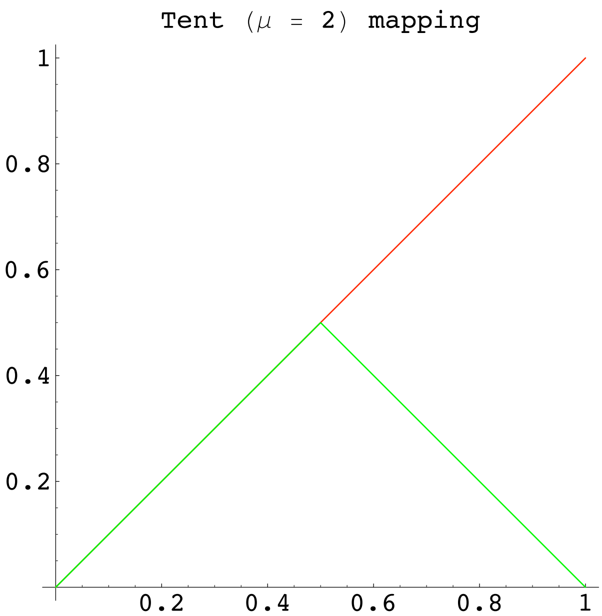
DEFINITION The family of **tent mappings** $\mu \longrightarrow T_\mu$ is obtained by putting

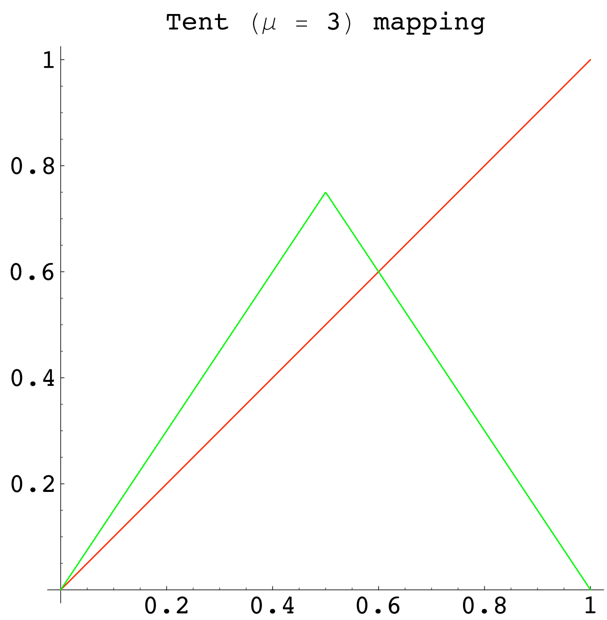
$$T_\mu(x) = \frac{\mu}{4}\,(1 - |2\,x - 1|),$$

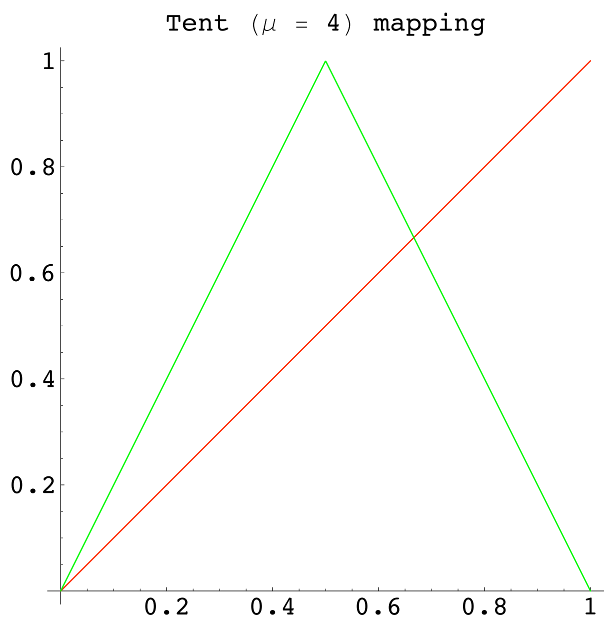
or

$$
T_\mu(x) = \begin{cases} \frac{\mu\, x}{2} & \text{for } 0 \;\le\; x \;\le\; 1/2 \\ \frac{\mu\,(1-x)}{2} & \text{for } 1/2 \;\le\; x \;\le\; 1. \end{cases}
$$









For $0 \leq \mu \leq 4$, it is easy to show that T_μ maps the interval $[0, 1]$ to itself. As μ increases, so does the complexity of the dynamics of the mapping T_μ.

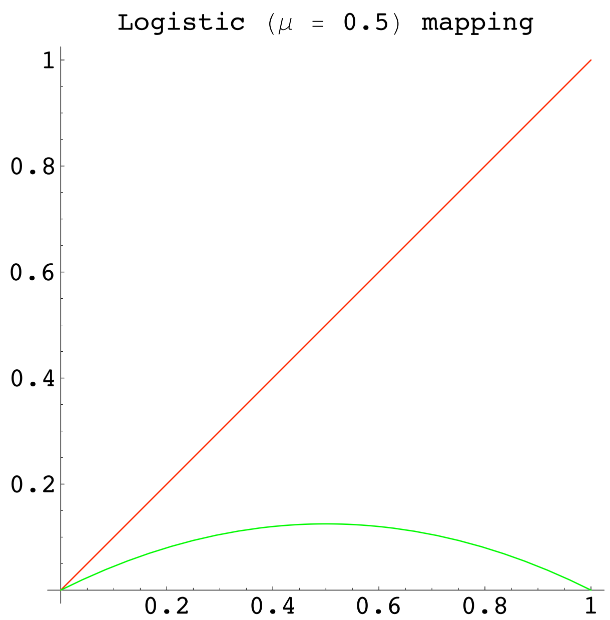
The two families Q_μ and T_μ share a number of common features. If f denotes a mapping in either of these two families, then

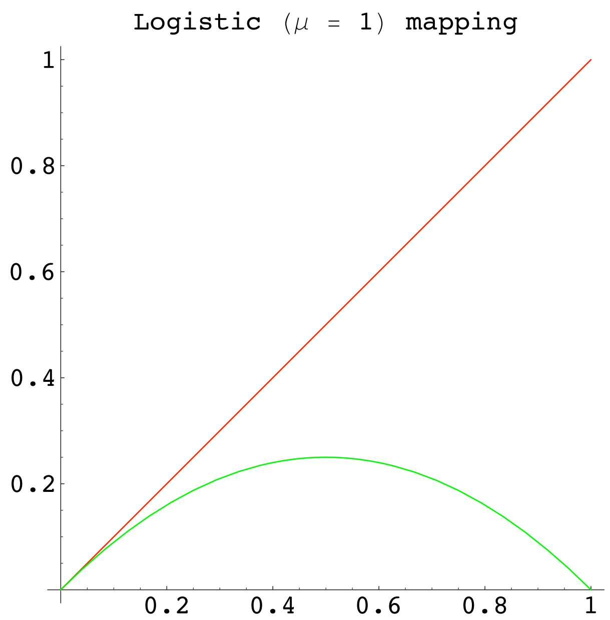
- The mapping satisfies $f(1/2 - x) = f(1/2 + x)$ for all x in $[0, 1/2]$ so the mapping f is symmetric about the line $x = 1/2$.
- The values of f increase steadily from $f(0) = 0$ at the left to the maximum value at $x = 1/2$ and then decrease steadily to $f(1) = 0$. So the maps are **unimodal**.

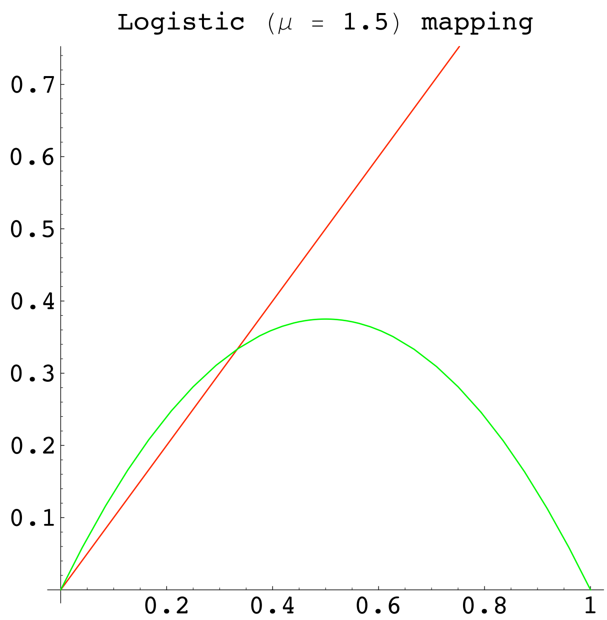
4.1 Families of Fixed Points

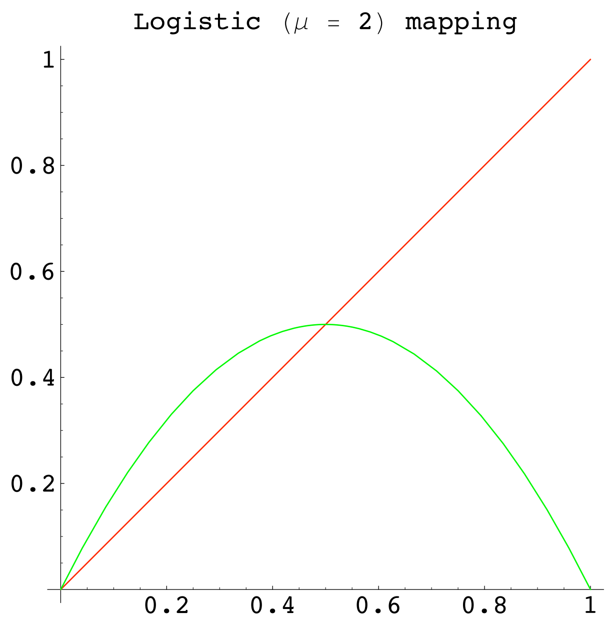
We now investigate the changes which occur in the fixed points of these mappings as the parameter μ is increased.

Fixed Points of Q_μ.









To determine the nature of the fixed points, we evaluate the derivatives of Q_μ. We have

$$Q_\mu{}'(x) = \mu - 2\,\mu\,x,$$

so $Q_\mu{}'(0) = \mu$. Thus

- if $\mu = 1$, $Q_\mu(x)$ is tangent at $x = 0$ to the line $y = x$,
- if $\mu > 1$, the graph rises above that of the function $\mathrm{id}(x) = x$.
- if $\mu < 1$, the graph drops below that of the function $\mathrm{id}(x) = x$.

Thus as μ increases past the value $\mu = 1$, $Q_\mu(x)$ acquires a second fixed point. Check this algebraically. The fixed points are solutions of the equation $Q_\mu(x) = x$, that is $\mu\, x\,(1-x) = x$. Solving we get

$$x = 0 \qquad \text{and} \qquad x = \frac{\mu - 1}{\mu}.$$

Thus if $\mu \le 1$, there is only one fixed point which is that at $x = 0$. But if $\mu > 1$ there is a second fixed point at $x = \frac{\mu-1}{\mu}$.

Now we consider the nature of these fixed points.

For $0 \le \mu < 1$, the only fixed point is at $x = 0$. Since

$$Q_\mu{}'(0) = \mu < 1,$$

then this fixed point is attracting.

If $\mu > 1$, $Q_\mu{}'(0) = \mu > 1$ so $x = 0$ is a repelling fixed point.

Now consider the fixed point $\frac{\mu-1}{\mu}$ for $\mu > 1$.

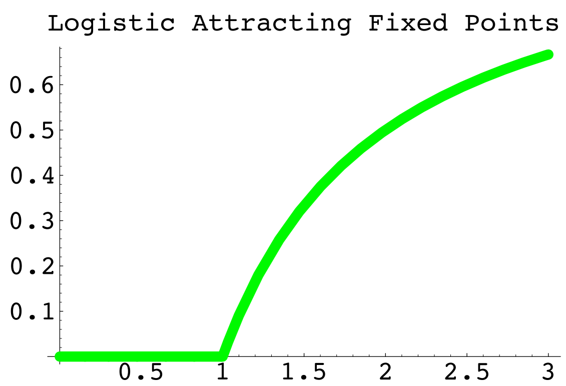
Since

$$Q_\mu{}'(\frac{\mu - 1}{\mu}) = 2 - \mu,$$

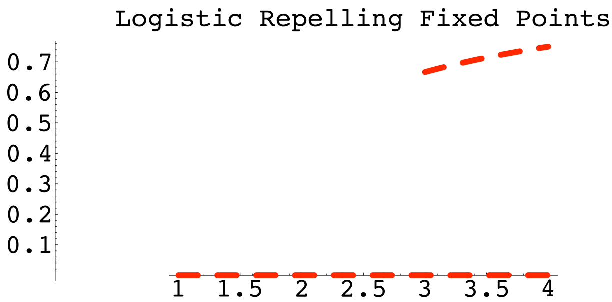
then for $1 < \mu < 3$, $|2-\mu| < 1$. This means that the fixed point at $\frac{\mu-1}{\mu}$ is attracting for $1 < \mu < 3$ and for $3 < \mu \le 4$ it is repelling.

Now we plot these two sets of fixed points.

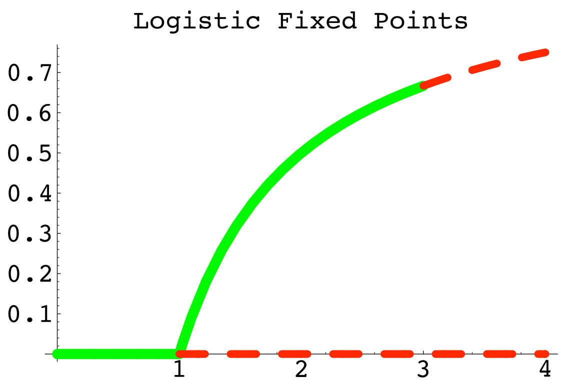
First the **attracting** fixed points.



Now the **repelling** fixed points.



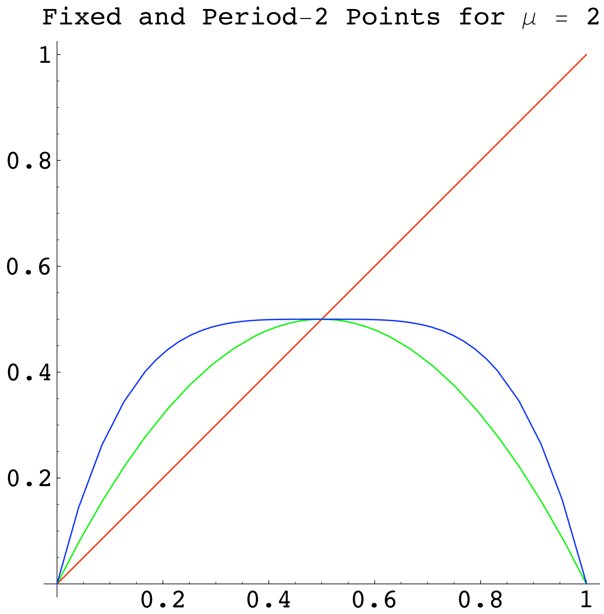
Combining the two sets this diagram shows the fixed points of the logistic family $\mu \longrightarrow Q_\mu(x) = \mu\, x\,(1-x)$. The dashed red graph gives the repelling fixed points and the solid green graph gives the attracting fixed points.



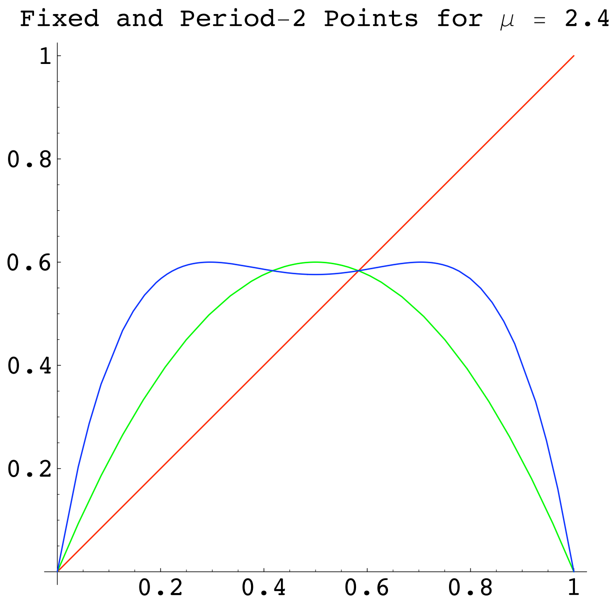
4.2 Periodic-2 Points for the Q_μ Family

The period-2 points of Q_μ are the fixed points of ${Q_\mu}^2(x)$. Some of the period-2 points of Q_μ are also fixed points of Q_μ. The novelty now is that there are fixed points of ${Q_\mu}^2(x)$. which are **not** fixed points of Q_μ. These are the points of **prime** period-2 for Q_μ.

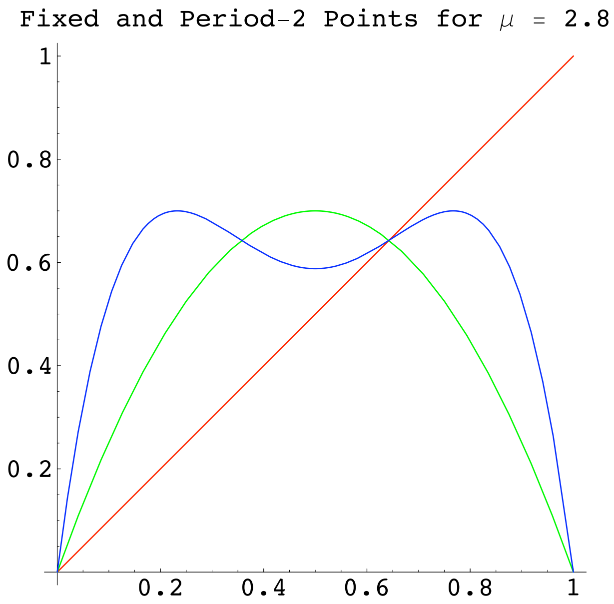
For $0 \le \mu < 3$, the mappings $Q_\mu(x)$ and ${Q_\mu}^2(x)$ have the same fixed points. The following diagrams illustrate this.



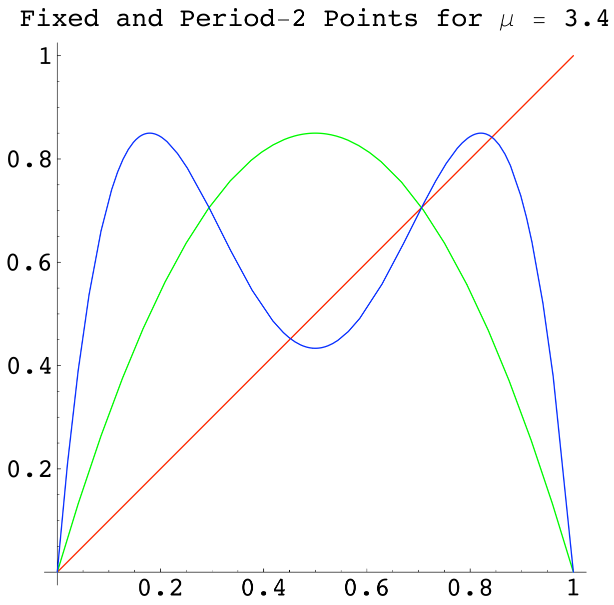
Fixed and Period-2 Points for μ = 2
1
0.8
0.6
0.4
0.2
0.2
0.4
0.6
0.8
1

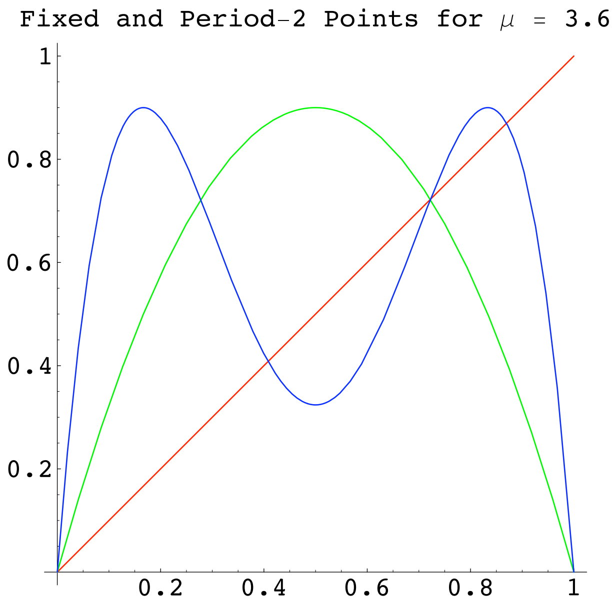


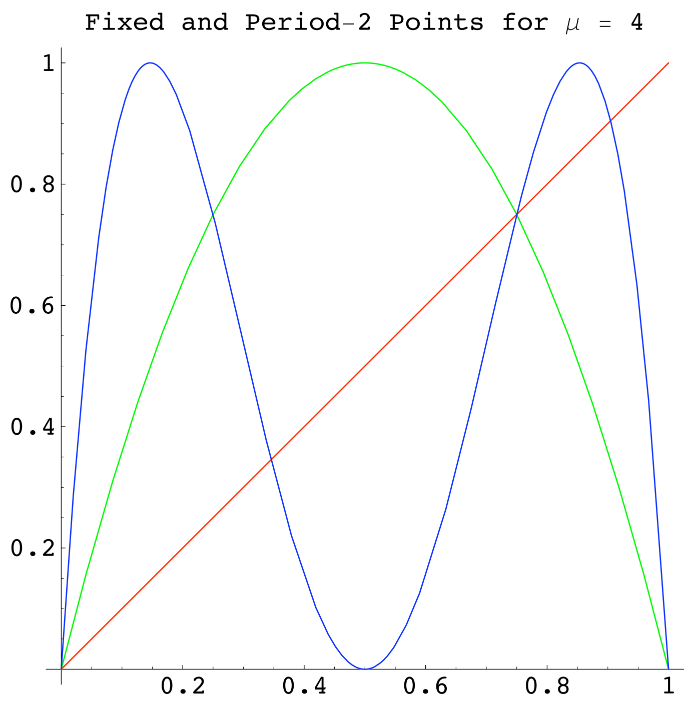
Fixed and Period-2 Points for μ = 2.4
1
0.8
0.6
0.4
0.2
0.2
0.4
0.6
0.8
1



For $3 < \mu \leq 4$, the mapping $Q_\mu{}^2(x)$ has 2 extra fixed points. They are not fixed points of Q_μ and so they are prime period-2 points. The following diagrams illustrate this.







The period-2 points of $Q_\mu(x)$ are solutions of ${Q_\mu}^2(x) = x$. It can be shown that the solutions are

$$x = 0, \quad x = \frac{\mu - 1}{\mu}, \quad x = \frac{\mu + 1}{2\,\mu} \pm \frac{\sqrt{(\mu+1)(\mu-3)}}{2\,\mu}.$$

Since $\mu > 0$, $\mu + 1 > 0$ so the last 2 solutions are real if and only if $\mu \geq 3$. Thus an extra pair of fixed points for ${Q_\mu}^2(x) = x$ appear for $\mu > 3$.

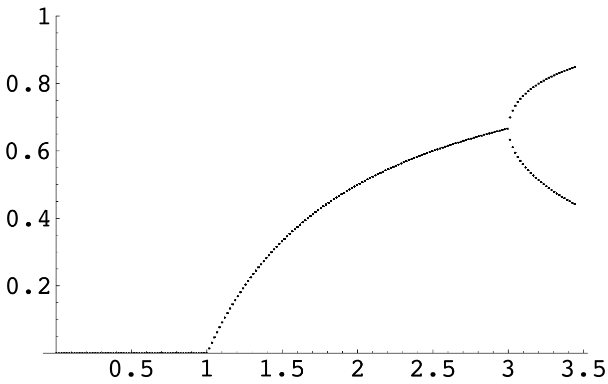
At the points

$$\frac{\mu + 1}{2\,\mu} \pm \frac{\sqrt{(\mu+1)(\mu-3)}}{2\,\mu},$$

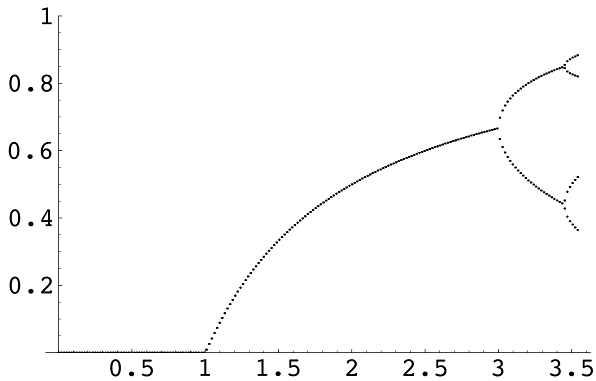
the derivative of $Q_\mu{}^2(x)$ has value

$$5-(\mu-1)^2.$$

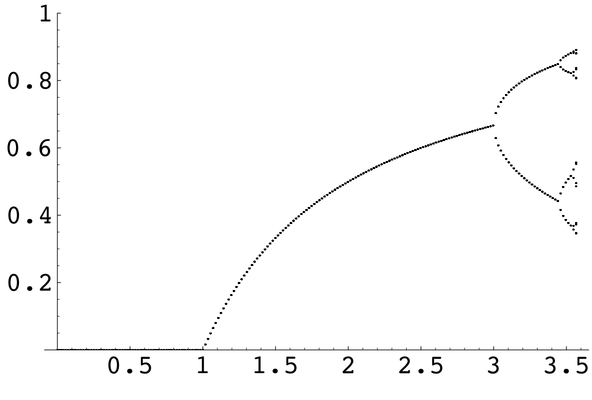
Thus these two points are attracting for $3 < \mu < 1+\sqrt{6}$ and repelling for $1+\sqrt{6} < \mu \leq 4$. So we get the following **bifurcation** diagram.



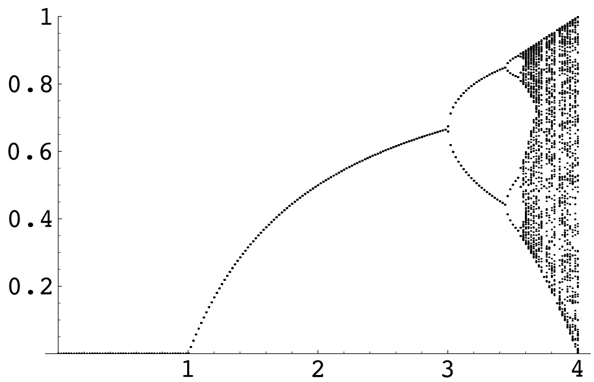
Since the numerical value of $1+\sqrt{6}$ is 3.44949, then at this value we see that there is a bifurcation when a period-4 orbit appears.



Shortly after that, period-8 attracting orbits appear.



This period doubling continues with the intervals between successive doublings decreasing to zero. The following well-known bifurcation diagram shows this behaviour.



4.3 Graphing Higher Order Iterates

Consider the function

$$f(x) = 4\,x\,(1 - x).$$

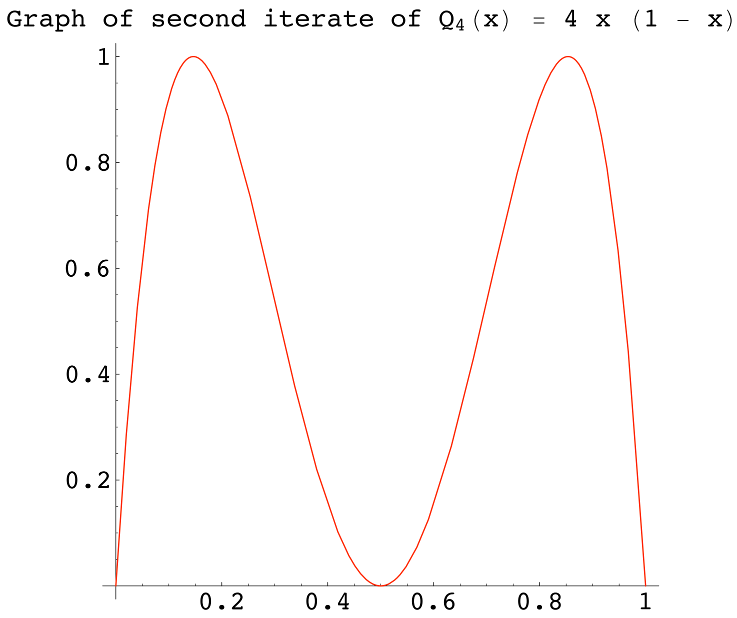
(This is the quadratic function $Q_4(x)$.)

To sketch $f^2(x)$, note that as x increases from 0 to 1/2, $f(x)$ values increase from 0 to 1 and as x increases from 1/2 to 1 then $f(x)$ values decrease form 1 to 0. Thus in the range 0 to 1/2,

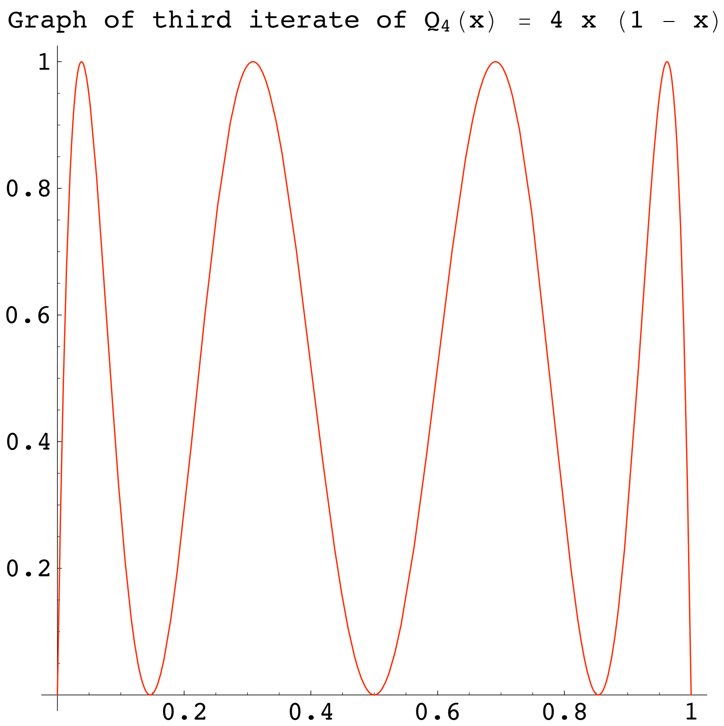
$$f^2(x) = f(f(x))$$

values go from 0 to 1 and back again to 0.

To graph on $[1/2, 1]$, use the fact that $f^2(x)$ is symmetric about $x = 1/2$. This follows from the symmetry of $f(x)$ about $x = 1/2$, *i.e.* $f(x-1/2) = f(x+1/2)$ for $0 < x \leq 1/2$. Then the symmetry of $f^2(x)$ about $x = 1/2$ means that the graph of $f^2(x)$ has 2 arches on $[0, 1]$.



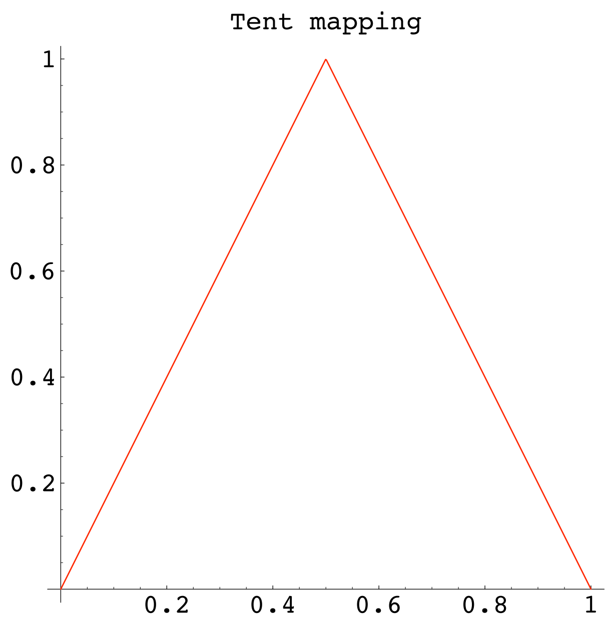
Since $f^3 = f^2 \cdot f$, the graph of f^3 will have 4 arches.



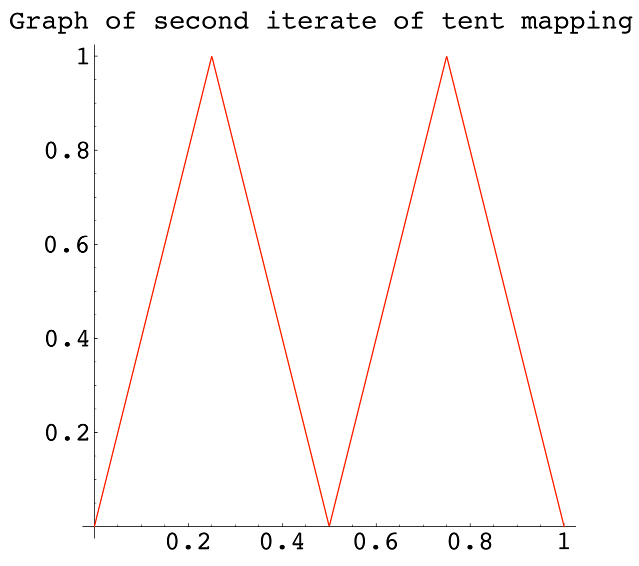
In general, f^n will have 2^{n-1} arches.

Now consider the tent mapping $g(x)$ given by

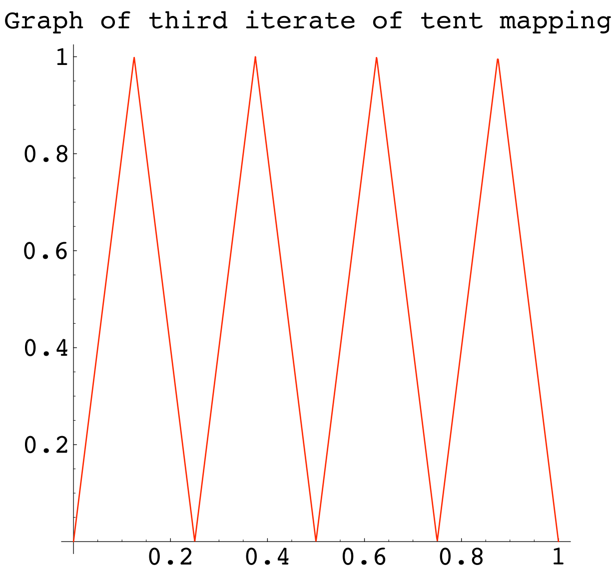
$$\begin{aligned} T_4(x) &= 2\,x \quad \text{if} \quad 0 \leq x \leq 1/2, \\ &= 2-2\,x \quad \text{if} \quad 1/2 \leq x \leq 1. \end{aligned}$$



Using arguments similar to the case of $f(x) = 4\,x\,(1-x)$ we see that ${T_4}^2(x)$ has a graph with 2 tents.



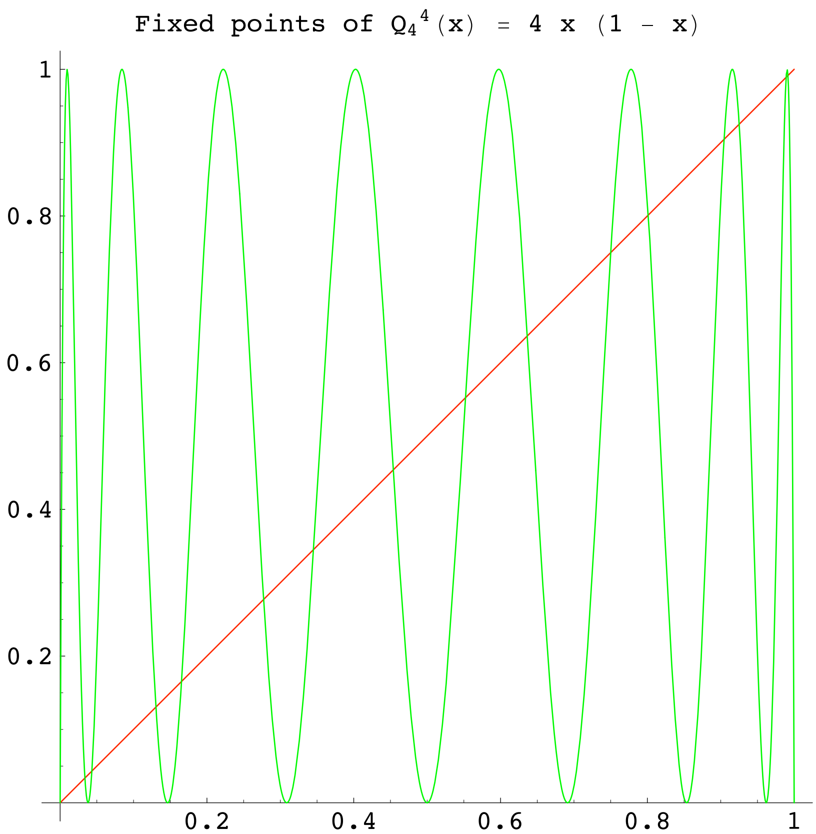
and that ${T_4}^3(x)$ has 8 tents.



By induction, show that ${T_4}^n$ has 2^{n-1} tents.

4.4 Density of Periodic Points

From the graphs of ${Q_4}^n(x)$ and ${T_4}^n(x)$, it is clear that these two maps have infinitely many periodic points as $n \longrightarrow \infty$. Consider ${Q_4}^n(x)$ where $Q_4(x) = 4\,x\,(1-x)$. Its graph (for $n = 4$) is



The graph of the diagonal line $y = x$ meets this graph in 16 points (it meets each arch twice). Thus ${Q_4}^4(x)$ has 2^4 fixed points or $Q_4(x)$ has 2^4 period-4 points. Since ${Q_4}^n(x)$ has 2^{n-1} arches, $Q_4(x)$ has 2^n period-n points. So as $n \longrightarrow \infty$, $2^n \longrightarrow \infty$ and thus the number of periodic points becomes infinite. Furthermore, each arch has 2 periodic points and the arches are uniformly located on the subintervals of length $\frac{1}{2^{n-1}}$, so the periodic points are spread in a dense fashion on the interval $[0, 1]$.

For the tent mapping,

$$\begin{aligned} T_4(x) &= 2\,x \quad \text{if} \quad 0 \;\leq\; x \;\leq\; 1/2, \\ &= 2 - 2\,x \quad \text{if} \;\; 1/2 \;\leq\; x \;\leq\; 1, \end{aligned}$$

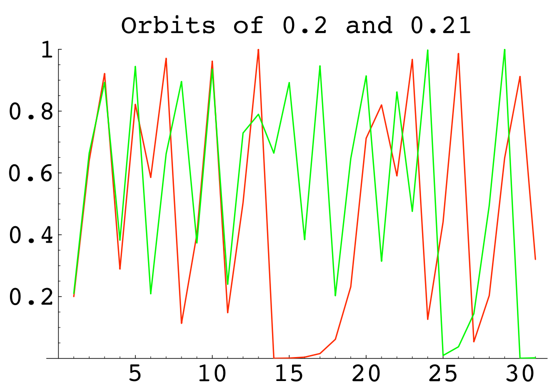
the periodic points are also dense.

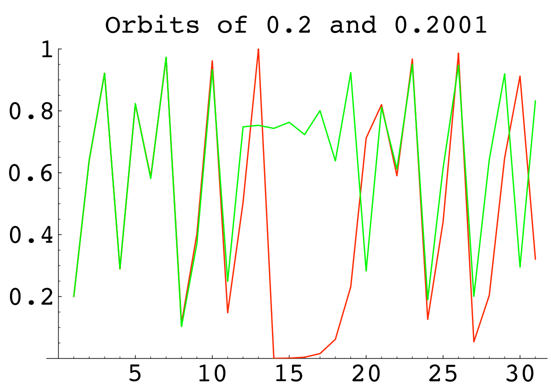
Furthermore, the periodic points of both maps, $Q_4(x)$ and $T_4(x)$ are repelling as the slope of the tangents to the graphs are clearly greater that 1, if the slope is positive, and less that -1 if the slope is negative.

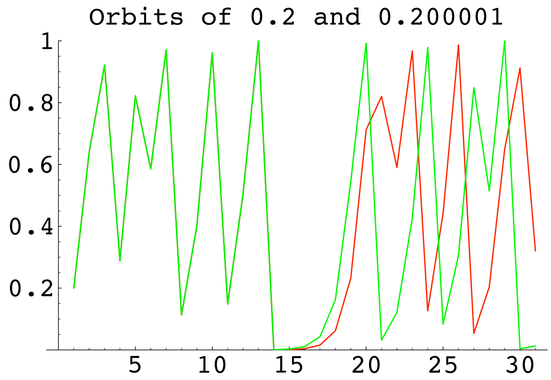
4.5 Diverging Iterates

We now demonstrate graphically how two orbits of iterates of $Q_4(x) = 4\,x\,(1-x)$, which have different seeds, diverge from each other.

Consider orbits starting with nearby seeds x_0 and y_0. We draw the graphs of 3 such pairs; 0.2 and 0.21, 0.2 and 0.2001, 0.2 and 0.200001.

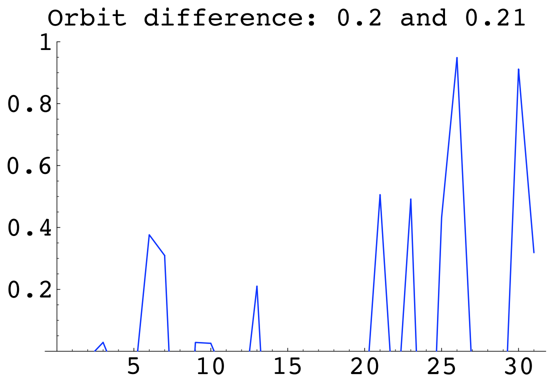


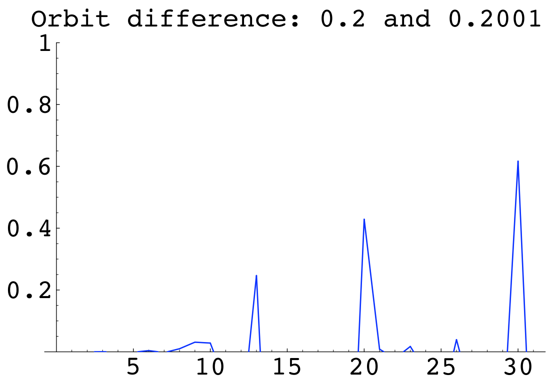


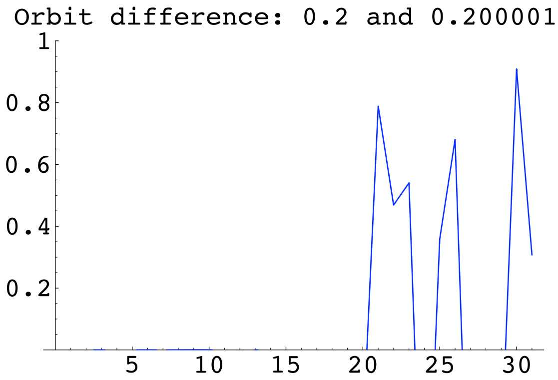


The sequences stay close initially but then diverge away from each other. They diverge eventhough the initial points are very close, although the divergence takes longer to occur.

It is easier to see how the iterates diverge if we graph the difference between the iterates , ${Q_4}^n(x_0)$ and ${Q_4}^n(y_0)$. These graphs are plotted below for each of the 3 cases above.







In the first case, the iterates separate at the third iteration, in the second case, iterates separate at the eight iterate and in the third case, the iterates separate at the twentieth iterate.

DEFINITION Let $f : \mathbf{S} \longrightarrow \mathbf{S}$ be a function defined on an interval $\mathbf{S}$ of $\mathbf{R}$. Then we say that f exhibits **sensitive dependence on initial conditions** if there exists a number $\delta > 0$ such that for all $x \in \mathbf{S}$ and all $\varepsilon > 0$, there is a $y \in \mathbf{S}$ and an $n \in \mathbf{N}$ such that

$$|x - y| < \varepsilon \quad \text{and} \quad |f^n(x) - f^n(y)| > \delta.$$

Practically speaking, sensitive dependence implies that if we are using an iterated function to model long-term behaviour (such as population growth, the weather, economic performance) and the function exhibits sensitive dependence, then any error in the measurement of the initial conditions may result in large differences between predicted behaviour and the actual behaviour of the system we are modeling. (This is the so-called *butterfly effect*.) Since all physical measurements include error, this condition severely limits the utility of our model.

4.6 Chaos

Until the 1960's most scientists and mathematicians thought that typical differential and difference equations did not exhibit the kind of behaviour we now call **chaos**. Now we know that this is not the case. There are many models whose behaviour is quite erratic, and this behaviour persists when the parameters are varied.

Roughly speaking, chaos means unpredictability.

EXAMPLE Consider the function

$$Q_2(x) = 2\,x\,(1 - x).$$

This function has 2 fixed points at $x = 0$ and $x = 0.5$. The point $x = 0$ is repelling and the point $x = 0.5$ is attracting. Using cobweb diagrams, one can show that the orbit of every x_0, with $0 < x_0 < 1$ tends to the fixed point 0.5. The point $x_0 = 1$ has an orbit that is eventually fixed. So we know the fate of all orbits.

In real life we rarely know the seed value for our orbit with complete accuracy. For example, we most likely do not know the precise value of the population at time zero; we probably miss a few individuals in our initial count. If our model is the simple quadratic function $Q_2(x)$, then this inaccuracy does not really matter. A small fluctuation in our initial seed in the interval $0 < x_0 < 1$ does not alter our predictions.

Now consider the orbits of the function

$$Q_4(x) = 4\,x\,(1-x).$$

This function has fixed points at $x = 0$ and at $x = 3/4$ and it has eventually fixed points at $x = 1/2$ and at $x = 1$. But, consider the fate of virtually any other orbit. We saw earlier that orbits of arbitrarily close initial pairs of seeds eventually diverge. We also saw that this function, $Q_4(x)$, has repelling periodic points which are dense on the interval [0, 1]. So, the fate of the other orbits is extremely unpredictable.

A further property of $Q_4(x)$ is that it has a **dense orbit**. This means that there is an orbit with seed x_0 such that if we take any subinterval J of [0, 1], no matter how small, then there is a point x_n of the orbit which is in J. Proving this result requires some results in topology.

Thus the function $Q_4(x)$ has the following properties. It has

- sensitive dependence on initial conditions,
- a dense set of periodic points,
- a dense orbit.

In a popular definition of chaos, due to the U.S. mathematician Robert Devaney, these three properties are taken as the essential ingredients of chaos. Thus $Q_4(x)$ is an example of a chaotic mapping.

The tent mapping

$$\begin{aligned} T_4(x) &= 2\,x \quad \text{if} \quad 0 \le x \le 1/2, \\ &= 2-2\,x \quad \text{if} \quad 1/2 \le x \le 1, \end{aligned}$$

is also a chaotic mapping. We discussed the density of periodic points earlier. Proving the other two properties requires some analysis and topology.

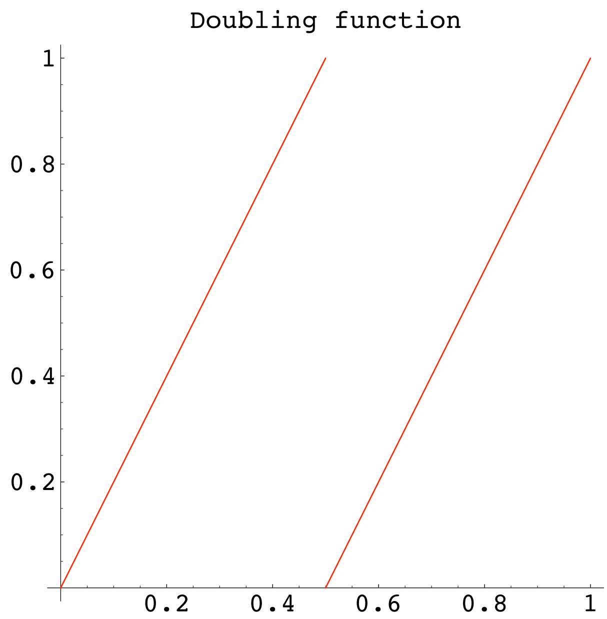
However, there is an example of a chaotic mapping which is easily shown to be chaotic. We will now discuss this mapping, called the **doubling mapping**.

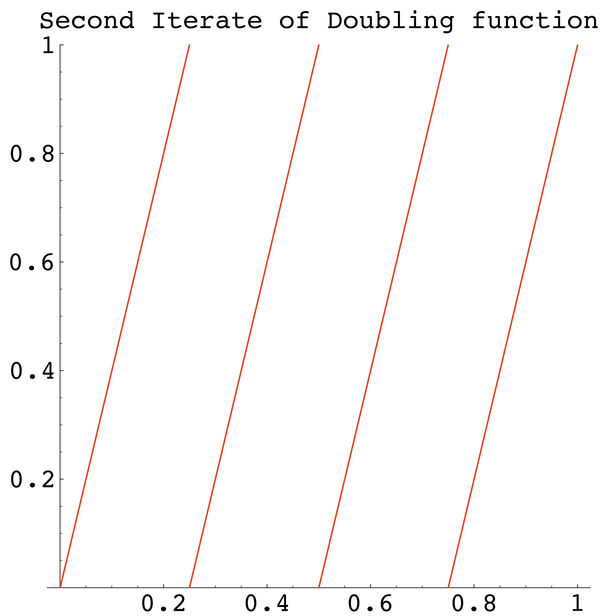
4.7 Doubling Mapping

The doubling mapping $D(x)$ is defined as follows

$$\begin{aligned} D_4(x) &= 2\,x \quad \text{if} \quad 0 \le x < 1/2, \\ &= 2\,x-1 \quad \text{if} \quad 1/2 \le x < 1, \end{aligned}$$

The graphs of $D(x)$ and its second iterate $D^2(x)$ are the following.





This function can be described in a convenient fashion if we represent numbers in the interval $[0, 1)$ in their binary representation.

If x_0 has binary representation

$$x_0 = 0.a_1a_2a_3a_4a_5\ldots$$

then

$$x_0 = \frac{a_1}{2^1} + \frac{a_2}{2^2} + \frac{a_3}{2^3} + \frac{a_4}{2^4} + \frac{a_5}{2^5} + \ldots$$

with each a_i equal to zero or one.

We now consider $D(x_0)$.

If $0 \ < \ x_0 \ < \ 1/2$ then $a_1 = 0$ so

$$\begin{aligned} x_0 &= 0.0a_2a_3a_4a_5\ldots \\ &= \frac{a_2}{2^2} + \frac{a_3}{2^3} + \frac{a_4}{2^4} + \frac{a_5}{2^5} + \ldots \end{aligned}$$

Thus

$$\begin{aligned} D(x_0) \ &= 2\,x \\ &= \frac{a_2}{2^1} + \frac{a_3}{2^2} + \frac{a_4}{2^3} + \frac{a_5}{2^4} + \ldots \\ &= 0.a_2a_3a_4a_5\ldots \end{aligned}$$

On the other hand, if $0.5 \ < \ x_0 \ < 1$, then $a_1 = 1$ so

$$x_0 = \frac{1}{2^1} + \frac{a_2}{2^2} + \frac{a_3}{2^3} + \frac{a_4}{2^4} + \frac{a_5}{2^5} + \ldots$$

and

$$\begin{aligned} D(x_0) \ &= 2\,x_0 - 1 \\ 1 - 1 &+ \frac{a_2}{2^1} + \frac{a_3}{2^2} + \frac{a_4}{2^3} + \frac{a_5}{2^4} + \ldots \\ &= 0.a_2a_3a_4a_5\ldots \end{aligned}$$

So in both cases ($a_0 = 1$ or $a_0 = 1$)

$$D(0.a_1a_2a_3a_4a_5\ldots) = 0.a_2a_3a_4a_5\ldots,$$

That is $D(x_0)$ simply drops or chops off the first digit in the binary expansion for x_0.

For example,

$$D(0.1001011\ldots) = 0.001011\ldots$$

and for the third iterate, $D^3(x)$

$$D^3(0.100100100\ldots) = 0.100100\ldots$$

Notation: We denote a repetition of a block in the binary expansion with an overbar. Thus

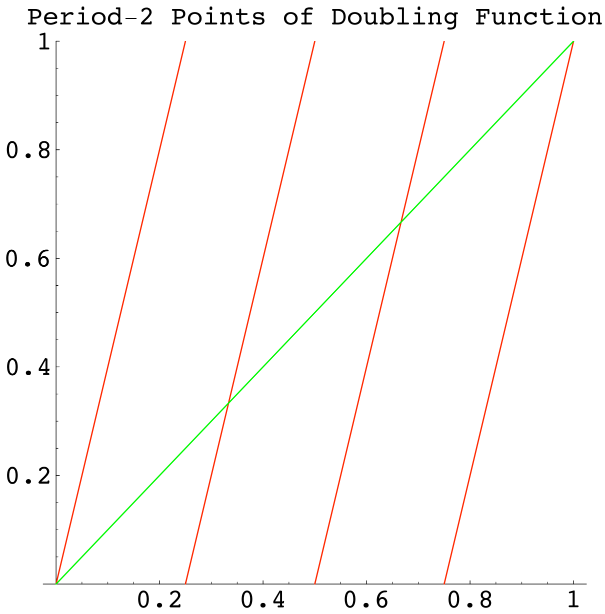
$$0.100100100100\ldots \ = \ 0.\overline{100},$$

$$0.10101010\ldots \ = \ 0.\overline{10}.$$

Now it is easy to write down the fixed points.

The fixed points are $0 = 0.\overline{0}$ and $1 = 0.\overline{1}$. The points of prime period-2 are

$$1/3 = 0.\overline{01} \qquad \text{and} \qquad 2/3 = 0.\overline{10}.$$



The points of prime period-3 are $0.\overline{010}$, $0.\overline{001}$, $0.\overline{011}$, $0.\overline{100}$, $0.\overline{101}$, $0.\overline{110}$.

The density of periodic points can now be seen as follows.
Suppose

$$x_0 = 0.a_1a_2a_3a_4\ldots$$

is any point of $[0,\,1)$, then the point

$$y = 0.\overline{a_1a_2a_3a_4\ldots a_n}$$

is a periodic point of period n and is within a distance $\frac{1}{2^n}$ of x_0.

Eventually periodic points are those whose binary expansion eventually repeats. So if

$$x_0 = 0.10010101111\ldots,$$

then x_0 is eventually fixed and $D^n(x_0) = 1$ for $n \geq 8$. If $y_0 = 0.010110\overline{10}$ then y_0 is eventually periodic with period-2 after 6 iterations.

Now consider two points x_0 and y_0 whose binary expansions are the same except in the $(n+1)^{th}$ position. Then

$$|x_0 - y_0| = \frac{1}{2^{n+1}},$$

and

$$|D^n(x_0) - D^n(y_0)| = \frac{1}{2}.$$

Thus we have sensitive dependence on initial conditions.

Finally we construct a seed x_0 whose orbit is dense. Consider

$$x_0 = 0.0100011011000100\ldots$$

where we have the 1-blocks "0", "1" at the beginning; next we have all possible 2-blocks, "00", "01", "10", and "11"; next we have all possible 3-blocks "000", "001", "100", "101", "010", "011", "110", "111", and so on.

Now we claim that the orbit of this point is dense. Consider a point $y = 0.y_1y_2y_3y_4y_5\ldots \in [0,\,1)$. The iteration of the mapping D chops off binary digits at the front of the expansion of x_0 so after a sufficient number of D applications, say k iterations, then the first n of what is left will coincide with the first n of the binary expansion of y. This means that $D^k(x_0)$ and y coincide in the first n binary digits. Thus the distance between $x_k = D^k(x_0)$ and y_0 is less than $\frac{1}{2^n}$. Now taking n sufficiently large, we get a point of the orbit of x_0 which is arbitrarily close to the point y_0. So we conclude that the orbit of x_0 is dense in $[0,\,1)$.

We have shown that the mapping $D(x)$ defined on the interval $[0,\,1)$ has the three properties required of a chaotic mapping.

Complex Dynamical Systems

1 Algebra and Geometry of Complex Numbers

A complex number is a number of the form

$$x + \imath\, y$$

where $\imath = \sqrt{-1}$. We call x the **real part** and y the **imaginary part**. Write $x = Re(z)$ and $y = Im(z)$ of $z = x + \imath\, y$. The **modulus** of $z = x + \imath\, y$ is $\sqrt{x^2 + y^2}$ and is denoted by $|z|$. The set of all complex numbers is denoted by **C**.

Addition and multiplication of complex numbers are defined in the natural way, remembering that $\imath\,\imath = \imath^2 = -1$.

$$(x + \imath\, y) + (u + \imath\, v) = (x + u) + \imath\,(y + v),$$

$$(x + \imath\, y)\,(u + \imath\, v) = (x\,u - y\,v) + \imath\,(x\,v + y\,u).$$

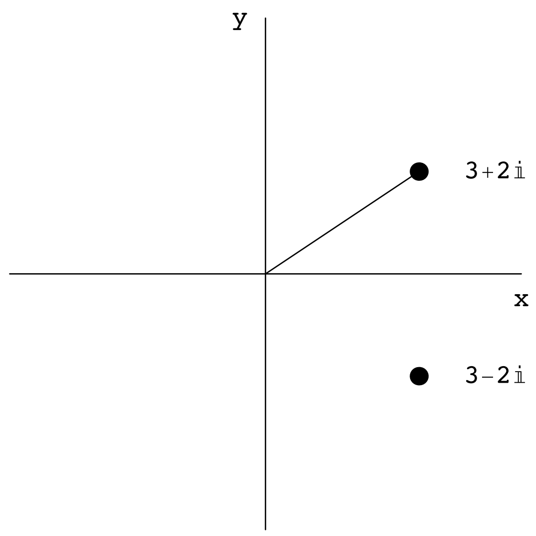
The complex conjugate $\overline{z}$ of $z = x + \imath\, y$ is given by $\overline{z} = x - \imath\, y$. Note that if we conjugate twice, we arrive back at the original complex number, $z = \overline{\overline{z}}$. Furthermore, we can extract the real and imaginary parts of a complex number by the use of conjugation:

$$2\,Re(z) = z + \overline{z}, \qquad 2\,Im(z) = z - \overline{z}.$$

Finally, the operation of taking complex conjugation commutes with the operations of addition and multiplication;

$$\overline{(w + z)} = \overline{w} + \overline{z} \qquad \overline{(w\,z)} = \overline{w}\,\overline{z}.$$

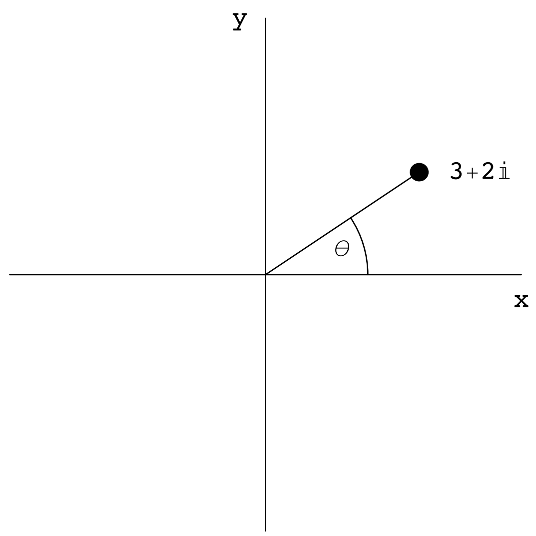
Complex numbers may be depicted geometrically as points in the plane. We simply place the complex number $z = x + \imath\, y$ at the point $(x,\, y)$ in the Cartesian plane. Thus $3 + 2\,\imath$ is placed at the point $(3,\, 2)$ and the complex conjugate $3 - 2\,\imath$ is placed at the point $(3,\, -2)$.



We call the plane with points identified as complex numbers the **Argand plane**.

An alternative method of describing points in the complex plane is the **polar representation** of a complex number. Given a complex number $z = x + \imath y$, its polar representation is determined by the modulus $|z| = \sqrt{x^2 + y^2}$, which represents the distance from the origin to the point z, and the **polar angle** of z, which is the angle between the positive x-axis and the ray from O, the origin, to z, measured in the counterclockwise direction.

We write $r = |z|$ and θ for the polar angle.



The polar representation of complex numbers makes it particularly easy to visualize multiplication of complex numbers. If $z = r\,(\cos(\theta) + \imath\,\sin(\theta))$ and $w = \rho(\cos(\phi) + \imath\,\sin(\phi))$ then

$$z\,w = r\,\rho(\cos(\theta + \phi) + \sin(\theta + \phi)).$$

So the modulus of the product is the product of the moduli, and the argument of the product is the *sum* of the arguments.

The exponential form of a complex number uses the fact that

$$e^{\imath\theta} = \cos(\theta) + \imath\,\sin(\theta).$$

There are a few different ways of understanding how one can arrive at this equation. Each of them relies on some knowledge from other areas of mathematics. Perhaps the simplest way is to write down the Taylor series for the three functions involved. We will not go into the details here. Using the exponential form, we can write

$$z = r\,(\cos(\theta) + \imath\,\sin(\theta)) = r\,e^{\imath\theta}.$$

Then, if $z = r\,e^{\imath\theta}$ and $w = \rho\,e^{\imath\phi}$ then

$$z\,w = r\,e^{\imath\theta}\,\rho\,e^{\imath\phi} = r\,\rho\,e^{\imath(\theta+\phi)}.$$

Thus, to multiply complex numbers, we simply multiply their moduli and add their polar angles.

Division of complex numbers is defined in a roundabout way. First, if $z = x + \imath\,y$, then $z\,\overline{z} = x^2 + y^2 = |z|^2$, so $z\,\overline{z}$ is always a non-negative real number. Then, given two complex numbers z and w, with $z \neq 0$, we get

$$\frac{w}{z} = \frac{w\,\overline{z}}{z\,\overline{z}} = \frac{w\,\overline{z}}{|z|^2}.$$

In exponential form, if $z = r\,e^{\imath\theta}$, then $\overline{z} = r\,e^{-\imath\theta}$ and so, with $w = \rho\,e^{\imath\phi}$ then

$$\frac{w}{z} = \frac{\rho}{r}\,e^{\imath(\phi-\theta)}.$$

This helps us visualize division of complex numbers.

1.1 Dynamics of Linear Complex functions

We now describe the dynamics of the simplest complex functions, namely, functions of the form

$$L_\alpha(z) = \alpha\,z,$$

where α is a complex number.

Clearly L_α has a fixed point at $z = 0$. We now describe the orbits of nonzero points under the iteration of L_α. So we consider sets of the form

$$\{z_0,\ z_1 = L_\alpha\,z_0 = \alpha\,z_0,\ z_2 = {L_\alpha}^2\,z_0 = \alpha^2\,z_0, \ldots, z_n = \alpha^n\,z_0, \ldots\}$$

Suppose $\alpha = \rho\,e^{\imath\psi}$ and $z_0 = r\,e^{\imath\theta}$. Then

$$\begin{aligned}
z_1 &= \rho\,e^{\imath\psi}\,r\,e^{\imath\theta} &&= \rho\,r\,e^{\imath(\psi+\theta)}\\
z_2 &= \rho^2\,e^{\imath 2\psi}\,r\,e^{\imath\theta} &&= \rho^2\,r\,e^{\imath(2\psi+\theta)}\\
z_3 &= \rho^3\,e^{\imath 3\psi}\,r\,e^{\imath\theta} &&= \rho^3\,r\,e^{\imath(3\psi+\theta)}
\end{aligned}$$

and, in general

$$z_n \;=\; \rho^n\, e^{\imath\, n\,\psi}\, r\, e^{\imath\,\theta} \;=\rho^n\, r\, e^{\imath(n\,\psi+\theta)}.$$

There are three cases to consider.

- $\rho \;<\; 1,$
- $\rho \;>\; 1,$
- $\rho \;=\; 1.$

Case $\rho \;<\; 1.$

If $\rho \;<\; 1,$ then $\rho^n \longrightarrow 0$ as $n \longrightarrow \infty$. Since $|e^{\imath\,(n\,\psi+\theta)}| = 1$ for all n, then $|z_n| = |\rho^n\, r\, e^{\imath\,(n\,\psi+\theta)}| = \rho^n\, r \longrightarrow 0$ as $n \longrightarrow \infty$, it follows that all the orbits of L_α tend to zero. Thus $z = 0$ is an attracting fixed point.

Case $\rho \;>\; 1.$

Since $\rho^n \longrightarrow \infty$ as $n \longrightarrow \infty$, all the orbits tend to infinity. Thus $z = 0$ is a repelling point.

REMARK. Note that the argument $n\,\psi + \theta$ of ${L_\alpha}^n(z_0) = z_n$ changes as n increases, provided $\psi \neq 0$. This means that the orbits spiral into or away from the origin according as $z = 0$ is attracting or repelling.

Case $\rho = 1.$

There are two important subcases depending on the argument ψ of α. Let's write ψ in the form

$$\psi = 2\,\pi\,\tau.$$

The two subcases are

i) τ rational,

ii) τ irrational.

Case $\rho = 1,$ and τ is rational :

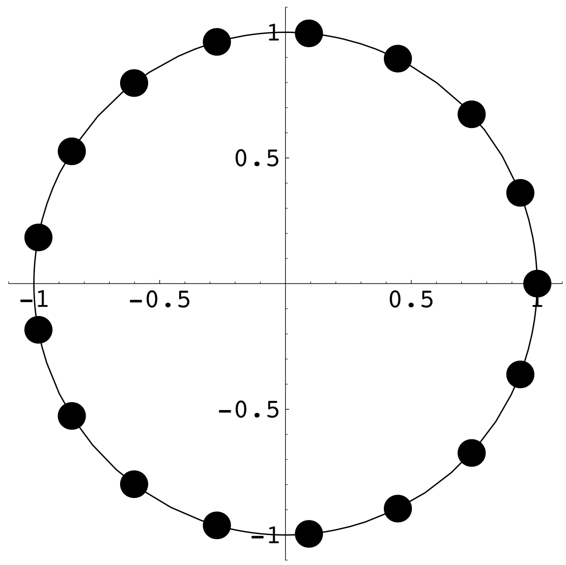
Let $\tau = \frac{p}{q}$ where $p,\, q \in \mathbf{Z}$, the set of all integers.

Then

$$\begin{aligned} {L_\alpha}^q(z_0) &= {L_\alpha}^q(r\, e^{\imath\,\theta}) \\ &= r\, e^{(2\,\pi\,\imath\, p/q)\, q + \imath\,\theta} \\ &= r\, e^{2\,\pi\,\imath\, p + \imath\,\theta} \\ &= r\, e^{\imath\,\theta}\, e^{2\,\pi\,\imath\, p} \\ &= r\, e^{\imath\,\theta} \\ &= z_0. \end{aligned}$$

Thus every point $z_0 \;\neq\; 0$ is periodic, with period q, under the action of L_α. Note that all points on the orbit of z_0 lie on the circle centered at $z = 0$ and with radius $|z_0|$.

In this case, L_α is called a **rational rotation**. Below is a graph of the orbit in the case that $q = 17$, and $r = 1$ so we see an orbit of period 17 on the circle with centre at $z = 0$ and radius 1.



Case $\rho = 1$, and τ is irrational :

Now L_α is multiplication by $e^{2\pi\imath\tau}$. Then if $z_0 = r\,e^{\imath\theta}$, there are no non-zero periodic points. We can show this as follows.

Suppose that for some k, if ${L_\alpha}^k(z_0) = z_0$, then $r\,e^{\imath\theta} = {L_\alpha}^k(r\,e^{\imath\theta}) = r\,e^{2\pi\imath\tau k + \imath\theta} \;=\; r\,e^{\imath(2\pi\tau k+\theta)}$. A periodic point would imply that for some integer m, we must have

$$\theta + 2\pi m = \theta + 2\pi k\tau.$$

Hence $\tau = \frac{m}{k}$, a contradiction, since τ is irrational.

We can prove more about the orbits of L_α in this case of τ being irrational.

Let $C_{|z_0|}$ be the circle with centre at $z = 0$ and radius $r = |z_0|$. A set S of $C_{|z_0|}$ is said to be **dense** in $C_{|z_0|}$ if every subarc of length ε, for every positive ε, contains a point of S. [This means that if the set S is plotted on a computer screen, the set S is indistinguishable from the circle $C_{|z_0|}$, no matter how fine the screen resolution is.] We now claim that when τ is irrational, the orbits are dense in the circle $C_{|z_0|}$. Choose any $\varepsilon > 0$ and then choose an integer k such that

$$k \;>\; \frac{2\pi\,|z_0|}{\varepsilon}.$$

The points of the orbit of z_0 under the mapping L_α lie on $C_{|z_0|}$. They are all distinct. (Why?) The circumference of the circle is $2\pi\,|z_0|$. Since

$$\frac{2\pi\,|z_0|}{k} \;<\; \varepsilon,$$

then in the first k points of the orbit, i.e. the part $\{z_0,\, z_1,\, z_2,\, z_3,\, \ldots, z_k\}$, there are at least two points which are closer than ε. Suppose these points are z_j and z_l. Then the arc between z_j and z_k has length less that ε and we assume that $j \;>\; l$. Consider now the function ${L_\alpha}^{j-l}$. We have

$${L_\alpha}^{j-l}(z_0) = e^{2\pi\imath\tau(j-l)}\,z_0,$$

so ${L_\alpha}^{j-l}$ simply rotates points through an angle $2\,\pi\,\tau\,(j-l)$.

Since

$$\begin{aligned} {L_\alpha}^{j-l}(z_l) &= {L_\alpha}^{j-l}(\,{L_\alpha}^{l}(z_0) \\ &= {L_\alpha}^{j}(z_0) \\ &= z_j, \end{aligned}$$

it follows that this function rotates points on the circle a distance smaller than ε. Hence the points

$${L_\alpha}^{j-l}(z_0),\ ({L_\alpha}^{j-l})^2(z_0) = {L_\alpha}^{2\,(j-l)}(z_0),\ \ldots\ {L_\alpha}^{n(j-l)}(z_0),\ \ldots$$

are arranged around the circle with the distance between successive points no larger that ε. It follows that the orbit of z_0 must enter any subarc whose length is less than ε. Since ε was arbitrary, it follows that the orbit of z_0 is dense in the circle $C_{|z_0|}$.

We may therefore summarise the dynamics of the linear complex mappings as follow.

PROPOSITION

Suppose that the mapping $L_\alpha(z)$ is given by $L_\alpha(z) = \alpha\,z$, where $\alpha = \rho\,e^{2\,\pi\,\imath\,\tau}$.

Then,

1 If $\rho < 1$, all orbits tend to the attracting fixed point $z = 0$.

2 If $\rho > 1$, all orbits tend to infinity except that of $z = 0$, which is a repelling fixed point.

3 When $\rho = 1$;

(i) If τ is rational, all orbits are periodic.

(ii) If τ is irrational, each orbit is dense on a circle centered at $z = 0$.

2 Julia Sets

Gaston Julia (1893 - 1978) was only 25 when he published his 199 page paper which made him famous in the mathematics centres of his day. As a French soldier in the First World War, Julia had been severely wounded, as a result of which he lost his nose. Between several painful operations, he carried on his mathematical research in a hospital. Later he became a distinguished professor at the Ecole Polytechnique in Paris.

Although Julia was a world famous mathematician in the 1920's, his work was essentially forgotten until B. Mandelbrot brought it back to light at the end of the 1970's through fundamental computer experiments. With the aid of the computer, Mandelbrot showed us that Julia's work is a source of some of the most beautiful fractals known today.

We will now study the Julia sets of complex functions. As we will see, the Julia set is the place where all the chaotic behaviour of a complex valued function occurs. We will consider only some quadratic functions of the form

$$Q_c(z) = z^2 + c.$$

2.1 The Squaring function

We begin with the squaring function

$$Q_0(z) = z^2.$$

As before, write our initial point z_0 as $z_0 = r\,e^{\imath\,\theta}$. Then the orbit under Q_0 is given by

$$\begin{aligned} z_0 &= r\,e^{\imath\,\theta} \\ z_1 &= r^2\,e^{\imath\,(2\,\theta)} \\ z_2 &= r^4\,e^{\imath\,(4\,\theta)} \end{aligned}$$

and, in general,

$$z_n = r^{2^n}\,e^{\imath\,(2^n\,\theta)}.$$

There are three possible fates for an orbit of z_0. If $r < 1$, we have

$$r^{2^n} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

Hence $|{Q_0}^n(z_0)| \longrightarrow 0$ as $n \longrightarrow \infty$. We have $Q_0(0) = 0$ and so $z = 0$ is an attracting fixed point, if $r < 1$.

If $r > 1$, then $r^{2^n} \longrightarrow \infty$ so $|{Q_0}^n(z_0)| \longrightarrow \infty$ as $n \longrightarrow \infty$. In this case, $z = 0$ is a repelling fixed point.

Finally, we consider the intermediate case when $r = 1$, that is when $|z_0| = 1$. Then $|Q_0(z_0)| = 1$ also, so Q_0 preserves the unit circle, $\{z : |z| = 1\}$, in the sense that the image of any point on the unit circle is again a point on the unit circle. We now prove some properties of the squaring mapping on the unit circle.

If $z_0 = e^{\imath\,\theta}$, then $Q_0(z_0) = e^{\imath(2\,\theta)}$, so Q_0 doubles angles on the unit circle.

First we show that the periodic points are dense on the unit circle. To do this, we must produce a periodic point in any arc of the form $\theta_1 < \theta < \theta_2$.

This means that we must find an n and a θ so that

$${Q_0}^n(e^{\imath\,\theta}) = e^{\imath\,\theta},$$

with $\theta_1 < \theta < \theta_2$. But

$${Q_0}^n(e^{\imath\,\theta}) = e^{\imath\,2^n\,\theta},$$

so we must solve

$$e^{\imath\,2^n\,\theta} = e^{\imath\,\theta}.$$

The angle θ solves this equation provided that

$$2^n\,\theta = \theta + 2\,k\,\pi$$

for some integers k, n. That is θ must satisfy

$$\theta = \frac{2\,k\,\pi}{2^n - 1}.$$

If we fix n and let k be an integer with $0 \leq k < 2^n - 1$, then the complex numbers with arguments

$$\frac{2\,k\,\pi}{2^n - 1}$$

are evenly distributed around the circle with arc length $2\,\pi/(2^n - 1)$ between successive points. If we now choose n so that

$$\frac{2\,\pi}{2^n - 1} < \theta_2 - \theta_1$$

we guarantee that there is at least one point with argument

$$\frac{2\,k\,\pi}{2^n - 1}$$

between θ_1 and θ_2. This point is therefore periodic with period n. Thus the periodic points are dense on the unit circle.

Next consider a seed

$$z_0 = e^{2\,\pi\,\imath\,\tau}$$

where τ is irrational. Using methods similar to those we used earlier for complex linear mappings, we can show that this seed has a dense orbit under the doubling mapping.

Finally, if we take an arc $\theta_1 < \theta < \theta_2$, its image is double the length. So after several iterations, the full circle is covered by ${Q_0}^n$ for some n. Thus we can find pairs of points arbitrarily close which are eventually mapped to diametrically opposite points on the circle. Thus the system has sensitive dependence on initial conditions.

So we have the following result.

THEOREM

The behaviour of the squaring mapping $Q_0(z) = z^2$ is as follows. It is

- a chaotic mapping on the unit circle.
- if $|z| < 1$, then $|Q_0(z)| \longrightarrow 0$ as $n \longrightarrow \infty$.
- if $|z| > 1$, then $|{Q_0}^n(z)| \longrightarrow \infty$ as $n \longrightarrow \infty$.

2.2 Examples of Julia Sets

An orbit of z under Q_0 is **bounded** if there exists some constant K such that

$$|{Q_0}^n(z)| < K,$$

for all n. Otherwise, the orbit is unbounded.

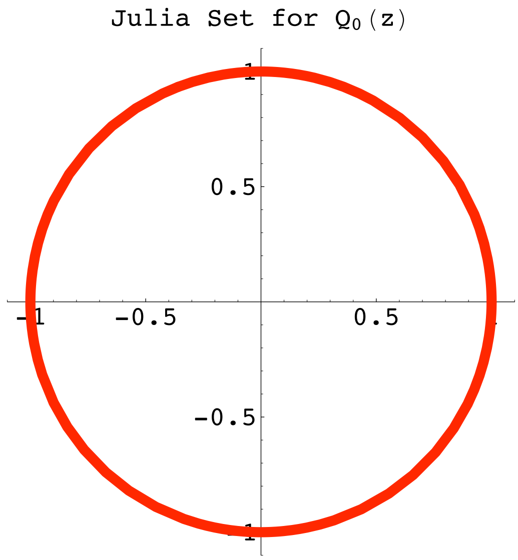
The orbit of any seed point inside, and on, the unit circle is bounded under Q_0; points outside the unit circle have unbounded orbits.

We now define **Julia Sets** of the quadratic functions $Q_c(z)$ given by

$$Q_c(z) = z^2 + c.$$

The **filled-in Julia set** K_c of the quadratic function Q_c is the set of all points whose orbits are bounded. The **Julia set** J_c of Q_c is the boundary of the filled-in Julia set, that is the points such that every neighbourhood contains points of the filled-in Julia set K_c and points which do not belong to the filled-in Julia set.

Thus the Julia set of $Q_0(z)$ is the unit circle.



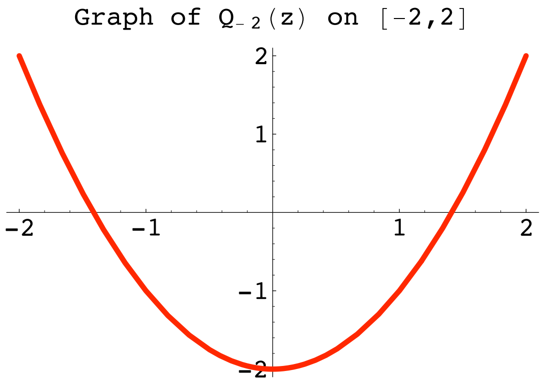
2.3 Quadratic function Q_{-2}.

We now consider a second quadratic complex function

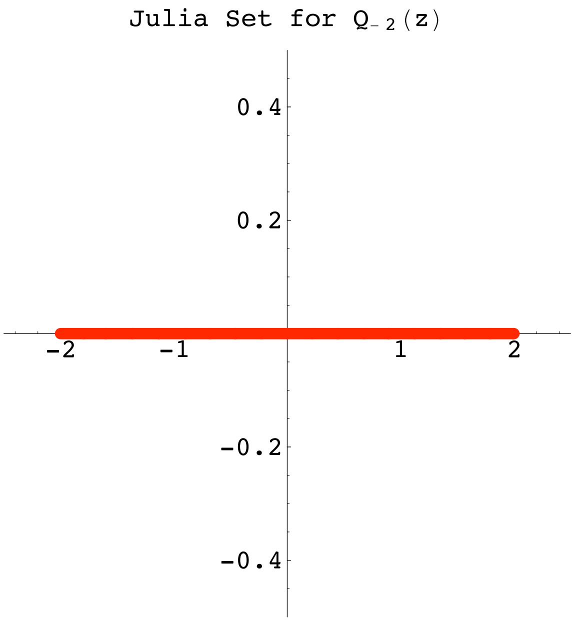
$$Q_{-2}(z) = z^2 - 2.$$

First consider the points in the closed interval $[-2, 2]$ of the real axis. Then Q_{-2} maps $[-2, 2]$ onto itself. To see this, let the points on the interval $[-2, 2]$ be denoted by α. Then if $-2 \le \alpha \le 2$, we have $0 \le \alpha^2 \le 4$ and $-2 \le \alpha^2 - 2 \le 2$, so ${Q_{-2}}^{-1}([-2, 2]) \subset [-2, 2]$. Clearly, the image of any point z of the interval $[-2, 2]$ is a point of $[-2, 2]$, that is $Q_{-2}([-2, 2]) \subset [-2, 2]$. Thus $Q_{-2}([-2, 2]) = [-2, 2]$.

The graph of $Q_{-2}(z)$ is the following.



From this we conclude that $[-2, 2]$ is invariant under Q_{-2}. Thus the filled in Julia set K_{-2} contains $[-2, 2]$ since the orbit of any point of this set remains in the set and so is bounded. We now show that K_{-2} is in fact $[-2, 2]$ by proving that the orbit of any complex number, not in this portion of the real axis, has unbounded orbit. Once this is proved, then since the boundary of the set $K_{-2} = [-2, 2]$ is the set itself, then the Julia set J_{-2} is also the interval $[-2, 2]$.



To prove our claim that the points in $\mathbf{C} \setminus [-2, 2]$ have unbounded orbits, we first set up a mapping H between the region R outside the unit disk in the complex plane and the set $\mathbf{C} \setminus [-2, 2]$. The set R is the set

$$R = \{z : |z| > 1\},$$

and we take a mapping

$$H(z) : R \longrightarrow \mathbf{C} \setminus [-2, 2],$$

defined by

$$H(z) = z + \frac{1}{z}.$$

We prove the following three properties of the mapping H :

(i) $H(z)$ is one-to-one,

(ii) $H(z) \in \mathbf{C} \setminus [-2,\, 2]$, if $z \in R$,

(iii) If $w \in \mathbf{C} \setminus [-2,\, 2]$, then $w = H(z)$ for some $z \in R$.

So $H(z) : R \longrightarrow \mathbf{C} \setminus [-2,\, 2]$ is a bijection.

Proof of (i):

Suppose z_1 and $z_2 \in R$ and $H(z_1) = H(z_2)$. Then

$$z_1 + \frac{1}{z_1} = z_2 + \frac{1}{z_2}$$

and so, with a little algebra, we get

$$z_1\, z_2 = 1.$$

If $z_1 \in R$, then $|z_2| = \frac{1}{|z_1|} < 1$ and so $z_2 \notin R$. Thus there is at most one point of R which is mapped by H to a point in $\mathbf{C} \setminus [-2,\, 2]$.

Proof of (ii): Next, suppose if possible that $H(z) \in [-2,\, 2]$ for some $z \in R$, that is for some z with $|z| > 1$.

If $H(z) \in [-2,\, 2]$, suppose that $H(z) = \alpha$ for some $\alpha \in [-2,\, 2]$. Thus $z + 1/z = \alpha$ where $\alpha^2 \leq 4$. Thus $z^2 + 1 = \alpha\, z$ and so

$$z = \frac{\alpha \pm \sqrt{\alpha^2 - 4}}{2} = \frac{\alpha}{2} \pm \imath \frac{\sqrt{4 - \alpha^2}}{2}.$$

Then

$$|z| = \sqrt{\left(\frac{\alpha}{2}\right)^2 + \left(\frac{\sqrt{4-\alpha^2}}{2}\right)^2} = \sqrt{1} = 1.$$

This is a contradiction, since $|z| > 1$. So we conclude that $H(z) \in \mathbf{C} \setminus [-2,\, 2]$ for $z \in R$.

Proof of (iii):

To show that $H(z)$ maps R *onto* $\mathbf{C} \setminus [-2,\, 2]$, let $w \in \mathbf{C} \setminus [-2,\, 2]$. Solve $H(z) = w$ to get two solutions

$$z = \frac{w}{2} \pm \frac{\sqrt{w^2 - 4}}{2}.$$

The product of both of these roots is 1 so exactly one of them lies in R or, both lie on the unit circle. In the latter case, it is easy to check that the value of $H(z_1) = H(z_2)$ belongs to $[-2,\, 2]$, where z_1 and z_2 are the roots. So we conclude that $H(z)$ maps R onto $\mathbf{C} \setminus [-2,\, 2]$ in a one-to-one fashion.

Next, observe that the mappings $H(z)$, $Q_0(z)$ and $Q_{-2}(z)$ are related as follows:

$$H \circ Q_0(z) = Q_{-2} \circ H(z).$$

To check this:

$$H \circ Q_0(z) = H(z^2) = z^2 + \frac{1}{z^2},$$

and

$$\begin{aligned}
Q_{-2} \circ H(z) &= Q_{-2}\left(z + \frac{1}{z}\right) \\
&= \left(z + \frac{1}{z}\right)^2 - 2 \\
&= z^2 + \frac{1}{z^2} + 2 - 2 \\
&= z^2 + \frac{1}{z^2} \\
&= H \circ Q_0(z)
\end{aligned}$$

The idea now is to use this property to establish a one-to-one correspondence between orbits of $Q_{-2}(z)$ on $\mathbf{C} \setminus [-2, 2]$ and orbits of $Q_0(z)$ on R. We know that the orbits of $Q_0(z)$ on R tend to infinity and so it follows that orbits of $Q_{-2}(z)$ on $\mathbf{C} \setminus [-2, 2]$ will also tend to infinity.

The correspondence between the orbits follows from showing that for any positive integer n,

$$Q_{-2}{}^n \circ H(z) = H \circ Q_0{}^n(z).$$

We prove this by induction. The case of $n = 1$ is the equation proved above. Now suppose that for some positive integer k

$$Q_{-2}{}^k \circ H(z) = H \circ Q_0{}^k(z).$$

Then

$$\begin{aligned}
Q_{-2}{}^{k+1} \circ H(z) &= Q_{-2}\left(Q_{-2}{}^k \circ H(z)\right) \\
&= Q_{-2} \circ \left(H \circ Q_0{}^k(z)\right), \quad \text{by induction hypothesis} \\
&= (Q_{-2} \circ H) \circ Q_0{}^k(z) \\
&= (H \circ Q_0) \circ Q_0{}^k(z), \\
&= H \circ Q_0{}^{k+1}(z).
\end{aligned}$$

Result now follows by induction.

We now show that all the orbits in $\mathbf{C} \setminus [-2, 2]$ under $Q_{-2}(z)$ tend to ∞. Let $w_0 \in \mathbf{C} \setminus [-2, 2]$ and let z_0 be the point $z_0 = H^{-1}(w_0)$. Then since $|z_0| > 1$, the orbit of z_0 under $Q_0(z)$ tends to ∞. Thus $Q_0{}^n(z_0) \longrightarrow \infty$. But

$$\begin{aligned}
w_n &= Q_{-2}{}^n(w_0) \\
&= Q_{-2}{}^n \circ H(z_0) \\
&= H \circ Q_0{}^n(z_0) \\
&= H(z_n) \\
&= z_n + \frac{1}{z_n} \\
&\longrightarrow \infty,
\end{aligned}$$

since $z_n \longrightarrow \infty$ and $1/z_n \longrightarrow 0$.

Thus all orbits in $\mathbf{C} \setminus [-2, 2]$ are unbounded.

We already saw that orbits of points in $[-2, 2]$ are bounded under $Q_{-2}(z)$. Thus

$$\begin{aligned} K_{-2} &= [-2, 2], && \text{the filled in Julia set, and} \\ J_{-2} &= [-2, 2], && \text{the Julia set.} \end{aligned}$$

3 The Mandelbrot Set

The filled in Julia sets for the family of quadratic mappings

$$Q_c : \quad z \longrightarrow z^2 + c$$

are very varied. We have seen that for the cases corresponding to $c = 0$ and $c = -2$ the filled in Julia sets are

$$\begin{aligned} K_0 &= \{z : |z| \leq 1\}, && \text{and} \\ K_{-2} &= \{z : z \text{ is real and } -2 \leq z \leq 2\}. \end{aligned}$$

For every value of $c \in \mathbf{C}$, there is a unique K_c. Drawing them and studying them requires a deep knowledge of complex variable theory. The filled in Julia sets seem to fall into one of two types, those that are connected (one piece) and those that are totally disconnected (or a dust). This dichotomy follows from the mathematical works of G. Julia and Pierre Fatou (1878 - 1929).

Around 1979, Benoit Mandelbrot had the idea of making a picture of this dichotomy in the complex plane. This led to the investigations of what we now call the **Mandelbrot set** which is

$$\mathcal{M} = \{c \in \mathbf{C} : \text{ the Julia set } J_c \text{ in connected.}\}$$

He coloured each point (pixel on a computer screen) in the plane black or white depending on whether the associated Julia set turned out to be connected or disconnected.

But how did he actually let the computer make the decision whether a parameter c belongs to the Mandelbrot set or not. He was one of the few people at that time who knew about the works of Julia and Fatou. In particular, he was aware of the fact that there is a tight interrelation between the dichotomy of Julia sets and the fate of the orbit of the point $z = 0$. He knew that

the filled Julia set K_c is connected if and only if the orbit of $z = 0$

$$0 \longrightarrow c \longrightarrow c^2 + c \longrightarrow (c^2 + c)^2 + c \longrightarrow \cdots$$

is bounded.

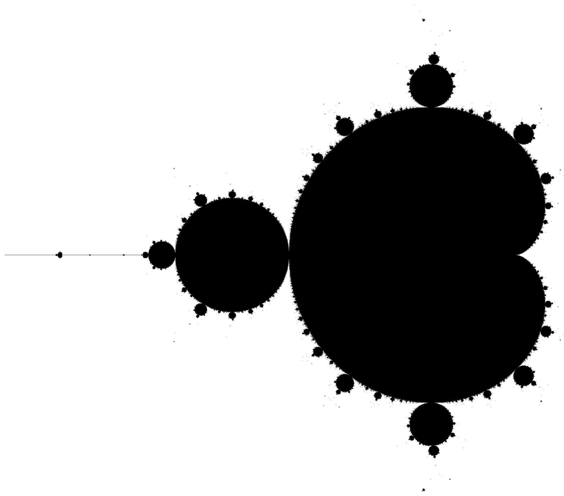
This fact provides an alternative definition for the Mandelbrot set. In 1979, Mandelbrot used the following as the definition of $\mathcal{M}$ in his computer experiments.

$$\mathcal{M} = \{z \in \mathbf{C} : c \longrightarrow c^2 + c \longrightarrow (c^2+c)^2 + c \longrightarrow \cdots \text{ remains bounded.}\}$$

It can be shown that everything outside the disk $\{z : |z| \leq 2\}$ is not part of the Mandelbrot set. From this we get an algorithm for computing the Mandelbrot set. We consider the square grid of points in the complex plane which is centred at the origin and bounded by the lines $x = \pm 2$ and $y = \pm 2$. For each point c in this grid, we compute the corresponding orbit of 0 under Q_c and ask whether or not this orbit tends to infinity. If the orbit does not escape, then our original point is in $\mathcal{M}$, so we will colour it black. If the orbit escapes, then we leave the original point white.

Algorithm for the Mandelbrot Set.

Choose a maximum number of iterations, N. For each point c in a grid, compute the first N points of the orbit of 0 *under Q_c. If $|{Q_c}^i(0)| > 2$ for some $i \leq N$ then stop iterating and colour c white. If $|{Q_c}^i(0)| \leq 2$ for all $i \leq N$, then colour c black.*



The striking feature of the Mandelbrot set are the buds which are lined up along the big, heart-shaped, central region. These buds have a meaning for the associated Julia sets. To examine this we need to discuss attractive and repelling fixed points and periodic points.

3.1 Fixed and Periodic Points

Suppose p is a fixed point for $Q_c(z)$, that is $Q_c(p) = p$. Then p is an **attracting** fixed point if there is a disk D of the form

$$D = \{z : |z - p| < \delta\}$$

about p in which the following condition is satisfied. If $z \in D$, then $Q_c{}^n(z) \longrightarrow p$ as $n \longrightarrow \infty$.
The point p is a **repelling** fixed point if there is a disk D of the form

$$D = \{z : |z - p| \ < \ \delta\}$$

about p in which the following condition is satisfied. If $z \in D$ but $z \neq p$, then there is an integer $n \ > \ 0$ such that $Q_c{}^n(z) \notin D$.

EXAMPLE
If $c = 0$, so $Q_c(z) = z^2$, then $z = 0$ is an attracting fixed point.

There is a simple test to decide whether a fixed point is attracting or repelling. The test involves evaluating the derivative of the mapping at the fixed point. We are interested in the family of functions $Q_c(z) = z^2 + c$ and the derivative of these functions is computed in the same way as for real functions;

$$\frac{d}{d\,z}\,(Q_c(z)) = 2\,z.$$

Given this, the fixed point can be classified as follows:

- it is **attractive** if the absolute value of the derivative at the fixed point is less that 1;
- it is **repelling** if the absolute value of the derivative at the fixed point is greater that 1;
- it is **indifferent** if the absolute value of the derivative at the fixed point equals 1.

EXAMPLES

1. If $Q_0(z) = z^2$, then
$$Q_0{}'(z) = 2\,z.$$
The point $z = 0$ is an attractive fixed point.

2. If $c = -0.5 + 0.5\,\imath$, then the fixed points of $Q_c(z)$ are $z_1 = 1.408 - 0.275\,\imath$ and $z_2 = -0.408 + 0.275\,\imath$. Then $|2\,z| = 2.869$ and $|2\,z_2| = 0.984$. Thus z_1 is a repelling fixed point and z_2 is an attracting fixed point.

3. If $c = 1$, then $Q_1(z) = z^2 + 1$. The fixed points are solutions of $z = z^2 + 1$ which has solutions
$$z = \frac{1 \ \pm \ \imath\sqrt{3}}{2}.$$
Both of these fixed points are repelling as
$$\left|Q_1{}'\left(\frac{1 \pm \imath\sqrt{3}}{2}\right)\right| = |1 \pm \imath\sqrt{3}| = 2.$$

The derivative criterion for repelling and attracting fixed points can be generalised to periodic points. Recall that a point q_0 is **periodic** for $Q_c(z)$ with period k if

$$Q_c{}^k(q_0) = q_0.$$

The orbit of q_0, that is,

$$\{q_0,\ Q_c(q_0),\ \dots,\ Q_c{}^k(q_0)\}$$

is called a **cycle**. The **multiplier** ρ of the cycle is the derivative of $Q_c{}^k(z)$ at q_0; using the chain rule,

$$\rho \;=\; \frac{d}{d\,z}\left(Q_c{}^k(z)\right)|_{z=q_0} \;=\; Q_c{}'(q_0).Q_c{}'(q_1).\dots Q_c{}'(q_{k-1}),$$

where $q_j = Q_c{}^j(q)$ for $j = 1, 2, \dots, k-1$. For example if q_0 has period 3,

$$Q_c{}^3(q_0) \;=\; q_0$$

that is

$$Q_c\left(Q_c\left(Q_c(q_0)\right)\right) \;=\; q_0.$$

Differentiate the left hand side to evaluate the multiplier ρ;

$$\begin{aligned}\frac{d}{d\,z}\left(Q_c{}^3(z)\right)|_{z=q_0} &= \frac{d}{d\,z}\,Q_c\left(Q_c\left(Q_c(z)\right)\right)|_{z=q_0}\\ &= Q_c{}'\left(Q_c{}^2(q_0)\right).Q_c{}'\left(Q(q_0)\right).Q_c{}'(q_0)\\ &= Q_c{}'(q_2).Q_c{}'(q_1).Q_c{}'(q_0).\end{aligned}$$

We call a cycle attracting if $|\rho| \;<\; 1$, and repelling if $|\rho| \;>\; 1$.

EXAMPLE

Let $c = -1$, so we consider the mapping $Q_{-1}(z) = z^2 - 1$. then

$$Q_{-1}{}^2(z) = \left(z^2-1\right)^2 - 1 = z^4 - 2\,z^2.$$

If $Q_{-1}{}^2(z) = z$, then

$$z^4 - 2\,z^2 = z.$$

Solving, we get that the roots are

$$0, \quad -1, \quad \frac{1+\sqrt{5}}{2}, \quad \frac{1-\sqrt{5}}{2}.$$

Check that the roots $\frac{1\pm\sqrt{5}}{2}$ are repelling.

The orbit of 0 is $\{0,\ 1\}$, a 2-cycle. Then the multiplier is

$$Q_{-1}{}'(0).Q_{-1}{}'(-1) \;=\; (0).(-2) \;=\; 0,$$

thus the 2-cycle $\{0,\ -1\}$ is an attracting cycle.

3.2 Components of the Mandelbrot Set

We now consider the components of the Mandelbrot set. Suppose Q_c has an attracting periodic orbit of some period k. Then the entire basin of attraction of this point lies in the filled in Julia set K_c, and so the Julia set is not a dust like set. By our earlier discussion, this means that c should lie in the Mandelbrot set. But where in the set $\mathcal{M}$ dies it lie?

To answer this, let us first consider the set of c-values for which Q_c has an attracting or indifferent fixed point. We denote this set by $\mathcal{C}_1$,

$$\mathcal{C}_1 \;=\; \{c \in \mathbf{C} : Q_c(z) \quad \text{has an attracting/indifferent fixed point}\}.$$

Let z_c be the corresponding attracting fixed point for Q_c when $c \in \mathcal{C}_1$. Then z_c satisfies the following equations

$$z_c{}^2 + c \;=\; z_c \tag{3.1}$$

$$|2\, z_c| \;\leq\; 1 \tag{3.2}$$

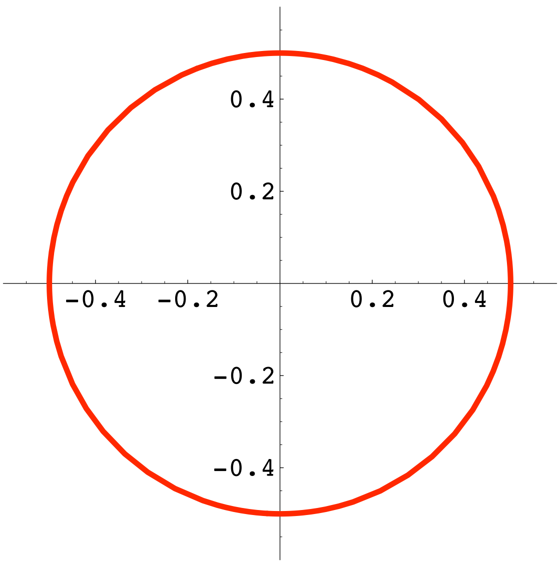
Equation (3.1) holds since z_c is a fixed point for Q_c and equation (3.2) holds because z_c is an attracting/indifferent fixed point.
Equation (3.2) tells us that z_c lies in the disk

$$\{z : |z| \;\leq\; \frac{1}{2}\}.$$

On the bounding circle

$$\{z : |z| = \frac{1}{2}\},$$

we have $|Q_c{}'(z_c)| = 1$, so z_c is an indifferent fixed point. Inside the circle, any fixed points are attractive.



Then the c-values for which z_c are indifferent fixed points are identified by writing points on the circle

$$\{z : |z| = \frac{1}{2}\}$$

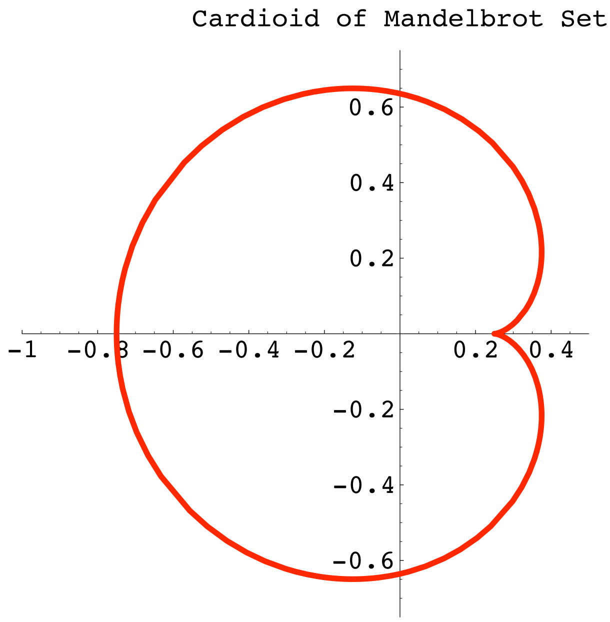
in polar form, that is $z_c = \frac{1}{2} e^{\imath\theta}$. But such points also satisfy equation (3.1) above, so

$$\left(\frac{e^{\imath\theta}}{2}\right)^2 - \frac{e^{\imath\theta}}{2} + c = 0,$$

that is

$$c = \frac{e^{\imath\theta}}{2} - \frac{e^{2\imath\theta}}{4}, \quad 0 \le \theta \le 2\pi.$$

This curve is the cardioid.



The interior of this region is the set of c-values for which Q_c has an attracting fixed point.

Next we turn to the c-values corresponding to attracting 2-cycles. First we get an equation for the points with minimal period-2. Such points must satisfy

$$\left(z^2 + c\right)^2 + c = z,$$

or

$$\left(z^2 + c\right)^2 + c - z = 0.$$

The fixed points for this c-value (that is the solutions of $z^2 + c - z = 0$) also satisfy this last equation. Thus the quadratic polynomial $z^2 + c - z$ must be a factor of the fourth degree polynomial $\left(z^2 + c\right)^2 + c - z$. In fact it can be checked that

$$\left(z^2 + c\right)^2 + c - z = \left(z^2 + z + 1 + c\right)\left(z^2 + c - z\right).$$

From this, it follows that the solutions z_1 and z_2 of

$$z^2 + z + 1 + c = 0$$

are precisely the points of period-2. In other words, we have

$$z_1{}^2 + c = z_2 \tag{3.3}$$

$$z_2{}^2 + c = z_1 \tag{3.4}$$

We now compute the derivatives of $Q_c{}^2(z)$ at the point z_1.

Since

$$Q_c{}^2(z) = \left(z^2 + c\right)^2 + c,$$

then

$$\begin{aligned} \frac{d}{d\,z}\left(Q_c{}^2(z)\right)|_{z=z_1} &= 2\,(z_1{}^2 + c)\,2\,z_1 \\ &= 2\,z_2\,2\,z_1, \quad \text{by equation (3.4)} \\ &= 4\,z_1\,z_2. \end{aligned}$$

Similarly

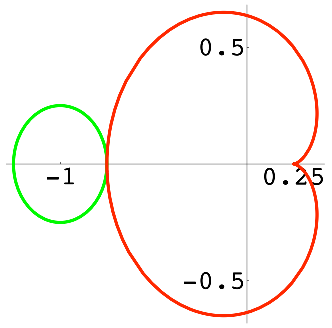
$$\frac{d}{d\,z}\left(Q_c{}^2(z)\right)|_{z=z_2} = 4\,z_1\,z_2.$$

Since z_1 and z_2 are the solutions of the quadratic equation $z^2 + z + 1 + c = 0$ then their product, $z_1\,z_2$, equals $1+c$. We computed above the derivative of $Q_c{}^2(z)$ at z_1 (and z_2), to have value $4\,z_1\,z_2$. Thus this derivative has value $4\,(1+c)$ and to ensure that this has absolute value less than 1, we need $|1+c| < 0.25$. It follows that the c-values for which there is an attracting 2-cycle lie in the set

$$\{c : |c+1| < 0.25\}.$$

This is a disk with radius 0.25 and centre at the point $z = -1$. It touches the cardioid bounding the set $\mathcal{C}_1$ at the point $z = -0.75$.

Period-1 and period-2 points of Mandelbrot set



The next big buds attached to the edge of the cardioid correspond to period-3 behaviour; then there are buds which have c-values belonging to attractive cycles of period 4, and so on.

The Mandelbrot set has been studied intensely since its discovery and much is known about it, although there is still more to be learned. It is known that it is connected, and the manner in which the various bulbs are connected is understood. The details of these results are well beyond the scope of this course. An interesting and accessible discussion of the properties of the Mandelbrot set can be found in Chapter 14 of the book *Chaos and Fractals* by Peitgen, Jürgens and Saupe. (UCC Library, 516.1 PEIT)

1. Given the real valued function

$$f(x) = x^2 - 1,$$

   (a) Find the fixed points of the function f;

   (b) Find the period-2 points of the function f;

   (c) Determine the nature of the fixed points and period-2 points found above. (That is, classify them as attracting, repelling or indifferent points.)

   (d) Consider the real-valued function

$$g(x) = x + x^2.$$

   Show that $x = 0$ is a fixed point and find its nature. Show using a cobweb diagram that it is repelling from one side and attracting from the other.

2. The logistic family of mappings is given by

$$Q_\mu(x) = \mu x(1 - x),$$

   where $0 \leq \mu \leq 4$ and $0 \leq x \leq 1$.

   (a) Motivate the use of the logistic equation as a model for population growth explaining the reasoning behind each of the three terms, μ, x and $(1 - x)$.

   (b) Find the fixed points of the system in terms of μ.

   (c) Find the nature of the fixed points (that is, classify them as attracting, repelling or indifferent points).

   (d) Suppose an infection enters the population introducing a rate of mortality that is proportional to the population at any time. Find the fixed points of the amended system in terms of μ and the new mortality rate.

3. Consider a linear complex function

$$L_\alpha(z) = \alpha z$$

   where z is a complex number and α is a constant and is also complex. Describe the dynamics of L_α in terms of fixed points and the presence or otherwise of periodic orbits. In particular, prove the following.

   (a) For $|\alpha| < 1$ all orbits tend to a unique limit (and find the limit);

   (b) For $|\alpha| > 1$ all orbits tend to infinity;

   (c) For $|\alpha| = 1$ there are two subcases: one where all orbits are periodic and one where each orbit is dense on a circle centred at $z = 0$. (For the dense orbits a plausibility argument will suffice.)

4. Newton's method for solving for the roots of a function $g(x)$ is given by the iterative procedure

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}$$

where $g'(x)$ is the derivative of $g(x)$. Newton's method is convergent if $x_{n+1} \to x_r$ where x_r is a root of $g(x)$, as $n \to \infty$.

   (a) Rewrite Newton's method in the form of an iterated function.

   (b) Show that if $g'(x) \neq 0$ then a root of $g(x)$ is a fixed point of the iterated function.

   (c) Find a condition on $g(x)$ and its derivatives guaranteeing convergence of the method.

1. Given the real valued function

$$f(x) = x^2 - 4x + 2,$$

   (a) Find the fixed points of the function f.

   (b) Find the period-2 points of the function f.

   (c) Are the fixed points and period-2 points found above attracting or repelling or indifferent points?

   (d) Find two eventually periodic points of $f(x)$ which are not periodic.

2. Consider the family of functions

$$T_\mu(x) = \begin{cases} \frac{\mu}{2}x & \text{if} \quad 0 \leq x \leq \frac{1}{2}, \\ \frac{\mu}{2}(1-x) & \text{if} \quad \frac{1}{2} \leq x \leq 1, \end{cases}$$

where μ is a positive number smaller or equal with 4.

   (a) i) Graph the function T_3^2.

   ii) Find the period-2 points of T_3.

   (b) Find the first 5 iterates of the point $x_0 = \frac{2}{11}$ under the function T_4.

   (c) Show that any positive rational number smaller than 1 is an eventually periodic point of the function T_4 from part b).

   (d) i) Find the local minimum and local maximum of the real valued function

$$g(x) = x^3 + x^2 - x + 1$$

   and draw its graph.

   ii) How many fixed points does the real valued function

$$f(x) = x^3 + x^2 + 1$$

   have? Are they repelling or attracting or indifferent points?

3. (a) Let $Q_{0.5i+0.25}(z) = z^2 + 0.5i + 0.25$. Find the fixed points of $Q_{0.5i+0.25}$ and determine if they are attracting, repelling or indifferent points.

   (b) Let

$$T(x) = \begin{cases} 2x & \text{if} \quad 0 \leq x < \frac{1}{2} \\ 2x - 1 & \text{if} \quad \frac{1}{2} \leq x \leq 1. \end{cases}$$

   i) If x has the binary representation $x = \frac{a_1}{2} + \frac{a_2}{2^2} + \frac{a_3}{2^3} + ...$ what is the value of $T(x)$?

   ii) Show that there are points in $[0, 1)$ which are not eventually periodic.

   iii) Sketch the graph of T^3.

4. (a) Find two points in the filled in Julia set $K_{-1+\frac{1}{4}i}$ for the mapping

$$Q_{-1+\frac{1}{4}i}(z) = z^2 - 1 + \frac{1}{4}i.$$

(b) Show that $e^{\frac{2\pi i}{5}}$ is a periodic point for the function $Q_0(z) = z^2$ and find its prime period. Is it a repelling, attracting or indifferent periodic point?

(c) Show that $-\frac{1}{16}$ is in the Mandelbrot set.

1. Given the real-valued function

$$f(x) = x^2 - 3x + 4,$$

   (a) How many fixed points does f have? Justify your answer.

   (b) Does f have any real period-2 point? Justify your answer.

   (c) Show that any fixed point of f is an indifferent point.

   (d) Find an eventually periodic point of $f(x)$ that is not periodic.

2. Consider the family of functions

$$T_\mu(x) = \begin{cases} \frac{\mu}{2}x & \text{if } \ 0 \le x \le \frac{1}{2}, \\ \frac{\mu}{2}(1-x) & \text{if } \ \frac{1}{2} \le x \le 1, \end{cases}$$

   where μ is a positive number, $\mu \le 4$.

   (a) i) Find all the eventually periodic points of T_2.
   ii) Graph the function T_4^2.

   (b) How many period-2 points does T_4 have? Justify your answer.

   (c) i) Does there exist a real-valued function f such that the orbit of 1 is : $1, 2, 3, 4, 2, 3, 2, 3, ...$? Justify your answer.
   ii) Write down a real-valued function f for which the orbit of 3 is given by the sequence: $x_n = n^3 + 3$.

3. (a) Let $Q_{\frac{1}{2}}(z) = z^2 + \frac{1}{2}$. Find the fixed points of $Q_{\frac{1}{2}}$ and determine if they are attracting, repelling or indifferent points.

   (b) Let

$$T(x) = \begin{cases} 3x & \text{if } \ 0 \le x < \frac{1}{3} \\ 3x - 1 & \text{if } \ \frac{1}{3} \le x < \frac{2}{3} \\ 3x - 2 & \text{if } \ \frac{2}{3} \le x \le 1 \end{cases}.$$

   i) If x has the ternary representation $x = \frac{1}{3} + \frac{2}{3^2} + \frac{2}{3^3} + \frac{1}{3^4}$, compute $T(x)$.
   ii) Show that all rational points in $[0, 1)$ are eventually periodic.
   iii) Sketch the graph of T^2.

4. (a) Show that $e^{2\pi i/5}$ is a periodic point for the function $f(z) = z^3$ and find its prime period. Is it a repelling, attracting or indifferent periodic point?

   (b) Find two points in the filled in Julia set K_{-1+2i} for the mapping

$$Q_{-1+2i}(z) = z^2 - 1 + 2i.$$

   (c) Give examples of a point in the Mandelbrot set and a point not in the Mandelbrot set. Justify your answer.

Section A

1. (a) Find the first 6 iterates of the point $x_0 = \frac{1}{7}$ under the action of the function

$$T(x) = \begin{cases} 3\,x & \text{if} \quad 0 \leq x < 1/3, \\ 3\,x - 1 & \text{if} \quad 1/3 \leq x < 2/3, \\ 3\,x - 2 & \text{if} \quad 2/3 \leq x \leq 1. \end{cases}$$

(b) Let $f : \mathbf{R} \longrightarrow \mathbf{R}$ be the function $f(x) = x^2 - x$, defined on the interval $[0, 2]$.

(i) Sketch the graphs of $f(x)$ and the graph of the function $d(x) = x$ on the same diagram.

(ii) Find the period-1 points of $f(x)$.

(iii) Find the period-2 points of $f(x)$.

(iv) Are there points of prime period-2?

(c) Find the first four iterates of the point $z_0 = 1+i$ under the action of the function $Q_{2\,i}(z) = z^2 + 2\,i$.

(d) Find all the fixed points of the complex valued function

$$f(z) = 3\,i\,z(2 - z),$$

and determine whether they are attracting, repelling, or neutral.

2. (a) Find all fixed points of the real valued function

$$f(x) = x^3 - 3\,x^2 + 2\,x,$$

and determine whether they are attracting, repelling, or neutral.

(b) The point $x_0 = 0$ lies on a periodic orbit of the function

$$f(x) = \frac{\pi}{2\sqrt{3}} \tan(\frac{\pi}{6} - x).$$

Determine the period of this orbit and decide if this orbit is attracting, repelling or neutral.

(c) Show that the point $z = e^{2\,\pi\,i/15}$ lies on a periodic cycle for $Q_0(z) = z^2$ What is the prime period of the cycle and is the cycle attracting, repelling or neutral?

(d) Find at least 3 points in the filled in Julia set K_{-20} for the mapping $Q_{-20}(z) = z^2 - 20$.

Section B

3. Consider the family of functions given by

$$T_\mu(x) = \begin{cases} \frac{\mu}{2}\,x & \text{if } 0 \le x \le 1/2, \\ \frac{\mu}{2}\,(1-x) & \text{if } 1/2 \le x \le 1. \end{cases}$$

(a) Sketch the graphs of $T_\mu(x)$ for $\mu = 2,\ 3,\ 4$.

(b) Sketch the graphs of ${T_4}^2(x)$ and ${T_4}^3(x)$.

(c) Find the fixed points of $T_3(x)$ and $T_4(x)$.

(d) Show that $\frac{2}{7}$ is a point of prime period-3 for $T_4(x)$. Give 2 other points of prime period-3 for this mapping.

(e) (i) Show that $x = 0$ is a fixed point of $T_\mu(x)$ for $0 \le \mu \le 4$. For which of these μ-values is $x = 0$ attracting/repelling?

(ii) Show that $x = \frac{\mu}{\mu+2}$ is a fixed point of $T_\mu(x)$ for $2 < \mu \le 4$. For which of these μ-values is $x = \frac{\mu}{\mu+2}$ attracting/repelling?

4. Consider the family of functions given by

$$F_\mu(x) = x^2 + \mu \quad \text{for} -2 \le x \le 2.$$

(a) Sketch the graphs of $F_\mu(x)$ for $\mu = \frac{1}{4},\ 0$ and -2.

(b) Find the fixed points of $F_{\frac{1}{4}}(x)$, $F_0(x)$ and $F_{-2}(x)$.

(c) Let $q_+ = \frac{1+\sqrt{1-4\mu}}{2}$ and $q_- = \frac{1-\sqrt{1-4\mu}}{2}$.

(i) Show that $x = q_+$ is a fixed point of $F_\mu(x)$ for $-2 \le \mu \le \frac{1}{4}$. For which of these μ-values is $x = q_+$ attracting/repelling?

(ii) Show that $x = q_-$ is also a fixed point of $F_\mu(x)$ for $-2 \le \mu \le \frac{1}{4}$. For which of these μ-values is $x = q_-$ attracting/repelling?

(iii) Use the results from parts (i) and (ii) to sketch a bifurcation diagram for the fixed points of the family $\mu \mapsto F_\mu$.

Section C

5. Let $f(z) = z^3$.
   (a) Find the fixed points of $f(z)$ and determine whether they are attracting, repelling or neutral.
   (b) Find the points of period-2 and determine whether they are attracting, repelling or neutral.
   (c) Show that $z = e^{2\pi i \theta}$, with $\theta = \frac{2}{13}$, is a repelling point of period 3.
   (d) Describe the points of period n. How many period-n points are there? Are they attracting, repelling or neutral?
   (e) Describe the orbits of points in $\mathbf{C} \setminus \{z : |z| = 1\}$.

6. (a) Let $f : \mathbf{R} \longrightarrow \mathbf{R}$ be the function $f(x) = x - x^2$.
      (i) Sketch the graph of $f(x)$.
      (ii) Prove that $f(x)$ has only one fixed point and that it is a neutral fixed point.
      (iii) Show that there are points arbitrarily close to the fixed point whose orbits converge to the fixed point.
      (iv) Show that there are also points arbitrarily close to the fixed point whose orbits diverge away from the fixed point.
   (b) Let x be a period-n point of a mapping $f : S \mapsto S$.
      (i) Suppose that p is the prime period of $x \in S$. Prove that n is an integer multiple of p.
      (ii) Prove that each of the points on the orbit of x also has period n.

Section A

1. (a) Find the first 4 iterates of the point $x_0 = -0.5$ under the action of the function

$$f(x) = x^2 - 2.$$

Sketch the graph of $f(x)$ on the interval $[-2, 2]$.

(b) Let $f : \mathbf{R} \longrightarrow \mathbf{R}$ be the function $f(x) = 3x^2 - 2$, defined on the interval $[-1, 1]$.

(i) Sketch the graph of $f(x)$ and the graph of the function $d(x) = x$ on the same diagram.

(ii) Find the period-1 points of $f(x)$.

(iii) Find the period-2 points of $f(x)$.

(iv) Which of the period-2 points are prime period-2?

(c) Iterate the complex valued function $Q_{-i}(z) = z^2 - i$ starting at the seed $z_0 = -i$. Describe the behaviour of the orbit. Do you think it tends to infinity or remains bounded? Explain your answer.

(d) Find all the fixed points of the complex valued function

$$f(z) = z^3 + z - 1,$$

and determine whether they are attracting, repelling, or neutral.

2. (a) Sketch, on the same diagram, the graphs of the function

$$f(x) = x^2 + 2x - 6,$$

and the function $d(x) = x$ defined on the interval $[-4, 2]$. Find all the fixed points of $f(x)$ and determine whether they are attracting, repelling, or neutral.

(b) The point $x_0 = 0$ lies on a periodic orbit of the function

$$f(x) = \frac{4\pi}{3\sqrt{3}} \cos(\frac{\pi}{6} - x).$$

Determine the period of this orbit and decide if this orbit is attracting, repelling or neutral.

(c) Show that the point $z = e^{10\pi i/7}$ lies on a periodic cycle for $Q_0(z) = z^2$ What is the prime period of the cycle and is the cycle attracting, repelling or neutral?

(d) Show that the points $\frac{-1\pm\sqrt{21}}{2}$ belong to the filled in Julia set K_{-6} of the mapping $Q_{-6}(z) = z^2 - 6$. Find at least 3 other points in K_{-6}.

Section B

3. Consider the doubling function given by

$$D(x) = \begin{cases} 2\,x & \text{if} \quad 0 \leq x < 1/2 \\ 2\,x - 1 & \text{if} \quad 1/2 \leq x \leq 1. \end{cases}$$

(a) Sketch the graphs of D, D^2 and D^3. What do you expect the graph of D^n to look like? How many fixed points does D^n have?

(b) Suppose that a point $x_0 \in [0,\, 1]$ has binary representation $0.a_1a_2a_3a_4a_5\ldots$, that is

$$x_0 = \frac{a_1}{2^1} + \frac{a_2}{2^2} + \frac{a_3}{2^3} + \frac{a_4}{2^4} + \frac{a_5}{2^5} + \cdots$$

What is the binary expansion of $D(x_0)$? Of $D^n(x_0)$?

(c) Find all period-2 points of the mapping $D(x)$ in the interval $[0,\, 1]$.

(d) What is the binary expansion of the points which are fixed by D^n? How many such points are there in $[0,\, 1]$?

(e) Find the binary expansion all points in $[0,\, 1]$ that are eventually fixed at 0 after n applications of D.

(f) Show that the periodic points are dense in $[0,\, 1]$.

(g) Find a seed point in $[0,\, 1]$ whose orbit in dense.

Justify your answers in each of the parts (a) to (g).

4. Let $f : \mathbf{R} \longrightarrow \mathbf{R}$ with $f(x) = \frac{1}{2}\,(x^2 + 1)$. Sketch the graph of $f(x)$ defined on $[-2,\, 2]$.

(a) Prove that f has only one fixed point and that it is a neutral fixed point.

(b) Prove that there are points arbitrarily close to the fixed point whose orbits converge to the fixed point.

(c) Prove that there are points arbitrarily close to the fixed point whose orbits diverge away from the fixed point.

Section C

5. Let $f(z) = z^2$.
   (a) Find the fixed points of f and determine whether they are attracting, repelling or neutral.
   (b) Find the points of period 2 and determine whether they are attracting, repelling or neutral.
   (c) Determine the points of period n. How many period-n points are there? Are they attracting, repelling or neutral?
   (d) Describe the orbits of points in $\mathbf{C} \setminus \{z : |z| = 1\}$.

6. Consider the mapping $Q_c : \mathbf{C} \longrightarrow \mathbf{C}$ given by $Q_c(z) = z^2 + c$.
   (a) Find all complex c-values for which $Q_c(z)$ has a fixed point z_0 with ${Q_c}'(z_0) = -1$. What are the corresponding fixed points?
   (b) Show that if $c \neq -\frac{3}{4}$ then the points

$$q_{\pm}(c) = -\frac{1}{2} \pm \frac{1}{2}\sqrt{-3 - 4c}$$

   lie on a 2-cycle.
   (c) Determine whether this 2-cycle is attracting or repelling in the two real cases $-5/4 < c < -3/4$ and $c > -3/4$.

Section A

1. (a) Find the first 6 iterates of the point $x_0 = 0.5$ under the action of the function

$$T_3(x) = \begin{cases} 1.5\,x & \text{if } 0 \le x \le 1/2, \\ 1.5\,(1-x) & \text{if } 1/2 \le x \le 1. \end{cases}$$

   (b) Let $f : \mathbf{R} \longrightarrow \mathbf{R}$ be the function $f(x) = 2\,x^2 - 6$, defined on the interval $[-2, 2]$.
      (i) Find the period-1 points of f.
      (ii) Find the period-2 points of f.
      (iii) What are the two distinct orbits of prime period-2?

   (c) Iterate the complex valued function $Q_{2\,i+4}(z) = z^2 + 2\,i + 4$ starting at the seed $z_0 = -2\,i$. Describe the behaviour of the orbit. Do you think it tends to infinity or remains bounded? Explain your answer.

   (d) Find all the fixed points of the complex valued function

$$f(z) = 3\,z^2 + z + 1,$$

   and determine whether they are attracting, repelling, or neutral.

2. (a) Find all fixed points of the real valued function

$$f(x) = x^3 - \sqrt{5}\,x^2,$$

   and determine whether they are attracting, repelling, or neutral.

   (b) Let $f : \mathbf{R} \longrightarrow \mathbf{R}$ be the function $f(x) = 4\,x(1-x)$.
      (i) Prove that each of the following points is an eventually fixed point of f : 1, 1/2.
      (ii) What is the orbit of each of the eventually fixed points found in (i)?
      (iii) Find another eventually fixed point of f.

   (c) Show that the point $z = e^{2\,\pi\,i/7}$ lies on a cycle of period 3 for $Q_0(z) = z^2$ Is this cycle attracting, repelling or indifferent?

   (d) Find some points in the filled in Julia set $K_{1+\frac{3}{4}\,i}$ for the mapping $Q_{1+\frac{3}{4}\,i}(z) = z^2 + 1 + \frac{3}{4}\,i$.

Section B

3. Newton's iterative procedure to find the roots of a given function $P(x)$ is given by

$$x_{n+1} = x_n - \frac{P(x_n)}{P'(x_n)}.$$

(a) Show that a root x of $P(x)$ for which $P'(x) \neq 0$ is a fixed point for Newton iteration.

(b) Suppose $P(x) = x^3 - 3\,x^2 + 2\,x$. Determine whether the fixed points for the corresponding Newton iteration are attracting or repelling.

4. Consider the tripling function given by

$$T(x) = \begin{cases} 3\,x & \text{if} \quad 0 \leq x < 1/3, \\ 3\,x - 1 & \text{if} \quad 1/3 \leq x < 2/3, \\ 3\,x - 2 & \text{if} \quad 2/3 \leq x \leq 1. \end{cases}$$

(a) Sketch the graph of T^2. What do you expect the graph of T^n to look like? How many fixed points should T^n have?

(b) Suppose that a point $x_0 \in [0,\, 1)$ has ternary representation $0.a_1a_2a_3a_4a_5\ldots$, that is

$$x_0 = \frac{a_1}{3^1} + \frac{a_2}{3^2} + \frac{a_3}{3^3} + \frac{a_4}{3^4} + \frac{a_5}{3^5} + \cdots$$

What is the ternary representation of $T(x_0)$? Of $T^n(x_0)$?

(c) Give the ternary representation of all points in $[0,\, 1)$ that are eventually fixed at 0 by T^n.

(d) Give the ternary representation of all points in the interval $[0,\, 1)$ that are periodic with period 2 under iteration by T.

(e) Show that the periodic points are dense in $[0,\, 1)$.

(f) Find a seed point x_0 in $[0,\, 1)$ whose orbit is dense.

Section C

5. (a) Find all complex c-values for which $Q_c(z) = z^2 + c$ has a fixed point z_0 with $Q_c{}'(z_0) = 1$.

   (b) Consider the fixed points of Q_c in the complex plane for the cases where c is real and

   (i) $c < 1/4$,

   (ii) $c = 1/4$,

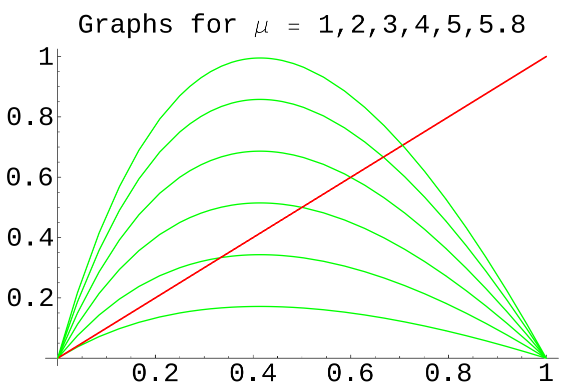
   (iii) $c > 1/4$.

   Determine whether these fixed points are attracting, repelling or indifferent.

   (c) Determine the set of all complex c-values at which the function $Q_c(z)$ has an attracting fixed point. Sketch this set of c-values in the plane. Where does this set meet the real axis?

6. The graphs shown are plotted from the formula

$$g_\mu(x) = \mu\, x \left(\frac{1-x}{1+x}\right), \quad 0 \le x \le 1 \quad \text{and} \quad \mu \ge 0.$$



   (a) Show that g_μ has its maximum value at the point $\sqrt{2}-1$ and that the maximum value of g_μ is $\mu\,(3-2\sqrt{2})$.

   (b) Deduce that g_μ maps $[0,\, 1]$ into itself for $0 \le \mu \le 3 + 2\sqrt{2}$.

   (c) Show algebraically that the fixed points are $x = 0$ and $x = \frac{\mu-1}{\mu+1}$ if $\mu \ge 1$.

   (d) Find the derivative $g_\mu{}'(x)$ and hence show that $g_\mu{}'(0) = \mu$ and

$$g_\mu{}'\left(\frac{\mu-1}{\mu+1}\right) = 1 - \frac{\mu}{2} + \frac{1}{2\,\mu}.$$

   (e) Use the results from parts (c) and (d) to sketch a bifurcation diagram for the fixed points of the family $\mu \mapsto g_\mu$.

Section A

1. (a) Find the first 10 iterates of the point $x_0 = \frac{7}{13}$ under the action of the function

$$f(x) = \begin{cases} 2\,x & \text{if } \ 0 \ \le \ x \ < \ 1/2, \\ 2\,x - 1 & \text{if } \ 1/2 \ \le \ x \ \le \ 1. \end{cases}$$

(b) Let $f : \mathbf{R} \longrightarrow \mathbf{R}$ be the function $f(x) = x^2 - 6$.

(i) Find the period-1 points of f.

(ii) Find the period-2 points of f.

(iii) What are the two distinct orbits of prime period-2?

(c) Iterate the complex valued function $Q_{3\,i+9}(z) = z^2 + 3\,i + 9$ starting at the seed $z_0 = -\,3\,i$. Describe the behaviour of the orbit. Do you think it tends to infinity or remains bounded? Explain your answer.

(d) Find all the fixed points of the complex valued function

$$f(z) = 2\,z(i - z),$$

and determine whether they are attracting, repelling, or neutral.

2. (a) Find all fixed points of the real valued function

$$f(x) = x^3 - x,$$

and determine whether they are attracting, repelling, or neutral.

(b) The point $x_0 = 0$ lies on a periodic orbit of the function

$$f(x) = 1 - x^3.$$

Determine the period of this orbit and decide if this orbit is attracting, repelling or neutral.

(c) Show that the point $z = -1 - i$ lies on a cycle of period 2 for the mapping $Q_{-i}(z) = z^2 - i$. Is this cycle attracting, repelling or neutral?

(d) Show that $z = -\,3$ belongs to the filled in Julia set K_{-12} for the mapping $Q_{-12}(z) = z^2 - 12$. Find two other points in K_{-12}.

Section B

3. Newton's iterative procedure to find the roots of a given function $P(x)$ is given by

$$x_{n+1} = x_n - \frac{P(x_n)}{P'(x_n)}.$$

(a) Show that a root x of $P(x)$ for which $P'(x) \neq 0$ is a fixed point for Newton iteration.

(b) Suppose $P(x) = x^3 + 3\,x^2 + 2\,x$. Determine whether the fixed points for the corresponding Newton iteration are attracting or repelling.

4. Consider the family of functions given by

$$T_\mu(x) = \begin{cases} \frac{\mu}{2}\,x & \text{if } 0 \leq x \leq 1/2 \\ \frac{\mu}{2}\,(1-x) & \text{if } 1/2 \leq x \leq 1. \end{cases}$$

(a) Sketch the graphs of $T_\mu(x)$ for $\mu = 2$, 4.

(b) Sketch the graphs of ${T_4}^2(x)$ and ${T_4}^3(x)$.

(c) Find the fixed points of $T_2(x)$ and $T_4(x)$.

(d) Show that 0.4 is a point of prime period-2 for $T_4(x)$. Give another point of prime period-2 for this mapping.

(e) (i) Show that $x = 0$ is a fixed point of $T_\mu(x)$ for $0 \leq \mu \leq 4$. For which of these μ-values is $x = 0$ attracting/repelling?

(ii) Show that $x = \frac{\mu}{\mu+2}$ is a fixed point of $T_\mu(x)$ for $2 < \mu \leq 4$. For which of these μ-values is $x = \frac{\mu}{\mu+2}$ repelling?

Section C

5. Let $f(z) = z^2$.
   (a) Find the fixed points of f and determine whether they are attracting, repelling or neutral.
   (b) Find the points of period-2 and determine whether they are attracting, repelling or neutral.
   (c) Show that $z_0 = e^{2\pi i\theta}$, with $\theta = \frac{1}{7}$, is a repelling point of period 3.
   (d) Describe the points of period n. How many period-n points are there? What is the nature of these points?
   (e) Describe the orbits of points in $\mathbf{C} \setminus \{z : |z| = 1\}$.

6. Let $f : \mathbf{R} \to \mathbf{R}$ be an affine mapping with slope a such that $|a| \neq 1$. Let p be the fixed point of the affine mapping. For the orbit, under f, of $x_0 \in \mathbf{R}$, prove that
   (a) if $|a| < 1$, then the orbit converges to the fixed point p,
   and that
   (b) if $|a| > 1$ then the sequence of distances of the orbit elements from p diverges monotonically to ∞, provided that $x_0 \neq p$.

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SUMMER EXAMINATION 2007

B.A. Degree

MATHEMATICAL STUDIES — MS3011
Dynamical Systems

Professor J.K. Langley
Professor J. Berndt
Dr. D.J. Hurley

Answer *one* question from each section

Time allowed: One and a half hours

Reading time of fifteen minutes is permitted prior to the commencement of this examination

Marks may be lost if not all your work is clearly shown or if you do not indicate where a calculator has been used

Section A

1. (a) Find the first 10 iterates of the point $x_0 = 0.25$ under the action of the function

$$f(x) = 2.4x(1 - x).$$

What is the long term behaviour of this orbit?

(b) Let $f : \mathbf{R} \longrightarrow \mathbf{R}$ be the function $f(x) = x^2 - 2$.

(i) Find the period-1 points of f.

(ii) Find the period-2 points of f.

(iii) What are the two distinct orbits of prime period-2?

(c) Iterate the complex valued function $Q_{1-i}(z) = z^2 + 1 - i$ starting at the seed $z_0 = i$. Describe the behaviour of the orbit. Do you think it tends to infinity or remains bounded? Explain your answer.

(d) Find all the fixed points of the complex valued function

$$f(z) = z^2 + z + 1,$$

and determine whether they are attracting, repelling, or neutral.

2. (a) Find all fixed points of the real valued function

$$f(x) = x^2 - 2\,x,$$

and determine whether they are attracting, repelling, or neutral.

(b) The point $x_0 = 0$ lies on a periodic orbit of the function

$$f(x) = -x^5 + 1.$$

Determine the period of this orbit and decide if this orbit is attracting, repelling or neutral.

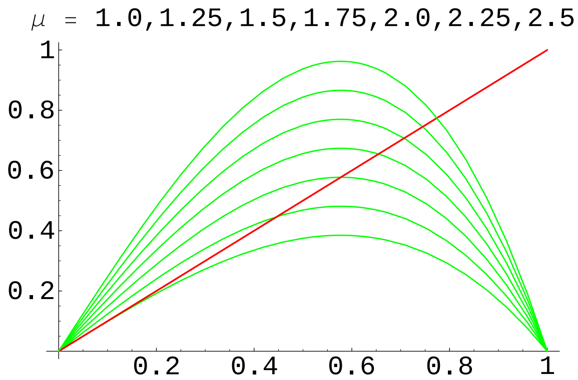
(c) Show that the point $z = -1+i$ lies on a cycle of period 2 for $Q_i(z) = z^2+i$. Is this cycle attracting, repelling or neutral?

(d) Show that $z = 2$ belongs to the filled in Julia set K_{-6} for the mapping $Q_{-6}(z) = z^2 - 6$. Find three other points in K_{-6}.

Section B

3. The graphs shown are plotted from the formula

$$g_\mu(x) = \mu x \left(1 - x^2\right), \quad 0 \le x \le 1 \quad \text{and} \quad \mu \ge 0.$$



(i) Show that $g_\mu(x)$ has its maximum value at the point $x = \frac{1}{\sqrt{3}}$ and that the maximum value of $g_\mu(x)$ is $\frac{2\mu}{3\sqrt{3}}$.

(ii) Deduce that $g_\mu(x)$ maps $[0,\,1]$ into itself for $0 \le \mu \le \frac{3\sqrt{3}}{2}$.

(iii) Show algebraically that the fixed points are $x = 0$, and $x = \sqrt{\frac{\mu-1}{\mu}}$ if $\mu \ge 1$.

(iv) Show that $g_\mu{}'(0) = \mu$ and

$$g_\mu{}'\left(\sqrt{\frac{\mu-1}{\mu}}\right) = 3 - 2\mu.$$

(v) Use the results from parts (iii) and (iv) to sketch a bifurcation diagram for the fixed points of the family $\mu \mapsto g_\mu$.

4. Consider the doubling function given by

$$D(x) = \begin{cases} 2x & \text{if} \quad 0 \le x < 1/2 \\ 2x - 1 & \text{if} \quad 1/2 \le x < 1. \end{cases}$$

(i) Sketch the graphs of D^2 and D^3. What do you expect the graph of D^n to look like? How many fixed points does D^n have?

Question 4 continued overleaf

(ii) Suppose that a point $x_0 \in [0,\, 1)$ has binary representation $0.a_1a_2a_3a_4a_5\ldots$, that is

$$x_0 = \frac{a_1}{2^1} + \frac{a_2}{2^2} + \frac{a_3}{2^3} + \frac{a_4}{2^4} + \frac{a_5}{2^5} + \cdots$$

What is the binary expansion of $D(x_0)$? Of $D^n(x_0)$?

(iii) Find all points in $[0,\, 1)$ that are eventually fixed at 0 after n applications of D.

(iv) Find all period-2 and period-3 points of the mapping $D(x)$ in the interval $[0,\, 1)$.

(v) How many points in $[0,\, 1)$ are fixed by D^n for each n?

(vi) Show that the periodic points are dense in $[0,\, 1)$.

(vii) Find a seed point in $[0,\, 1)$ whose orbit in dense.

Section C

5. Let $f(z) = z^2$.

   (a) Find the fixed points of f and determine whether they are attracting, repelling or neutral.

   (b) Find the points of period 2 and determine whether they are attracting, repelling or neutral.

   (c) Show that $z = e^{2\pi i\theta}$, with $\theta = \frac{1}{5}$, is a repelling point of period 4.

   (d) Describe the points of period n. How many period-n points are there? Are they attracting, repelling or neutral?

   (e) Describe the orbits of points in $\mathbf{C} \setminus \{z : |z| = 1\}$.

6. Let $f : S \to S$ be a differentiable function, with continuous derivative, defined on an interval S. Suppose that $p \in S$ is a fixed point of f and that $|f'(p)| < 1$. Prove the following.

   (a) There is a positive number $a < 1$ and an open-in-S interval I such that for all $x \in I$,

   $$|f(x) - f(p)| \leq a\,|x - p|.$$

   (b) If $x_0 \in I$, then the orbit

   $$\{x_0,\, x_1,\, x_2,\, \ldots,\, x_n,\, \ldots\}$$

   converges to the fixed point p.

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AUTUMN EXAMINATION 2007

B.A. Degree

MATHEMATICAL STUDIES — MS3011
Dynamical Systems

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Answer *one* question from each section

Time allowed: One and a half hours

Reading time of fifteen minutes is permitted prior to the commencement of this examination

Marks may be lost if not all your work is clearly shown or if you do not indicate where a calculator has been used

Section A

1. (a) Find the first 12 iterates of the point $x_0 = 0.85$ under the action of the function

$$f(x) = 2.2\,x(1 - x^2).$$

What is the long term behaviour of this orbit?

(b) Let $f : \mathbf{R} \longrightarrow \mathbf{R}$ be the function $f(x) = x^2 - 12$.

(i) Find the period-1 points of f.

(ii) Find the period-2 points of f.

(iii) What are the two distinct orbits of prime period-2?

(c) Iterate the complex valued function $Q_{1+i}(z) = z^2 + 1 + i$ starting at the seed $z_0 = -i$. Describe the behaviour of the orbit.

(d) Find all the fixed points of the complex valued function

$$f(z) = 4\,z^2 + z + 1,$$

and determine whether they are attracting, repelling, or neutral.

2. (a) Find all fixed points of the real valued function

$$f(x) = x^2 + 2\,x - 12,$$

and determine whether they are attracting, repelling, or neutral.

(b) The point $x_0 = 0$ lies on a periodic orbit of the function

$$f(x) = \frac{\pi}{2}\cos x.$$

Determine the period of this orbit and decide if this orbit is attracting, repelling or neutral.

(c) Show that the point $z = e^{2\pi i/3}$ lies on a cycle of period 2 for $Q_0(z) = z^2$. Is this cycle attracting, repelling or neutral?

(d) Define what it means to say that a point belongs to the filled in Julia set K_{-12} of the mapping $Q_{-12}(z) = z^2 - 12$. Find four points belonging to K_{-12} and justify your answer.

Section B

3. Consider the family of functions given by

$$T_\mu(x) = \begin{cases} \frac{\mu}{2}\, x & \text{if } 0 \le x \le 1/2 \\ \frac{\mu}{2}\,(1-x) & \text{if } 1/2 \le x \le 1. \end{cases}$$

(a) Sketch the graphs of $T_\mu(x)$ for $\mu = 1, 2, 3, 4$.

(b) Sketch the graphs of ${T_4}^2(x)$ and ${T_4}^3(x)$.

(c) Find the fixed points of $T_2(x)$ and $T_4(x)$.

(d) Show that 0.4 is a point of prime period-2 for $T_4(x)$. Give another point of prime period-2 for this mapping.

(e) (i) Show that $x = 0$ is a fixed point of $T_\mu(x)$ for $0 \le \mu \le 4$. For which of these μ-values is $x = 0$ attracting/repelling?

(ii) Show that $x = \frac{\mu}{\mu+2}$ is a fixed point of $T_\mu(x)$ for $2 < \mu \le 4$. For which of these μ-values is $x = \frac{\mu}{\mu+2}$ repelling?

4. (a) Let $x_0, x_1, x_2,$ be three distinct points in the domain of a mapping f. Suppose, furthermore, that f maps the set of these three points into itself and that f has no period-2 points. What are the possible orbits for x_0 under f?

(b) Let $f : \mathbf{R} \longrightarrow \mathbf{R}$ with $f(x) = \frac{1}{2}\,(x^2 + 1)$.

(i) Prove that f has only one fixed point and that it is a neutral fixed point.

(ii) Show that there are points arbitrarily close to the fixed point whose orbits converge to the fixed point.

(iii) Show that there are points arbitrarily close to the fixed point whose orbits diverge away from the fixed point.

Section C

5. Consider the mapping $Q_c : \mathbf{C} \longrightarrow \mathbf{C}$ given by $Q_c(z) = z^2 + c$.

   (a) Find all complex c-values for which $Q_c(z)$ has a fixed point z_0 with ${Q_c}'(z_0) = -1$. What are the corresponding fixed points?

   (b) Show that if $c \neq -\frac{3}{4}$ then the points

$$q_{\pm}(c) = -\frac{1}{2} \pm \frac{1}{2}\sqrt{-3 - 4c}$$

   lie on a 2-cycle.

   (c) Determine whether this 2-cycle is attracting or repelling in the two real cases $-5/4 < c < -3/4$ and $c > -3/4$.

6. Consider the mapping $Q_c : \mathbf{C} \longrightarrow \mathbf{C}$ given by $Q_c(z) = z^2 + c$. Let $\mathcal{C}_1$ be the set given by

$$\mathcal{C}_1 = \{c \in \mathbf{C} : Q_c(z) \quad \text{has an attracting/neutral fixed point}\}.$$

   (a) If $c \in \mathcal{C}_1$ and z_c is the corresponding fixed point, show that z_c satisfies

$$\begin{aligned} {z_c}^2 + c &= z_c \\ |2\,z_c| &\leq 1. \end{aligned}$$

   (b) Find a parametric equation of the boundary of $\mathcal{C}_1$. Use this to sketch the set $\mathcal{C}_1$.