MATH6038: Linear Algebra Test Solutions

1. Use Gaussian elimination to solve the system of linear equations given by

$$\begin{aligned} x + y + z &= 3 \\ 4x + 5y - 6z &= 6 \\ 2x - y &= 8 \end{aligned}$$

[9 Marks]

Solution: First we put the system in augmented matrix form:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 4 & 5 & -6 & 6 \\ 2 & -1 & 0 & 8 \end{array}\right] \xrightarrow{r_2 \to r_2 - 4r_1,\, r_3 \to r_3 - 2r_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & -10 & -6 \\ 0 & -3 & -2 & 2 \end{array}\right]$$

$$\xrightarrow{r_3 \to r_3 + 3r_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & -10 & -6 \\ 0 & 0 & -32 & -16 \end{array}\right]$$

$$\begin{aligned} \Rightarrow z &= \frac{1}{2} \\ \Rightarrow y &= -6 + 10z \\ &= -1 \\ \Rightarrow x &= 3 - y - z \\ &= 3 + 1 - \frac{1}{2} = \frac{7}{2}. \end{aligned}$$

The solution set is

$$(x, y, z) = \left(\frac{7}{2}, -1, \frac{1}{2}\right).$$

2. Use Gaussian elimination methods to determine A^{-1} where

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & -1 \\ -1 & -2 & 2 \end{pmatrix}.$$

[9 Marks]

Solution: We set up the array:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 2 & 3 & -1 & 0 & 1 & 0 \\ -1 & -2 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{r_2 \to r_2 - 2r_1,\, r_3 \to r_3 + r_1} \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{r_3 \to r_3 + r_2} \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{r_3 \to \frac{1}{2} r_3,\, r_2 \to r_2 - r_3',\, r_1 \to r_1 + r_3'} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{3}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right]$$

$$\xrightarrow{r_1 \to r_1 - r_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & -\frac{3}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right]$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 2 & 0 & 1 \\ -\frac{3}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

3. Use determinants to decide if the following homogeneous system of linear equations has either the trivial solution alone or non-trivial solutions:

$$\begin{aligned} x - 8y + 3z &= 0 \\ -x - 6y + 4z &= 0 \\ 2x - 2y - z &= 0 \end{aligned}$$

Answer part (a) or part (b) as appropriate.

(a) If the system has just the trivial solution, write the system as a matrix equation and explain why the trivial solution is the unique solution.

(b) If the system has non-trivial solutions solve the system using Gaussian elimination, identifying the free variable and writing your solution set in terms of a parameter t.

[12 Marks]

Solution: We calculate

$$\begin{aligned} \det A &= \begin{vmatrix} 1 & -8 & 3 \\ -1 & -6 & 4 \\ 2 & -2 & -1 \end{vmatrix} \\ &= \left[\begin{array}{ccc|cc} 1 & -8 & 3 & 1 & -8 \\ -1 & -6 & 4 & -1 & -6 \\ 2 & -2 & -1 & 2 & -2 \end{array}\right] \\ &= (6 - 64 + 6) - (-36 - 8 - 8) \\ &= 0, \end{aligned}$$

which implies[1] that the linear system has non-trivial solutions.

(b) We put the system into augmented matrix form:

$$\left[\begin{array}{ccc|c} 1 & -8 & 3 & 0 \\ -1 & -6 & 4 & 0 \\ 2 & -2 & -1 & 0 \end{array}\right] \xrightarrow{r_2 \to r_2 + r_1,\, r_3 \to r_3 - 2r_1} \left[\begin{array}{ccc|c} 1 & -8 & 3 & 0 \\ 0 & -14 & 7 & 0 \\ 0 & 14 & -7 & 0 \end{array}\right]$$

$$\xrightarrow{r_2 \to r_2/(-14),\, r_3 \to r_3 + r_2'} \left[\begin{array}{ccc|c} 1 & -8 & 3 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

[1] via $\det A = 0 \Leftrightarrow A$ not invertible

Now we have that the number of variables is three and the number of non-zero rows is two so there must be $3 - 2 = 1$ parameter. There is nothing saying that z must be something in particular so we may let $z = t$ for $t \in \mathbb{R}$ be the parameter/free variable:

$$\begin{aligned} \Rightarrow y &= \frac{1}{2}t \\ \Rightarrow x &= 8y - 3z \\ &= 4t - 3t = t, \end{aligned}$$

so we have a solution set

$$(x, y, z) = \left\{ \left(t, \frac{1}{2}t, t \right) : t \in \mathbb{R} \right\}.$$