Rounding Error Examples

In this section we see the effect of rounding error on your calculation. The lesson I want ye to learn here is to keep as many decimal places in your *intermediate* calculations as possible — only round when you are presenting your answer. If writing down intermediate steps you can write rounded figures but keep the full precision on your calculator.

1. If a loaded beam has a bending moment given by

$$M(x) = \frac{1}{5}x^3 - 9x^2 + 40x,$$

estimate the error in the calculation of the bending moment if we round to one decimal place at

(a) $x = 1$.

(b) the location of the maximum bending moment, $x_{\text{max}} = 15 - 5\frac{\sqrt{57}}{3}$.

Solution:

(a) If we are rounding to one decimal place, then the error in x, $\Delta x = 0.05$. Now we have

$$\Delta M \approx dm = \left|\frac{dM}{dx}\right| \Delta x.$$

Now we differentiate to find

$$\begin{aligned}
\frac{dM}{dx} &= \frac{1}{5}(3x^2) - 18x + 40 \\
&= \frac{3}{5}x^2 - 18x + 40\bigg|_{x=1} \\
&= \frac{3}{5} - 18 + 40 \\
&= 22.6,
\end{aligned}$$

so that the rounding of x to the nearest 0.1 causes an error of

$$\Delta M \approx dM = |22.6|(0.05) = 1.13 \text{ kN m}$$

in the calculation of $M(1)$. In fact, because

$$M(1) = \frac{1}{5}(1)^3 - 9(1)^2 + 40(1) = 40 - 9 + 0.2 = 31.5,$$

the rounding error is of the order of 3%.

(b) Again we have $\Delta M \approx \left|\frac{dM}{dx}\right| \Delta x$. However at the maximum we have $M'(x_{\max}) = 0$ so that

$$\Delta M \approx dm = 0 \cdot (0.05) = 0.$$

Hence rounding is safe in this case.

2. Estimate the error in the calculation of[1]

$$Q = 40(p)^{39}(1-p).$$

if $p \approx 0.98$ is rounded to two decimal places.

Solution: If we are rounding to one decimal place, then the error in p, $\Delta p = 0.005$. Now we have

$$\Delta Q \approx dQ = \left|\frac{dQ}{dp}\right| \Delta p$$

Now we differentiate $Q = Q(p)$ using the product rule

$$\begin{aligned}
\frac{dQ}{dp} &= \overbrace{40p^{39}}^{u}\overbrace{(-1)}^{v'} + \overbrace{(1-p)}^{v}\overbrace{40(39p^{38})}^{u'} \\
&= -40p^{39} + (1-p)(1560p^{38})\Big|_{p=0.98} \\
&= -40(0.98)^{39} + (1-0.98)(1560(0.98)^{38}) \\
&= -3.71262311
\end{aligned}$$

so that the rounding of p to 0.01 causes an error of

$$\Delta Q \approx dQ = |-3.71262311| 0.005 = 0.01856311555 \approx 0.02$$

in the calculation of $Q(0.98)$. In fact, because

$$Q(0.98) \approx 0.36,$$

the rounding error is of the order of 6%!!

3. When doing mortgage payment calculations quantities such as

$$A(P, n, i) = \frac{Pi\left(1 + \frac{i}{12}\right)^n}{\left(1 + \frac{i}{12}\right)^n - 1},$$

occur. By rounding off the $i/12$ term, for e.g. $i = 7\% = 0.07$ say using the approximation

$$0.006 \approx 0.0058\dot{3} = \frac{0.07}{12}$$

[1] we will see quantities such as this in the final chapter

the errors can 'propagate' in significant ways. For example

$$\left(1+\frac{0.07}{12}\right)^{360} \approx 8.1165$$

$$(1+0.006)^{360} \approx 8.61535$$

It doesn't sound too much but for the example of a thirty year mortgage for E300,000 this little but manifests as a difference of more than E40 a month, that is over E485 per year and more than E14,500 over the lifetime of the mortgage. All over a fourth decimal place!! Beware of this so called 'round-off error' and where possible use exact numbers.

Exercises for Rounding Error:

1. Find the approximate affect of rounding a number a close to ten to one decimal place when calculating a^{10}.

2. If a loaded beam has a bending moment given by

$$M(x) = x^3 - 2x^2 + 8x,$$

estimate the error in the calculation of the bending moment if we round to one decimal place at $x = 1$.

3. A student wants to find the square root of a number close to 400 but rounds to the nearest unit. Find the approximate error in his calculation.