MATH7021: Assignment 1 — Linear Systems [73 Marks]

February 19, 2015

- I want you to read the next page carefully. When you start getting down to actually doing the assignment, read the problems carefully
- You should start right away although you have only two full weeks to complete the assignment. The hand in date is 15:00 Friday 6 March. If you have the assignment completed on Thursday hand it to me in class otherwise drop it in A283.
- We have covered everything in class that you need for this assignment.
- Yes you can ask me questions about the assignment before, after and sometimes during class (if ye are working on a problem). Also you can ask me questions via email.
- If you are having problems using Microsoft Excel then you are to email me ASAP and arrange to visit me in my office. You will bring your work on a memory key and we will sort it out
- The assignment is worth 15%. The question on the final paper will be worth *more than* 25% of the 70% — which is 17.5% — on offer on the final paper. Do a good job with this assignment and you should do well on the Linear Systems question on the paper and you might just have more than 32.5% in the bag before we go onto more technical stuff.
- MOST IMPORTANTLY — you are welcome to work together. All of you have different constants so none of you will have the same answers. Therefore you cannot copy each other. The best way to learn is by teaching.
- Work submitted after 15:00 on Friday 6 March will be assigned a mark of zero. Hand up whatever you have on time.

Instructions

When you open *MATH7021A1 - Student Data*, you should see a list of numbers that you are supposed to use in the questions. All of the c_i and P are to be taken as these constants. Everyone has different constants and this is to personalise each of your assignments and allow you to collaborate without copying.

There is also an *Excel* file called *MATH7021A1 - Your Name*. This file, which should be saved as "*MATH7021A1 - Your Actual Name*", will provide evidence of calculations that you do to support your assignment submission and will be emailed to me the same day that you hand up your assignment. I have made a start on the questions that can be done on Excel (and hence this file gives a hint for some questions!). Only Section 4: Partial Pivoting *has* to be done using Excel. You can see examples of Gaussian Elimination done in the Excel file *MATH7021A1 - GE Examples*.

Questions with a * *can* be done with the help of Excel. There is a chance if you do them a strange way you introduce rounding errors so be careful. Outside of Section 4: Partial Pivoting you don't *have* to use Excel if you don't want to. My advice is to do Section 4 first and if you find Excel helpful, use it whenever you can (because it is faster if you are careful). If you struggle with the Excel in Section 4 then you may well want to avoid it in the first three Sections. As far as I am concerned it makes no sense to do Sections 1 to 3 first without Excel and then start Section 4 and you find yourself a whiz on Excel and regret not doing the earlier sections with it.

If you do use Excel to do questions make sure to stretch the cells so that Excel doesn't cheat! If the cells are too narrow, Excel will display 0.5 as 1 for example. I have tried as much as possible to spread out the cells in *MATH7021A1 - Your Name* but be careful!

As before, anyone who has problems with Excel is to contact me ASAP by email so they can be straightened out.

Note that if you are doing Gaussian Elimination by hand you must use exact fractions and square roots rather a decimal approximation. If you are using rounding when doing Gaussian Elimination you have to do pivoting — however I don't want any pivoting done until Section 4. The questions from Sections 1 to 3 that can be done on Excel shouldn't involve any rounding (i.e. they have halves and fifths that are *equal* to 0.5 and 0.2 — as opposed to thirds for example, that are only approximately equal to 0.333).

One other hint: occasionally you will want to make a one and there is no number there! For example:

$$\left[\begin{array}{cc|c} 0 & * & * \\ k & * & * \end{array}\right]$$

with $k \neq 0$. In this case you need to swap rows. This might happen in Question 2 (a) and (c).

I advise that you do the questions out roughly first because small mistakes are inevitable.

1 2×2 Linear Systems

1.1 The Solution Space for a 2×2 Linear System [4 Marks]

A general 2×2 linear system is given by

$$\begin{aligned} a \cdot x + b \cdot y &= e \\ c \cdot x + d \cdot y &= f \end{aligned}$$

where a, b, c, d, e and f are constants and x and y are the variables/unknowns. If $b \neq 0$ and $d \neq 0$ we can write the equations in the form:

$$\begin{aligned} y &= m_1 x + c_1 \\ y &= m_2 x + c_2 \end{aligned}$$

Either *explain geometrically* or *prove* that:

(a) if $m_1 \neq m_2$ then the solution is unique

[1 Mark]

(b) if $m_1 = m_2$ and $c_1 \neq c_2$ then there are no solutions

[2 Marks]

(c) if $m_1 = m_2$ and $c_1 = c_2$ then there are an infinite number of solutions.

[1 Mark]

Note that every student has the same problem here. This gives a good opportunity for collaboration but remember collaboration does not mean one student solving the problem and everyone else copying that student's work. I demand *originality of presentation* here and you should at least understand what you hand up. If you are unsure of what I mean by this please email me immediately as if I have students who have clearly copied the answer *word-for-word* from another student they will all be sharing the marks.

1.2 Illegal 'Row-Operations' [6 Marks]

Consider a 2×2 linear system given by

$$\begin{aligned} x + 2y &= c_1 \\ 3x + 4y &= c_1 + 10 \end{aligned}$$

Written in augmented matrix form this is given by:

$$\left[\begin{array}{cc|c} 1 & 2 & c_1 \\ 3 & 4 & c_1 + 10 \end{array}\right]. \tag{1}$$

(a) Suppose we do the 'row operation' $r_2 \to r_2 \times r_1$:

$$\left[\begin{array}{cc|c} 1 & 2 & c_1 \\ 3 & 8 & c_1^2 + 10c_1 \end{array}\right].$$

This corresponds to the linear system:

$$\begin{aligned} x + 2y &= c_1 \\ 3x + 8y &= c_1^2 + 10 \end{aligned} \tag{2}$$

i. * Using Gaussian Elimination, solve the linear system (1).

[2 Marks]

ii. * Using Gaussian Elimination, *Wolfram Alpha* or some other method, solve the linear system (2).

[1 Mark]

iii. What can you conclude about 'row operations' of the form $r_i \to r_i \times r_j$?

[1 Mark]

(b) Starting with the linear system (1), suppose we do the row operation $r_2 \to r_2 + 1$:

$$\left[\begin{array}{cc|c} 1 & 2 & c_1 \\ 4 & 5 & c_1 + 11 \end{array}\right].$$

This corresponds to the linear system:

$$\begin{aligned} x + 2y &= c_1 \\ 4x + 5y &= c_1 + 11 \end{aligned} \tag{3}$$

i. Using Gaussian Elimination, *Wolfram Alpha* or some other method, solve the linear system (3). This question is *not appropriate* for *Excel*.

[1 Mark]

ii. What can you conclude about 'row operations' of the form $r_i \to r_i + k$?

[1 Mark]

1.3 Cramer's Rule [4 Marks]

Consider the Linear System:

$$\begin{aligned} (c_1+1)\cdot x + (c_2+1)\cdot y &= 5 \\ -c_3\cdot x + (c_4+1)\cdot y &= 10 \end{aligned} \tag{4}$$

(a) Using Gaussian Elimination, solve the linear system (4) for y **only**. This question is *not appropriate* for *Excel*.

[1 Mark]

(b) Solve for y using *Cramer's Rule*.

[2 Mark]

(c) Do your answers to part (a) and (b) agree?

[1 Mark]

2 Gaussian Elimination: Abstract Problems

(a) * Use Gaussian Elimination to solve:

$$\begin{aligned} x + y + z + w &= c_1 \\ x + 2y + z + 2w &= c_2 \\ x + y + z &= c_3 \\ x + 4y + 2z + 3w &= c_4 \end{aligned}$$

Note there is no reason why the entries of the matrix of coefficients nor your solutions should be whole numbers. You can *check* your answer using *Wolfram Alpha*.

[3 Marks]

(b) Use Gaussian Elimination to solve:

$$\begin{aligned} 2x + y + 2z - w &= c_1 \\ 3x + 2y + 4z &= c_2 \\ x + 5y + 2z + w &= c_3 \\ -2x + 3y - 2z + w &= c_3 - c_2 \end{aligned}$$

Note there is no reason why the entries of the matrix of coefficients nor your solutions should be whole numbers. This question is *not* suitable for Microsoft Excel.

[3 Marks]

(c) * Use Gaussian Elimination to solve:

$$\begin{aligned} 4x + 2y + z - 9w &= c_1 \\ 2x + y + w &= c_2 \\ x - z &= 10 \\ x + y + z + w &= c_4 \end{aligned}$$

Note there is no reason why the entries of the matrix of coefficients nor your solutions should be whole numbers.

[3 Marks]

3 Applied Problems

3.1 A Staffing Problem [8 Marks]

On a construction job a firm must employ forty workers. There are three types of workers: Engineers, Skilled & Labourers. Engineers are paid €35,000 per year, Skilled €30,000 per year and Labourers €25,000 per year and the firm has budgeted €1.25 million in wages.

(a) Identify variables x, y and z.

[1 Mark]

(b) Write down a linear system describing the problem.

[2 Marks]

(c) * Solve the linear system, identifying any parameters, if any.

[2 Mark]

(d) Find the maximum number of Labourers and hence the maximum number of Engineers.

[1 Mark]

(e) If guidelines mean that the number of Labourers should be twice the number of Skilled workers, then how many Engineers, Skilled and Labourer workers are there?

[2 Marks]

The warning given at the bottom of Q. 1.1 also applies here.

3.2 A Traffic Flow Problem [10 Marks]

Consider the following traffic flow problem:

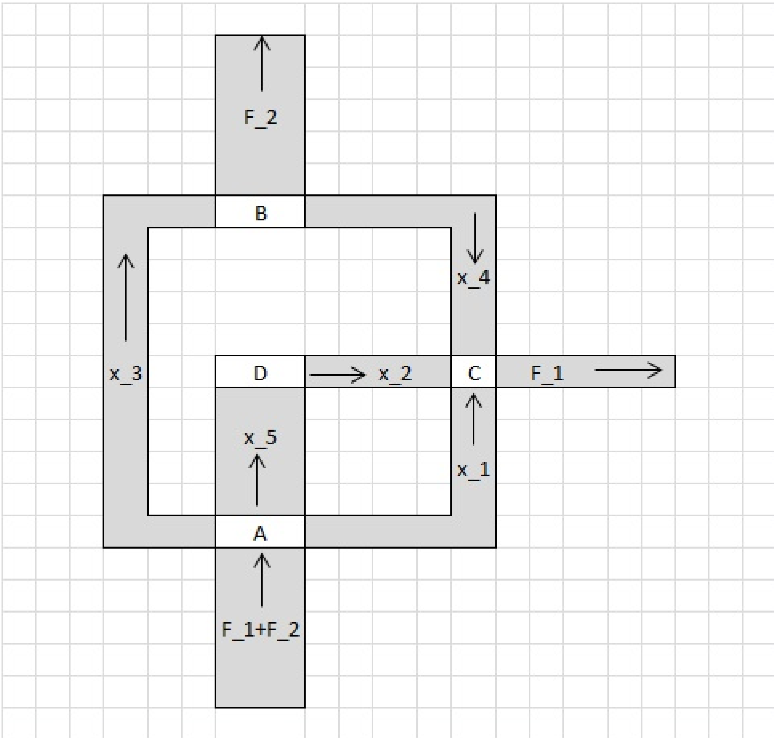

Figure 1: There are three different routes for those going from South at A to East at C. Take $F_1 = 400 + 40c_1$ and $F_2 = 10 + 10c_2$

(a) Write down the linear system governing the traffic flow.

[3 Marks]

(b) * Use Gaussian Elimination to solve the system.

[3 Marks]

(c) If you assume that the speed of traffic is inversely proportional to the traffic flow, then the time taken to travel a route is given by:

$$\text{time} = \text{distance} \times \text{traffic flow}.$$

Take the following distances:

route	distance
$A \to B$	15
$B \to C$	9
$A \to D \to C$	9
$A \to C$ direct	9

If the time taken to travel from $i \to j \to k$ is the same as the time taken to travel $i \to j$ *plus* the time taken to travel $j \to k$, write, in terms of your parameters, expressions for the time taken to travel:

i. $T_1 := A \to B \to C$

ii. $T_2 := A \to D \to C$

iii. $T_3 := A \to C$ direct

[2 Marks]

(d) Investigate if there are values of the parameters such that

$$T_1 = T_2 = T_3.$$

Note that the flows don't have to be whole numbers.

[2 Marks]

3.3 Truss Systems: the Method of Joints [21 Marks]

In this question we investigate the *Method of Joints*. The theory says that if you compare the number of beams and reactions $(b + r)$ and twice the number of joints $(2j)$ you have (with some subtleties I won't go into:)

- $b + r = 2j$ implies that the system is *determinate*
- $b + r > 2j$ implies that the system is *indeterminate*
- $b + r < 2j$ implies that the system is *unstable*

(a) Consider the following truss:

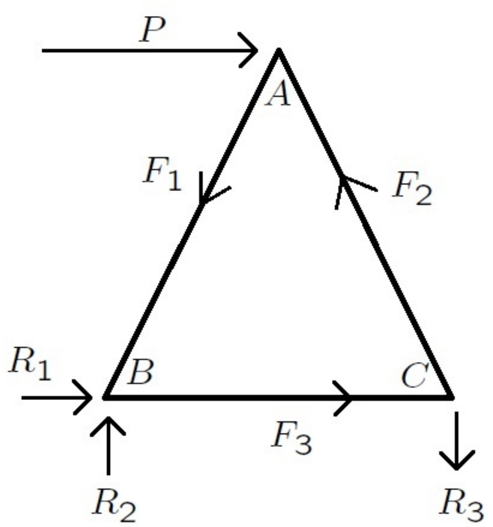

Figure 2: There is an external force of P on the truss as shown. The truss is fixed at B so there are two reactions there. There is a roller at C so only one reaction as shown.

In this case we have $b + r = 3 + 3 = 6$ and $2j = 6$ so that $b + r = 2j$ suggesting that the system is determinate.

The equations governing the forces and reactions, using the method of joints[1], are:

$$\begin{aligned}
\frac{1}{2}F_1 + \frac{1}{2}F_2 &= P \\
\frac{\sqrt{3}}{2}F_1 - \frac{\sqrt{3}}{2}F_2 &= 0 \\
\frac{\sqrt{3}}{2}F_1 - R_2 &= 0 \\
\frac{1}{2}F_1 - F_3 - R_1 &= 0 \\
\frac{1}{2}F_2 - F_3 &= 0 \\
\frac{\sqrt{3}}{2}F_2 - R_3 &= 0.
\end{aligned}$$

If we put this into augmented matrix form (with F_i in the first three columns and the R_i in the last three) and perform Gaussian Elimination we get:

$$\left[\begin{array}{cccccc|c}
1 & 1 & 0 & 0 & 0 & 0 & 2P \\
0 & 1 & 0 & 0 & 0 & 0 & P \\
0 & 0 & 1 & 1 & 0 & 0 & \frac{P}{2} \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & \frac{\sqrt{3}}{2}P \\
0 & 0 & 0 & 0 & 0 & 1 & \frac{\sqrt{3}}{2}P
\end{array}\right]$$

i. Hence, solve the linear system. You can *check* your answer by inputting the original system into *Wolfram Alpha.*

[1 Mark]

ii. Is there a

- unique solution,
- an infinite number of solutions or
- no solutions?

[1 Mark]

[1] If we assume stability, at any joint the sum of the forces is zero... just decompose into horizontal and vertical components

(b) Consider the following truss:

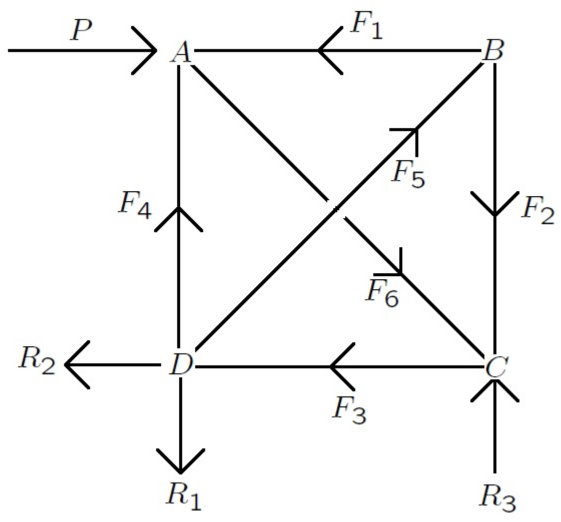

Figure 3: There is an external force of P on the truss as shown. The truss is fixed at D so there are two reactions there. There is a roller at C so only one reaction as shown. The middle beams are not pinned to each other.

In this case we have $b + r = 6 + 3 = 9$ and $2j = 8$ so that $b + r > 2j$ suggesting that the system is indeterminate.

The equations governing the forces and reactions, using the method of joints is:

$$\begin{aligned}
R_1 - F_4 - \frac{1}{\sqrt{2}} F_5 &= 0 \\
R_2 + F_3 - \frac{1}{\sqrt{2}} F_5 &= 0 \\
R_3 - F_2 - \frac{1}{\sqrt{2}} F_6 &= 0 \\
F_2 - \frac{1}{\sqrt{2}} F_5 &= 0 \\
F_1 - \frac{1}{\sqrt{2}} F_5 &= 0 \\
F_3 - \frac{1}{\sqrt{2}} F_6 &= 0 \\
F_4 - \frac{1}{\sqrt{2}} F_6 &= 0 \\
F_1 - \frac{1}{\sqrt{2}} F_6 &= P
\end{aligned}$$

If we put this into augmented matrix form (with the R_i in the first three columns and the F_i in the last six) we get:

$$\left[\begin{array}{ccccccccc|c}
1 & 0 & 0 & 0 & 0 & 0 & -1 & -\frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & +\frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & P
\end{array}\right]$$

i. The first seven rows are in reduced form. Apply the following elementary row operations to the last row to put the system in reduced row form:

$$r_8 \to r_8 - r_4\,, \text{ and then } r_8 \to r_8 \times \sqrt{2}.$$

This question is *not* suitable for Microsoft Excel.

[3 Marks]

ii. Solve the linear system.

[2 Marks]

ii. Is there a

- unique solution,
- an infinite number of solutions or
- no solutions?

[1 Mark]

(c) Consider the following truss:

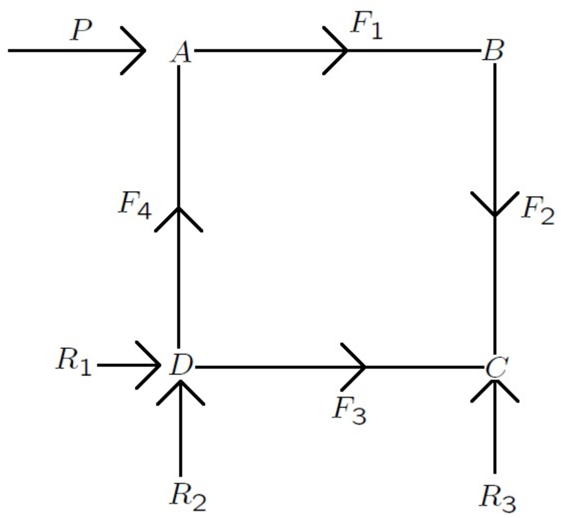

Figure 4: There is an external force of P on the truss as shown. The truss is fixed at D so there are two reactions there. There is a roller at C so only one reaction as shown.

In this case we have $b + r = 4 + 3 = 7$ and $2j = 8$ so that $b + r < 2j$ suggesting that the system is unstable.

The equations governing the forces and reactions, using the method of joints, when put in augmented matrix form (first the F_i and then the R_i) are:

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -P \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

i. * Apply the following elementary row operations to put the system in (almost) reduced row form[2]:

$$r_8 \to r_8 - r_1\,,\ r_3 \to r_3 - r_2\,,\ r_5 \to r_5 - r_4\,,\ r_7 \to r_7 - r_6.$$

[2 Marks]

ii. What can you conclude?

[2 Marks]

iii. What assumption was made to allow us to write down the equations governing the truss system?

[2 Marks]

iv. In this case we have

"assumption" implies "equations" implies "nonsense".

Therefore the "assumption" must have been incorrect. Hence, comment on the stability of the truss.

[2 Mark]

(d) Complete the following. I use the word *suggests* because things aren't always as straightforward as the above three examples. Please see the Microsoft Word Document *MATH7021A1 - Complete Me* for a digital version.

A truss system composed of b ____ and r ____ subject to an external ____ has $b+r$ ____.

Using the method of ____, for each ____ in a truss system we can write down two ____. Therefore if there are j joints in the truss system the associated linear system has $2j$ ____.

If $b+r = 2j$, then the number of ____ equals the number of ____. This suggests that when the linear system is brought into row-reduced form we have a ____ solution and so the truss system is determinate.

If $b + r > 2j$ then the number of ____ is greater than the number of ____. This suggests that when the linear system is brought into row-reduced form we have that the number of ____ is greater than the number of non-zero rows. We know from class that the number of ____ is equal to the number of ____ less the number of ____ rows so in this case we have at least one ____ so there are an ____ number of solutions. In this case the truss system is said to be indeterminate.

If $b+r < 2j$ then the number of ____ is less than the number of ____ so that when the linear system is brought into row-reduced form there is a chance that there are ____ solutions. In this case the assumption that the truss system is ____ leads to an absurdity so that we must conclude that the truss system is ____.

[5 Marks]

4 Partial Pivoting [11 Marks]

4.1 Abstract Example

Using three-decimal-place rounding, use *Microsoft Excel* to solve the linear system:

$$\begin{pmatrix} 1.80 & 2.52 & 4.50 \\ 1.60 & 5.12 & 5.44 \\ 2.50 & 6.50 & 9.25 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

[3 Marks]

4.2 Heat Distribution Problem

A plate is in the ratio 3 : 3. The top-boundary of the plate has a temperature of P, the right-boundary has a temperature of $P/10$, the left boundary has a temperature of $P/5$ and the bottom-boundary has a temperature of $2P$.

(a) Using an appropriate grid, find a linear system whose solution approximates the heat distribution of the plate at internal grid-points using the Mean-Value Property.

[3 Marks]

(b) Use *Microsoft Excel* to do Gaussian Elimination with partial pivoting to approximately solve the linear system. Use three places of decimals.

[3 Marks]

(c) Use *Wolfram Alpha* to find the exact solution of the linear system derived in part (a). What was the biggest discrepancy between the exact solution and the approximate solution found in part (b).?

[2 Marks]