MATH7016: 20% Written Assessment 1 [Ex 60 Marks]

Name:

1. Consider an initial value problem

$$\frac{dy}{dt} = F(t, y)\,; \;\; y(t_0) = y_0.$$

Give a reason why a numerical method might have to be used to approximate values of $y(t)$.

[2 Marks]

Solution:

2. Consider the following:

> the displacement, $x(t)$, of a of a body falling under gravity subject to a drag force proportional to the speed-squared. The initial displacement and initial speed are both zero.

Write this second order initial value problem as an equivalent system of first order initial value problems.

[7 Marks]

Solution:

3. Calculate the first four terms of the Maclaurin Series of $y(x) = e^x$.

[4 Marks]

Solution:

4. Consider an initial value problem

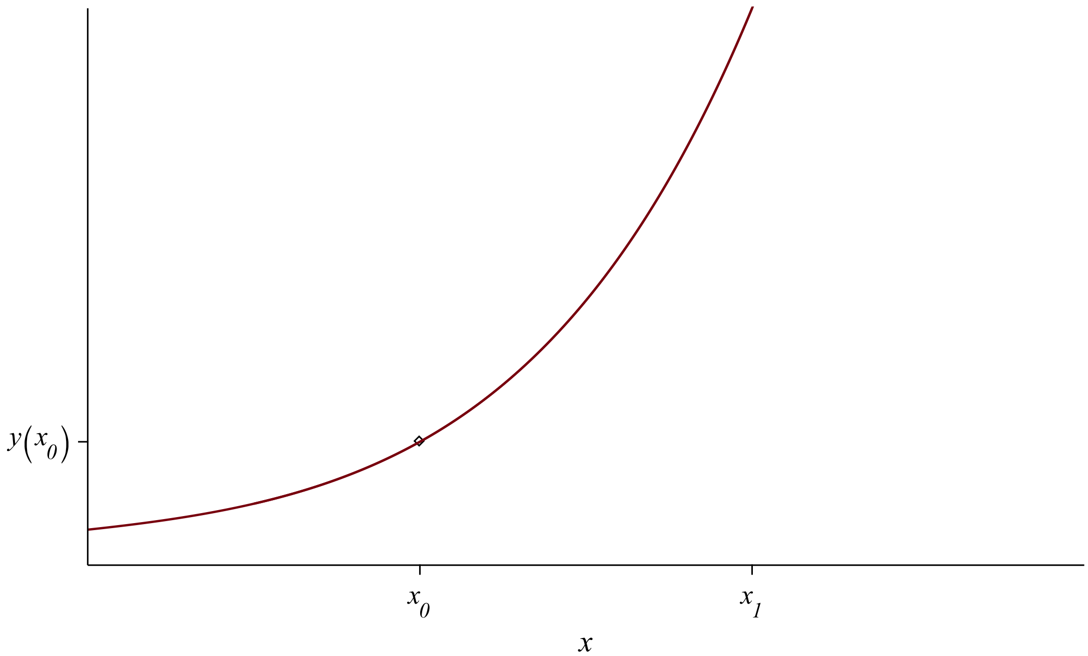
$$\frac{dy}{dx} = F(x, y)\,; \;\; y(x_0) = y_0.$$

Euler's Method uses the tangent to the curve at $x = x_0$ to approximate the y-value at $x = x_1 = x_0 + h$.

(a) Use
$$y - y_1 = m(x - x_1) \tag{1}$$
to find the equation of the tangent at $x = x_0$ in terms of x_0, y_0 and $F(x_0, y_0)$.

(b) Find the value of y at $x = x_0 + h$ in terms of $y(x_0)$, h, and $F(x_0, y_0)$.

(c) Draw the tangent at $(x_0, y(x_0))$, $y(x_1)$ as well as y_1. Also show the error, $\Delta y = |y(x_1) - y_1|$.



(d) Show, with the aid of a diagram, how a large second derivative causes problems for Euler's Method.

[11 Marks]

Solution:

Q. 3 Solution Continued:

5. Consider an initial value problem, with the derivative independent of y:

$$\frac{dy}{dx} = F(x)\,; \;\; y(x_0) = y_0.$$

Using Heun's Method with a step-size of h, it can be shown that an upper bound for the *local error*:

$$\varepsilon = \frac{\|y'''\|_{\max} h^3}{12},$$

where $\|y'''\|_{\max}$ is the maximum of the third derivative between $x = x_0$ and $x = x_1$.

(a) What does it mean to say that the local error is $\mathcal{O}(h^3)$?

(b) Produce a *rough* argument that shows if we use Heun's Method to approximate $y(x_n) = y(x_0 + n \cdot h)$, that the *global error* is $\mathcal{O}(h^2)$.

(c) What is the effect on the global error if we quarter the step-size?

[6 Marks]

Solution:

6. Consider an initial value problem

$$\frac{dy}{dt} = -0.06\sqrt{y(t)}\,; \quad y(0) = 3.$$

This models the height, y (in metres), of liquid in a cylindrical tank a time t (in minutes) after opening a valve at the base.

(a) Use Heun's Method with a step-size of 0.5 to approximate $y(1.5)$.

(b) If the exact solution is given by $y(t) = (\sqrt{3} - 0.03t)^2$, find the error in the approximation.

[14 Marks]

Solution:

7. Consider the initial value problem

$$\frac{d^2\theta}{dt^2} + 0.1\sin(\theta(t)) = 0\,; \;\; \theta(0) = \pi/4\,, \;\; \theta'(0) = 0.$$

This models the angular displacement, θ (in radians), of a pendulum a time t (in seconds) after being released from rest from a $45°$ C angle.

Use Euler's Method with a step-size of 0.5 to approximate $\theta(1)$.

[16 Marks]

Solution:

Rough Work:

Useful Formulae

A tables page will also be provided.

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \cdots$$

$$y(x) = y(a) + y'(a)(x-a) + \frac{y''(a)}{2!}(x-a)^2 + \frac{y'''(a)}{3!}(x-a)^3 + \cdots$$

$$y_{i+1} = y_i + h \cdot F(x_i, y_i)$$

$$y_{i+1} = y_i + h \cdot y'_i + \frac{h^2}{2} \cdot y''_i$$

$$\begin{aligned} y^0_{i+1} &= y_i + h \cdot F(x_i, y_i) \\ y_{i+1} &= y_i + h \cdot \frac{F(x_i, y_i) + F(x_{i+1}, y^0_{i+1})}{2} \end{aligned}$$

Runge-Kutta Notation

$$y_{i+1} = y_i + k_1 \cdot h$$

where

$$k_1 = F(x_i, y_i)$$

$$y_{i+1} = y_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right) \cdot h$$

where

$$\begin{aligned} k_1 &= F(x_i, y_i) \\ k_2 &= F(x_i + h, y_i + k_1 h) \end{aligned}$$